1-HYPERBOLIC GRAPHS

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Abstract. The shortest-path metric of a graph $G = (V, E)$ is called *δ-hyperbolic* if for any four vertices $u, v, w, x \in X$ the two larger ones of the three sums $d(u,v)+d(w,x), d(u,w)+d(v,x), d(u,x)+d(v,w)$ differ by at most δ. In this note, we characterize the graphs with 1-hyperbolic metrics in terms of a convexity condition and forbidden isometric subgraphs.

Key words. trees, graphs, 4-point condition, hyperbolicity.

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Introduction. It is well known that a metric space (X, d) embeds into a tree network (with positive real edge lengths), that is, d is a *tree metric,* if and only if the "classical" 4-point condition holds, that is, for any four points u, v, w, x the two larger ones of the sums

$$d(u,v) + d(w,x), \quad d(u,w) + d(v,x), \quad d(u,x) + d(v,w) \tag{1}$$

are equal; cf. [1, 6] for a comprehensive bibliography. In the case that d is the shortest-path metric of a graph G, this condition is satisfied exactly when G is a block graph; see [4, 16].

Every 4-point metric d (tree-realizable or not) has a canonical representation in the rectilinear plane; cf. [2, 10]. For Fig. 1 it is stipulated that the three distance sums in the list (1) are ordered from small to large, thus implying $\xi \leq \eta$. Then η is half the difference of the largest and the smallest sum, while ξ is half the largest minus the medium sum. In data analysis, the ratio ξ/η would thus (locally) measure the deviation from tree-likeness; cf. [11].

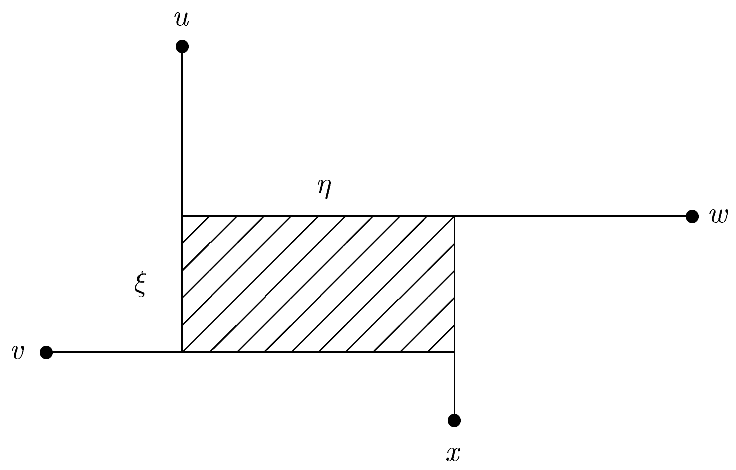

FIG 1. Realization of a 4-point metric in the rectilinear plane.

For graph metrics the deviation from a tree metric may be measured directly by ξ (rather than ξ/η) since ξ can only take values from $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. More specifically, following a general notion of δ-hyperbolic metric spaces due to M. Gromov, we say that the metric d of a graph G is *δ-hyperbolic* (or *tree-like with defect at most δ*) if and only if the difference 2ξ between largest and medium distance sums for any four vertices u, v, w, x does not exceed δ, that is,

$$(\beta_\delta) \qquad \begin{gathered} d(u,v) + d(w,x) \leq d(u,w) + d(v,x) \leq d(u,x) + d(v,w) \\ \text{implies } d(u,x) + d(v,w) - d(u,w) - d(v,x) \leq \delta. \end{gathered}$$

δ-Hyperbolic metric spaces play an important role in geometric group theory; see for example [12, 13, 14].

In this note, we characterize the graphs with 1-hyperbolic metrics, i.e., the graphs satisfying condition (β_1). All graphs $G = (V, E)$ occuring here are simple, connected, without loops or multiple edges, but not necessarily finite. We will write $u \sim v$ if the vertices u and v are adjacent in G. A shortest path between two vertices x, y of a cycle C of G is called a *bridge* of C if its length is smaller than the distance between x and y measured along C. A cycle C is called *well-bridged* if for any vertex $x \in C$ either there exists a bridge from x to some vertex of C or two neighbours of x from C are adjacent. We can now state the main result.

Theorem. *A graph G is 1-hyperbolic if and only if all cycles $C_n, n \neq 5$, of G are well-bridged and none of the graphs in Fig. 2 occur as isometric subgraphs of G :*

