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CONDORCET AND MEDIAN POINTS OF SIMPLE RECTILINEAR POLYGONS*

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Abstract—Let P be a simple rectilinear polygon with N vertices, endowed with a rectilinear metric, and let the location of n users in P be given. There are a number of procedures to locate a facility for a given family of users. If a voting procedure is used, the chosen point x should satisfy the following property: no other point y of the polygon P is closer to an absolute majority of users. Such a point is called a *Condorcet point*. If a planning procedure is used, such as minimization of the average distance to the users, the optimal solution is called a *median point*. We prove that Condorcet and median points of a simple rectilinear polygon coincide and present an $O(N+nlogN)$ algorithm for computing these sets. If all users are located on vertices of a polygon P, then the running time of the algorithm becomes $O(N+n)$. Copyright © 1996 Elsevier Science Ltd

Keywords: Computational geometry, Condorcet point, median point, rectilinear polygon, rectilinear distance.

1. INTRODUCTION

In location theory, the best location of a facility that has a given number of users can be obtained in different ways. On the one hand, if a planning procedure is used, such as the minimization of the average distance to the users, the optimal solution is called a *median point*. On the other hand, what would be the location of the facility resulting from a voting procedure in which each user prefers to have the facility as close as possible to him? The chosen point has to satisfy the following property: no other feasible location is closer to an absolute majority of users. Such a point is called a *Condorcet point*.

The comparison of these two decision making procedures was studied in both fundamental models in location theory: the discrete case dealing with locations on networks and graphs (e.g. Bandelt, 1985; Hansen and Labbé, 1988; Hansen and Thisse, 1981; Hansen *et al.*, 1992; and Labbé, 1985) and the continuous case dealing with locations in normed spaces (e.g. Durier, 1989; Wendell and McKelvey, 1981; and Wendell and Thorson, 1974). The standard framework of continuous location theory is a two-dimensional space (Plastria, 1993); the Euclidean norm used initially has been replaced by other norms, particularly, L_1-norm. The Condorcet points do not exist in general—equilateral triangles in the Euclidean plane or the triangle network are standard counterexamples. It is well-known that in a normal plane Condorcet points exist for all distributions of users if and only if the unit ball is a

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parallelogram (*cf.* Durier, 1989; Wendell and McKelvey, 1981; and Wendell and Thorson, 1974). Another characteristic of such spaces is that Condorcet points and median points coincide (*cf.* Durier, 1989; and Wendell and Thorson, 1974). As shown in Hansen and Thisse (1981) the same property holds for all tree networks. In Bandelt (1985), those networks on which Condorcet points and median points always coincide are characterized. Moreover, in this paper, Bandelt presents a complete characterization of those networks on which no Condorcet paradox occurs, i.e., for each distribution of users there exists at least one Condorcet point. In Hansen and Labbé (1988), a polynomial algorithm for determining the set of Condorcet points of a network is given.

This paper focuses on generating the set of Condorcet and median points of a simple rectilinear polygon under the assumption that all travel occurs according to the rectilinear metric. We can consider this problem as another kind of constrained facility location problem where we want, for example, to describe the set of optimal locations of the facility (resulting from a voting procedure) in an urban region or of a some service on a polygonal building floor. Then the rectilinear metric is often a reasonable approximation of travel behavior. The related problems of facility location with regions forbidden for location and travel in the case of rectilinear distance are considered in Larson and Sadiq (1983) and Batta *et al.* (1989). In the first paper, it is shown how to reduce the median problem to a similar problem on a special grid-like network.

In Chepoi and Dragan (1994), we present an $O(N+nlogN)$ time algorithm for finding a median point of a simple rectilinear polygon P with N vertices. If all n users are located in the vertices of P then the running time becomes $O(n+N)$. In this paper, we develop an algorithm of the same complexity for determining the whole set of Condorcet points of a simple rectilinear polygon. We show that in such polygons Condorcet points and median points coincide. The proof is based on some geometric properties of simple rectilinear polygons, in particular on the fact that they are median spaces.

The paper is organized as follows. In the next section, we present the problem formulation. In Section 3, we recall some facts about rectilinear polygons needed to justify the relationships between Condorcet and median points that are presented in Section 4. In Section 5, we show how to compute the Condorcet/median points, and the algorithm description and a simple example are given in Sections 6 and 7.

2. PROBLEM FORMULATION

Let P be a simple rectilinear polygon in the plane R^2 (i.e. a simple polygon having all edges axis-parallel) with N edges. A *rectilinear path* is a polygonal chain consisting of axis-parallel segments lying inside P. The length of a rectilinear path in the L_1-metric equals the sum of the length of its constituent segments. In other words, the length of a rectilinear path in the L_1-metric is equal to its Euclidean length. For any two points u and v in P, the *rectilinear distance* between u and v, denoted by $d(u,v)$, is defined as the length of the minimum length rectilinear path connecting u and v. Consider the problem of locating a single facility on a simple rectilinear polygon P on which a given finite number of users are located. Two users may be located at the same point. Let $\pi(x)$ be the total number of all users located at a point x. The demand is thus described by a *weight function* π from P to the set of nonnegative integers. The polygon P is partitioned into three sets with respect to any pair x, y of points:

$$[x \succ y] = \{z \in P : d(x,z) < d(y,z)\},$$
$$[y \succ x] = \{z \in P : d(y,z) < d(x,z)\},$$
$$[x \sim y] = \{z \in P : d(x,z) = d(y,z)\}.$$

If a voting procedure is used to solve the facility location problem, the chosen point x should satisfy the following property:

No other point y of the polygon P is closer to an absolute majority of users, i.e. $\pi[y \succ x] \leqslant \pi(P)/2$ for all points $y \in P$

(For a $S \subseteq P$ define $\pi(S)$ as $\pi(S) = \sum_{p_i \in S} \pi(p_i)$, where $p_1, p_2, \ldots, p_n$ are the points of P where the users are located.) Such a point x is called a *Condorcet point*, (see Bandelt, 1985; Durier, 1989; Hansen and Thisse, 1981; Hansen and Labbé, 1988; Hansen *et al.*, 1992; Labbé, 1985; Wendell and McKelvey, 1981; and Wendell and Thorson, 1974). Denote by $Cond(P)$ the set of all Condorcet points of the polygon P. The weighted distance sum of a point x with respect