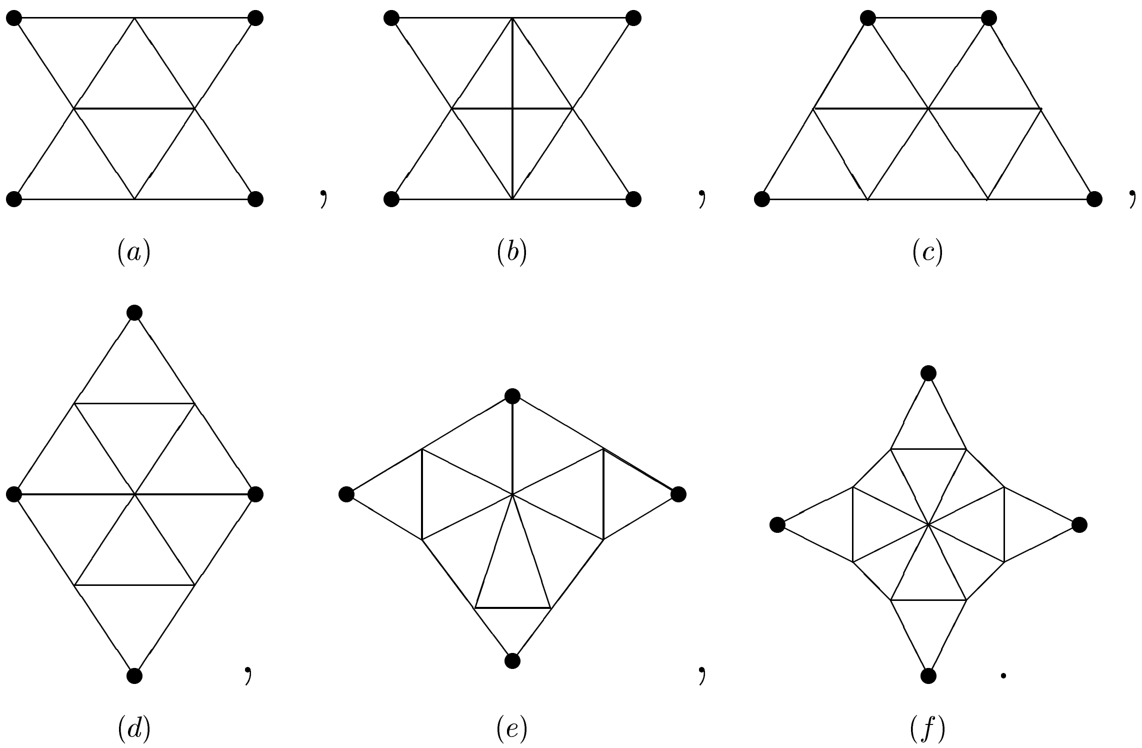

Fig. 2. Forbidden isometric subgraphs.

Since bridged graphs [15] have convex balls and thus satisfy the equivalent conditions of Fact 1, the Theorem covers the characterization of 1-hyperbolic bridged graphs given by Koolen and Moulton [17]. In the case of bridged graphs, the proof of the Theorem simplifies in that only "Case 1" of that proof needs to be considered.

Preliminaries. In this section, we recall some results used in the proof of the Theorem. The *distance* $d(u,v)$ between two vertices u, v of a graph G is the length of any shortest (u,v)-path. A subset A of vertices of G is *convex* if the *interval*

$$I(u,v) = \{x \in X : d(u,x) + d(x,v) = d(u,v)\}$$

between any two vertices u and v of A lies entirely in A. The *ball* $B_k(x)$ of center x and radius $k \geq 0$ consists of all vertices of G at distance $\leq k$ from x.

FACT 1. [15, 19] *For a graph G, the following conditions are equivalent:*

(i) *all balls of G are convex;*

(ii) *all cycles C_n, $n \neq 5$, of G are well-bridged;*

(iii) *G does not contain isometric cycles of length $n > 5$, and for any two vertices x, y the neighbours of x from the interval $I(x, y)$ are pairwise adjacent.*

Next we consider another condition involving four points, which is related to (β_i) for $i \leq 1$:

(α_i) if $v \in I(u, w)$ and $w \in I(v, x)$ such that v, w are adjacent,
then $d(u, x) \geq d(u, v) + d(w, x) + 1 - i$.

Conditions (α_i) and (β_j) come up with a near-isometric embedding φ of a graph G (with shortest-path metric d) into a tree network T (with metric d'), where the absolute error is bounded by some $\epsilon > 0$, that is,

$$|d(u, v) - d'(\varphi(u), \varphi(v))| \leq \epsilon$$

for all vertices u, v of G. If G admits such a representation, then it satisfies (α_i) and (β_j) with $i = \lfloor 3\epsilon \rfloor$ and $j = \lfloor 4\epsilon \rfloor$. Near-isometric tree embeddings were investigated in [7, 9], where it is shown that the error bound ϵ can be chosen to be 2 for chordal graphs and otherwise be bounded by a polynomial linear in the maximum length of induced cycles. A variant of this embedding concept would additionally require that $d' \leq d$; in this case (α_i) and (β_i) would hold with $i = \lfloor 2\epsilon \rfloor$.

Condition (α_0) was first introduced in [18] for metric spaces; in the case of graphs it characterizes the Ptolemaic graphs [8]. All chordal graphs satisfy (α_1) [8], but the latter class is in fact larger:

FACT 2. [20] *A graph G satisfies (α_1) if and only if all cycles C_n, $n \neq 5$, are well-bridged and G does not contain the graph of Fig. 2c as an isometric subgraph.*

It is clear by Facts 1 and 2 that all balls in graphs are convex provided (α_1) holds. On the other hand, condition (β_1) implies (α_1). Indeed, let v, w be adjacent vertices such that $v \in I(u, w)$ and $w \in I(v, x)$. The distance sums $d(u, w) + d(v, x)$ and $d(u, v) + d(w, x)$ differ by 2. If (β_1) is fulfilled, then necessarily $d(u, x) + d(v, w) \geq d(u, w) + d(v, x) - 1$. Therefore $d(u, x) \geq d(u, v) + d(w, x)$, hence (α_1) is verified. Consequently, in view of Fact 2, we can reformulate the characteristic condition in the Theorem in terms of (α_1) and five forbidden isometric subgraphs:

COROLLARY. *A graph G is 1-hyperbolic if and only if it satisfies (α_1) and none of the graphs of Fig. 2 a,b,d,e,f occur as isometric subgraphs.*

Three not necessarily distinct vertices u, v, w of a graph G are said to form a *metric triangle* uvw of type (k_1, k_2, k_3) if the intervals $I(u, v), I(v, w)$, and $I(w, u)$ pairwise intersect only in the common end vertices and $d(u, v) = k_1, d(v, w) = k_2, d(w, u) = k_3$. If $k_1 = k_2 = k_3 = k$, then this metric triangle is called *equilateral* of *size* k; when $k = 0$ this metric triangle is *degenerate,* consisting of a single vertex. A metric triangle uvw is a *quasi-median* of the triplet x, y, z if the following distance equations are satisfied:

$$d(x, y) = d(x, u) + d(u, v) + d(v, y),$$
$$d(y, z) = d(y, v) + d(v, w) + d(w, z),$$
$$d(z, x) = d(z, w) + d(w, u) + d(u, x).$$

If this quasi-median is an equilateral triangle, let $q(x, y, z)$ denote the size of the metric triangle uvw. Note that in [5] quasi-medians were defined more restrictively, by admitting only equilateral triangles. If the size $q(x, y, z)$ equals zero, then the (degenerate) quasi-median (triangle) is also referred to as a *median* (vertex).

We conclude this section with a somewhat technical observation, but which is quite useful in the proof of the Theorem.

LEMMA 1. *Given $1 \leq k \leq l$, let u, v, w, x be vertices of a graph G such that*

$$d(u, v) + d(w, x) + j = d(u, w) + d(v, x) + i = d(u, x) + d(v, w),$$

for which the sum of all distances between u, v, w, x is minimal relative to the constraints $i \geq k$ and $j \geq l$. If all quasi-medians of these vertices are equilateral triangles, then

$$d(u, v) + d(u, w) = d(v, w) + q(u, v, w),$$
$$d(u, w) + d(w, x) = d(u, x) + q(u, w, x),$$
$$d(u, v) + d(v, x) = d(u, x) + q(u, v, x),$$
$$d(v, x) + d(w, x) = d(v, w) + q(v, w, x),$$

whence

$$q(u,v,w) + q(u,w,x) = 2d(u,w) - j,$$
$$q(u,v,w) + q(u,v,x) = 2d(u,v) - i,$$
$$q(v,w,x) + q(u,w,x) = 2d(w,x) - i,$$
$$q(v,w,x) + q(u,v,x) = 2d(v,x) - j,$$

and finally

$$q(u,v,w) + q(u,v,x) + q(u,w,x) + q(v,w,x) = 2[d(u,x) + d(v,w) - i - j].$$

Proof. Suppose the first of the asserted equalities fails. Then necessarily we find a neighbour u' of u in $I(u,v) \cap I(u,w)$. Substituting u by u' reduces the smaller two distance sums for u, v, w, x by 1 and the largest one by at most 1, while the sum of all distances between u, v, w, x decreases, contrary to the minimality assumption. Since we may interchange either u and x or v and w or $\{u, x\}$ with $\{v, w\}$, we obtain the subsequent three equations by symmetry. Adding up the first two equations yield the asserted equality for $q(u,v,w) + q(u,w,x)$, and so forth. The final equation is obtained from the sum of the second set of equations. □

Properties of graphs satisfying (α_1)**.** Throughout this section, G is a graph in which (α_1) holds.

LEMMA 2. *If uvw is a metric triangle of G with $d(u,v) = 1$, then $d(w,u) = d(w,v) \leq 2$.*

Proof. Let $u' \in I(w,u)$ and $v' \in I(w,v)$ be neighbours of w. From the initial hypothesis we conclude that $u \in I(u',v)$ and $v \in I(v',u)$. Applying (α_1) we obtain

$$d(u',v') \geq d(u',u) + d(v,v') = d(u,w) + d(v,w) - 2 = 2d(u,w) - 2.$$

Since $d(u',v') \leq 2$, it follows immediately that $d(u,w) \leq 2$ as well. □

LEMMA 3. *If $x, y \in I(u,v)$ are neighbours of u, then $x \sim y$. Furthermore, x and y have a common neighbour v' one step closer to v.*

Proof. Let $d(u,v) = k+1$. Since $x, y \in B_k(v), u \notin B_k(v)$, and the ball $B_k(v)$ is convex, necessarily x and y are adjacent. Let $v' \in I(x,v) \cap I(y,v)$ be closest to x and y, thus giving a metric triangle xyv'. From Lemma 2 we know that $d(x,v') = d(y,v') \leq 2$. If v' is adjacent to x and y, we are done. Now suppose $d(x,v') = d(y,v') = 2$. Denote by x' and y' some common neighbours of v', x and v', y, respectively. Since $x', y' \in I(v', v)$, the vertices x' and y' are adjacent. We obtain a 4-cycle (x, y, y', x'), which must have at least one diagonal, say $x' \sim y$. Then $x' \in I(x,v) \cap I(y,v)$, contrary to the choice of v'. □

LEMMA 4. *Every metric triangle uvw of G is of type* $(1,1,1), (1,2,2), (2,1,2), (2,2,1)$, *or* $(2,2,2)$.

Proof. Assume $d(u,v) \leq d(v,w) \leq d(w,u)$. We proceed by induction on $k = d(u,v) + d(v,w) + d(w,u)$. The case $d(u,v) = 1$ is covered by Lemma 2. We distinguish two further cases.

Case 1: $d(u,v) = 2$.

Assume by way of contradiction that $d(u,w) \geq 3$. Pick a common neighbour x of u and v. If $d(x,w) > d(u,w)$, then (α_1) yields $d(v,w) \geq d(w,u) + 1$, contrary to the maximality of $d(u,w)$. Therefore $d(x,w) = d(u,w) \geq 3$. By Lemma 2 we can find a neighbour w' of w such that $w' \in I(w,u) \cap I(w,x)$. If $d(x,w) = d(v,w)$, there exists another neighbour $w'' \in I(w,v) \cap I(w,x)$ of w. On the other hand, if $d(v,w) < d(x,w)$, then $v \in I(x,w)$, hence every neighbour $w'' \in I(w,v)$ of w belongs to $I(w,x)$. In both cases, we have found a vertex $w'' \in I(w,v) \cap I(w,x)$ adjacent to w and distinct from w'. Since $w', w'' \in I(w,x)$, Lemma 3 yields $w' \sim w''$. Now, observe that $w' \in I(w'',u)$ and $w'' \in I(w',v)$. By condition (α_1), we have

$$2 = d(u,v) \geq d(u,w') + d(w'',x) = d(u,w) + d(v,w) - 2,$$

whence $d(u,w) = 2$, contrary to our assumption.

Case 2: $d(u,v) \geq 3$.

Pick a neighbour x of w in the interval $I(u,w)$. If $d(x,v) > d(w,v)$, then (α_1) yields $d(u,v) \geq d(u,w) + d(v,w) - 1 > d(u,w)$, contrary to the maximality of $d(u,w)$. Thus $d(x,v) = d(w,v)$. Since this distance exceeds 2, we can find a neighbour y of v in $I(v,w) \cap I(v,x)$ by virtue of Lemma 2. For a quasi-median $uv'x'$ of the triplet u, v, x, we obtain

$$\begin{aligned} (2) \qquad & d(u,w) - d(u,x') - 1 + d(x',v') + d(u,v) - d(u,v') \\ &= d(x,x') + d(x',v') + d(v',v) = d(x,v) = d(w,v) \leq d(u,w), \end{aligned}$$

and therefore

$$d(u,v) \leq d(u,x') + d(u,v') - d(x',v') + 1. \tag{3}$$

We can then apply the induction hypothesis to the metric triangle $uv'x'$ because

$$3 \leq d(u,v') + d(v',x') + d(x',u) \leq d(u,v) + d(v,x) + d(x,u) < k.$$

Using this hypothesis and the inequality $d(u,v) \geq 3$, from (3) we deduce that $d(x',u) = d(u,v') = 2$ and

$$4 \leq d(u,v) + d(v',x') \leq 5.$$

If the upper bound 5 is attained, then we infer from (2) that uvw is equilateral (of size ≥ 3). Pick a neighbour y' of v in $I(v,v')$, which is necessarily distinct from y. Since $y, y' \in I(x,v)$, Lemma 3 implies $y \sim y'$. Since uvw is a metric triangle, $d(y',w) > d(y,w)$ and $d(y,u) > d(y',u)$. By condition (α_1) we thus infer

$$d(u,w) \geq d(u,y') + d(y,w) = d(u,v) + d(v,w) - 2 > d(v,w),$$

a contadiction.

Finally, if the lower bound 4 is attained above, then $x' \sim v' \sim v$ and hence $v' \sim y$ by Lemma 3 applied to $I(v,x)$. Moreover,

$$d(v',w) = d(v,w) = d(w,u) - 1$$

by (2), and therefore

$$d(v',w) - 1 = d(y,w) = d(w,x').$$

Consequently, $x' \sim y$ by Lemma 3. Since $uv'x'$ is a metric triangle, any neighbours $v'' \in I(u,v')$ and $x'' \in I(u,x')$ of u are different. From $v'', x'' \in I(u,y)$ and Lemma 3 it follows that v', x', x'', v'' constitutes a 4-cycle. By condition (α_1) either $v' \sim x''$ or $x' \sim v''$, which however is impossible. □

LEMMA 5. *Let u, v, w, x be vertices of G such that $u \in I(v,w), v \in I(u,x)$, and $u \sim v$. Then the equality $d(w,x) = d(w,u) + d(x,v)$ holds if and only if there exist vertices $u' \in I(u,w)$ and $v' \in I(v,x)$ such that $u' \sim u$ and $v' \sim v$ as well as $d(u',v') = 2$; in particular, u' and v' lie on a common shortest path between w and x.*

Proof. If the condition if fulfilled, then the equality holds trivially. Now assume the converse and, without loss of generality, $I(w,x) \cap I(w,u) = \{w\}$ and $I(x,w) \cap I(x,v) =$

$\{x\}$. We wish to show that $d(u,w) = d(v,x) = 1$, which settles the proof. Observe that $I(u,x) \cap I(u,w) = \{u\}$ and $I(v,w) \cap I(v,x) = \{v\}$ since

$$d(w,u) + d(u,x) = d(w,v) + d(v,x) = d(w,x) + 1.$$

Therefore the quasi-medians of the triplets u,w,x and v,w,x have the form uwx' and $vw'x$, respectively. From Lemma 4 we know that $d(u,w) \leq 2$ and $d(v,x) \leq 2$. Suppose by way of contradiction that $d(u,w) + d(v,x) \geq 3$.

First, let $d(u,w) = 1$ and $d(v,x) = 2$, so that $d(w,x) = 3$. Then any quasi-median $vw'x$ of v,w,x is a metric triangle of type $(1,2,2)$, whence $v \sim w' \sim w$. Since $u,w' \in I(w,v)$, Lemma 3 yields $u \sim w'$. Therefore $v,w' \in I(u,x)$ have a common neighbour x' with x by Lemma 3. As a consequence x' belongs to $I(x,w) \cap I(x,v)$, contrary to our initial assumption.

Finally, let $d(u,w) = d(v,x) = 2$, so that $d(x,w) = 4$. Then any quasi-median $vw'x$ of v,w,x is of type $(1,2,2)$. Therefore $u,w' \in I(v,w)$ have a common neighbour w'' with w, thus conflicting with the initial assumption on $I(w,x) \cap I(w,u)$. $\square$

Proof of the Theorem. Each of the six forbidden graphs violates the condition (β_1), as indicated in Fig. 2. Since (β_1) implies (α_1), from Fact 2 we deduce that in a 1-hyperbolic graph all cycles $C_n, n \neq 5$, are well-bridged.

Conversely, assume that G is a graph whose all cycles of lenght $\neq 5$ are well-bridged and none of the graphs from Fig. 2 occurs as an isometric subgraph. By Fact 1, all balls of G are convex, and by Fact 2, G obeys condition (α_1). Suppose by way of contradiction that condition (β_1) is violated. Select a quartet u,v,w,x minimizing the total distance sum

$$\Sigma(u,v,w,x) = d(u,x) + d(v,w) + d(u,v) + d(u,w) + d(v,x) + d(w,x).$$

Then, say, we have

$$d(u,v) + d(w,x) + j = d(u,w) + d(v,x) + i = d(u,x) + d(v,w), \tag{4}$$

with $i \geq 2$ and $j \geq 2$. As in the proof of Lemma 1, the minimality of $\Sigma(u,v,w,x)$ guarantees that

$$\begin{aligned} I(u,v) \cap I(u,w) = \{u\}, \quad & I(v,u) \cap I(v,x) = \{v\}, \\ I(w,u) \cap I(w,x) = \{w\}, \quad & I(x,v) \cap I(x,w) = \{x\}. \end{aligned} \tag{5}$$

We distinguish two main cases.

Case 1: All quasi-medians of triplets from u, v, w, x are equilateral metric triangles.

If all triplets actually have medians, then u, v, w, x would induce a 4-cycle by Lemma 1, which however is impossible in view of (α_1). Therefore assume without loss of generality that some quasi-median of the triplet u, v, w is an equilateral metric triangle $uv'w'$ of size $k \in \{1, 2\}$. We assert that $i = j = 2$. Suppose by way of contradiction that $d(u,w) + d(v,x) > d(u,v) + d(w,x)$. Pick a neighbour u' of u in the interval $I(u, w')$. Then the ball $B_k(v')$ includes u, w' and thus u' by convexity, whence $d(v, u') = d(v, u)$ by (5). The quartet u', v, w, x also violates condition (β_1), therefore $\Sigma(u', v, w, x) \geq \Sigma(u, v, w, x)$, which implies $d(u', x) > d(u, x)$. By (α_1), we have $d(x, w) \geq d(u, x) + d(u, w) - 1$. Since $d(v, w) = d(v, u) + d(u, w) - k$, we obtain $d(x, w) + d(u, v) - k \geq d(u, x) + d(v, w) - 1$, which contradicts (4) because $k \leq 2$. This proves the assertion $i = j = 2$. From Lemma 1 we immediately infer that the distances $d(u, v), d(u, w), d(x, v), d(x, w)$ are no larger than 3.

If $q(u, v, w) = 1$, then Lemma 1 implies that $q(u, v, x), q(u, w, x), q(v, w, x)$ must be odd and thus equal to 1 as well, whence $d(u, w) = d(u, v) = d(w, x) = d(v, x) = 2$. Let $uv'w'$ be a quasi-median of u, v, w and $xv''w''$ be a quasi-median of x, v, w. Since $d(u, x) = d(v, w) = 3$, the vertices v', w', v'', w'' are pairwise different and belong to the interval $I(v, w)$. Then, by Lemma 3 and (α_1), these four vertices induce a 4-cycle with at least one diagonal, so that together with u, v, w, x they induce the graph of Fig. 2a or 2b as an isometric subgraph, contrary to the initial assumption.

Therefore the quasi-median $uv'w'$ of u, v, w has size 2. Pick vertices t and y with $u \sim t \sim v'$ and $u \sim y \sim w'$. Then $d(v', y) = d(w', t) = 2$ because $B_2(v')$ and $B_2(w')$ are convex. If $d(t, x) > d(u, x)$, then (α_1) implies

$$\begin{aligned} d(v,x) &\geq d(v,t) + d(u,x) \\ &= d(u,v) - 1 + d(u,w) + d(v,x) + 2 - d(v,w) \text{ by } (4) \\ &= d(v,x) + 3, \end{aligned}$$

a contradiction. Therefore $\Sigma(t, v, w, x) < \Sigma(u, v, w, x)$, whence the former quartet satisfies (β_1), yielding $t \in I(u, x)$. Analogously, $y \in I(u, x)$ holds, so that, by Lemma 3, t and y are adjacent and have a common neighbour z such that $d(x, z) = d(u, z) - 2$. If $d(v, t) < d(v, z)$, then (α_1) implies (as above)

$$d(v,x) \geq d(v,t) + d(z,x) = d(v,t) + d(u,x) - 2 = d(v,x) + 1,$$

a contradiction. We conclude that both t and z belong to $I(v, y)$ and have a common neighbour r such that $d(r, v) = d(u, v) - 2$. Similarly, there exists a vertex s with $y \sim s \sim z$

and $d(s,w) = d(u,w) - 2$. Then urs is a quasi-median of u,v,w, and we may thus assume $v' = r$ and $w' = s$. The six vertices t,u,v',w',y,z induce what is called a 3-*sun* [7]. Without loss of generality we may assume

$$d(z,x) \leq d(w',x) \leq d(v',x) \leq d(z,x) + 1,$$

so that three cases have to be distinguished.

Subcase 1.1: $d(z,x) = d(v',x)$.

Since $v',z \in I(t,x)$ and $w',z \in I(y,x)$, we find vertices $x',x'' \in I(z,x)$ by Lemma 3 such that $z \sim x' \sim v'$ and $z \sim x'' \sim w'$; moreover, x' and x'' are different (because v' and w' are not adjacent) and thus adjacent, having a common neighbour x''' in $I(z,x)$ at distance 2 from z. Then $u,t,y,v',z,w',x',x'',x'''$ induce the graph of Fig. 2d as an isometric subgraph.

Subcase 1.2: $d(w',x) < d(v',x)$.

Then as above we derive from (α_1) that

$$d(v,x) \geq d(v,v') + d(z,x) = d(v,t) + d(z,x) - 1 = d(v,x),$$

whence equality holds throughout. In particular, v',v,x have a quasi-median $v'vp$ of size 1 as $v' \in I(u,v)$ and $d(u,v) \leq 3$. Then p and z, as members of $I(v',x)$, are adjacent and have a common neighbour $x' \in I(v',x)$ with $d(v',x') = 2$. Since $x' \in I(z,x)$, we have $d(x',u) = 3$. If x' is adjacent to w', then v,x',w',u would violate (α_1). Therefore, by Lemma 3, z and w' have a common neighbour $x'' \in I(y,x)$ such that $x'' \neq x'$ and $d(u,x'') = 3$. Further, x' and x'' are adjacent and have a common neighbour $x''' \in I(z,x)$ where $d(u,x''') = 4$. Necessarily, $d(v,x''') = 3$ because $v'vp$ is a quasi-median of v',v,x. Then the vertices $u,t,y,v',z,w',v,p,x',x'',x'''$ induce the graph of Fig. 2e. This subgraph is in fact isometric because $d(v,x'') < 3$ would conflict with (α_1) for the quartet v,x'',w',u.

Subcase 1.3: $d(z,x) < d(w',x)$.

Then, as in the preceding subcase, v',v,x and w',w,x have quasi-medians $v'vp$ and $w'wq$, respectively, of size 1 such that $p \sim z \sim q$. Note that $d(u,x) \leq 4 = d(v,w)$ in view of Lemma 1. By Lemma 3, z has neighbours $x',x'' \in I(z,x)$ in common with p and q, respectively. If x' and w have some common neighbour q', then $q' \sim w'$ because $q',w' \in I(w,v)$. Necessarily, z and q' are adjacent because otherwise z,x',q',w' would induce a 4-cycle. Consequently, $u,t,y,v',z,w',v,p,x',q',w$ induce the graph of Fig. 2e as an isometric subgraph. Therefore we can now assume $d(x',w) = 3$ and, analogously, $d(x'',v) = 3$. In particular, x' and x''

are different, whence by Lemma 1 the three vertices x', x'', and x are adjacent. Finally, $u, t, y, v', z, w', v, p, x', x'', q, w, x$ induce the graph of Fig. 2f as an isometric subgraph.

Case 2: Some quasi-median of a triplet from u, v, w, x is not equilateral.

Then, by Lemma 4, we may assume without loss of generality that u, v, w has a quasi-median $uv'w'$ of type $(2,1,2)$ or $(1,2,2)$, see the two principal subcases below. When a common neighbour is selected for each of the two pairs at distance 2 in this quasi-median, then a 5-cycle arises, which must be induced because induced 4-cycles are forbidden.

Subcase 2.1: $d(v', w') = 1$.

Since $\Sigma(v', v, w, x) < \Sigma(u, v, w, x)$, the quartet v', v, w, x must satisfy (β_1), which implies $v' \in I(u, x)$ in view of (4). Analogously, we infer that $w' \in I(u, x)$. Hence, by Lemma 3, any two neighbours of u in $I(u, v') \cup I(u, w') \subseteq I(u, x)$ are adjacent, whence u, v', w' cannot lie on an induced 5-cycle, yielding a contradiction.

Subcase 2.2: $d(u, v') = 1$.

If $d(u, x) < d(v', x)$, then by (α_1) we have

$$\begin{aligned} d(v, x) &\geq d(v, v') + d(u, x) = d(u, v) + d(u, x) - 1 \\ &= d(u, v) + d(u, w) + d(v, x) - d(v, w) + i - 1 \quad \text{by} \quad (4) \\ &= d(v, x) + i, \end{aligned}$$

a contradiction. Therefore $\Sigma(v', v, w, x) < \Sigma(u, v, w, x)$, so that the quartet v', v, w, x must satisfy (β_1). Consequently, we obtain $i = 2$ and $v' \in I(u, x)$. Pick some vertex y with $u \sim y \sim w'$ and a common neighbour z of v' and w' at minimum distance to x. As these five vertices induce a 5-cycle and $v' \in I(u, v) \cap I(u, x)$, neither $I(u, v)$ nor $I(u, x)$ can contain y by Lemma 3, that is, $d(u, v) \leq d(y, v)$ and $d(u, x) \leq d(y, x)$. We also have $d(z, x) \leq d(v', x)$ and hence $d(w', x) \leq d(u, x)$, otherwise by (α_1) we would get

$$\begin{aligned} d(w, x) &\geq d(w, z) + d(v', x) = d(w, u) + d(u, x) - 2 \\ &= d(w, u) + d(u, v) + d(w, x) - d(v, w) + j - 2 \quad \text{by} \quad (4) \\ &= d(w, x) + j - 1, \end{aligned}$$

a contradiction. Hence, if $d(u, x) < d(y, x)$, then $u, w' \in I(y, x)$, which however is in conflict with Lemma 3. We conclude that $d(u, x) = d(y, x)$. Now, if $d(u, v) = d(y, v)$, then

$\Sigma(y,v,w,x) < \Sigma(u,v,w,x)$ but (β_1) is violated for y,v,w,x. Therefore $d(y,v) = d(u,v)+1$. Finally, as $d(z,x) \le d(u,x)-1$, the parameter

$$k = d(z,x) + d(w',x) - 2d(u,x) + 4$$

can only take the values 1, 2, or 3, which leads to the following distinction of subcases 2.2.k.

Subcase 2.2.1: $d(z,x) = d(u,x) - 2$.

Then $d(w',x) = d(u,x) - 1$. Further, by (α_1) and (4),

$$d(v,x) \ge d(v,v') + d(z,x) = d(v,v') + d(u,x) - 2 = d(v,x),$$

whence equality holds throughout. Lemma 5 thus provides us with neighbours $v'' \in I(v',v)$ of v' and $x' \in I(z,x)$ of z such that $d(v'',x') = 2$. Note that $d(v'',w') = 3 = d(u,x')$. Since the quartet u,v'',w',x' violates (β_1) and satisfies $\Sigma(u,v'',w',x') = 14 \le \Sigma(u,v,w,x)$, we conclude that $v'' = v, w' = w$, and $x' = x$. Pick a common neighbour p of v and x at minimum distance to y, and then a neighbour $q \in I(p,y)$ of p. First assume that $p \sim v'$. Then $p \sim z$ because induced 4-cycles are forbidden. Since $\{u,w\} \nsubseteq I(p,y)$ in view of Lemma 3, we have $d(p,y) = 2$. Then as q belongs to $I(y,v) \cap I(y,x)$ it must be adjacent to u,w (by Lemma 3) and hence to v',z as well. Consequently, u,v,v',q,w,p,z,x induce the graph of Fig. 2b as an isometric subgraph.

Therefore we can assume $d(p,v') = 2 = d(p,z)$, so that, as $\{p,z\} \nsubseteq I(u,x)$ and $\{p,v'\} \nsubseteq I(v,w)$ by Lemma 3, we obtain $d(p,u) = d(p,w) = 3$. Thus, $y \in I(u,w) \subseteq B_3(p)$ by convexity. If $d(p,y) = 2$, then $p,v' \in I(v,y)$ would have to be adjacent by Lemma 3, contrary to our assumption. Therefore $d(p,y) = 3$. Then $v,x,y \in B_2(q)$ and thus $u,w \in B_2(q)$ by convexity. Hence as $q,v \in I(p,u)$ and $q,x \in I(p,w)$ we obtain q as a common neighbour of v and x such that $d(q,y) < d(p,y)$, contrary to the choice of p.

Subcase 2.2.2: $d(z,x) = d(w',x) = d(u,x) - 1$.

Since $\Sigma(z,v,w,x) < \Sigma(u,v,w,x)$, the quartet z,v,w,x must satisfy (β_1), which implies $j=2$. By virtue of the triangle inequality and employing (4) as above, we obtain

$$\begin{aligned} d(w,x) &\le d(w,w') + d(w',x) = d(w,z) + d(v',x) - 1 \\ &= d(w,x) + j - 2 = d(w,x), \end{aligned}$$

whence equality holds throughout. In particular, $w' \in I(w,u) \cap I(w,x)$, so that $w = w'$ by (5). Since z was chosen as a common neighbour of v' and $w' = w$ closest to x, the triplet

v', x, w has a quasi-median $v'x'w$ of type $(2,2,2)$ by Lemma 4. Pick vertices p and q with $v' \sim p \sim x'$ and $w \sim q \sim x'$. By the choice of z, neither p is adjacent to w nor q is adjacent to v'. By convexity of balls, $p \in I(v', x') \subseteq B_2(w)$. Hence, if $d(p, y) = 3$, then $u, w \in I(p, y)$ would have to be adjacent, which is impossible. Therefore $d(p, y) = 2$, and by Lemma 3 applied to $I(x', y)$, we infer that $p \sim q$ and there exists a common neighbour t of p, q, w, y.

If $d(v, t) = d(v, w)$, then t is not adjacent to u because otherwise $t \sim v'$ and hence $t \in I(v, w)$ would follow. Therefore $\Sigma(u, v, t, x) = \Sigma(u, v, w, x)$ holds and u, v, t, x violate (β_1). Since $uv't$ is a quasi-median of u, v, t such that p is a common neighbour of v' and t with $d(p, x) = d(u, x) - 2$, we can substitute w by t and z by p. Thus, we are back in Subcase 2.2.1.

Therefore we can assume $d(v, t) = d(v, w) - 1 = d(v, y) - 1$. Then $t, u \in I(v, y)$ and consequently, $t \sim u$ by Lemma 3 and hence $t \sim v'$. Now, however, $uv'w$ is no longer a quasi-median of u, v, w, yielding a contradiction.

Subcase 2.2.3: $d(z, x) = d(w', x) - 1 = d(u, x) - 1$.

We can apply (α_1) to the quartet w, w', z, x and infer as above

$$\begin{aligned} d(w, x) &\geq d(w, w') + d(z, x) = d(w, z) + d(v', x) - 1 \\ &= d(w, x) + j - 2, \end{aligned}$$

whence $j = 2$ and equality holds throughout. By Lemma 5 we thus find neighbours $x' \in I(x, z)$ of z and $w'' \in I(w, w')$ of w such that x' and w'' have some common neighbour t. Then as $z \in I(v, w'')$ either $d(v, x') = d(v, z)$ or $d(v, x') = d(v, z) + 1$ holds. If the latter is true, (α_1) applied to the quartet v, z, x', x yields as above

$$\begin{aligned} d(v, x) &\geq d(v, z) + d(x', x) = d(v, v') + d(u, x) - 1 \\ &= d(v, x) + i - 1, \end{aligned}$$

a contradiction. Thus the former alternative is true. Therefore $t, w' \in I(v, w'')$, and we infer by Lemma 3 that $t \sim w'$ and hence $t \sim z$. Then $B_2(t)$ contains u by convexity because $v', y \in B_2(t)$. This, however, implies $t \in I(w, u) \cap I(w, x)$, contrary to (5).

This final contradiction shows that G is indeed 1-hyperbolic, which completes the proof.

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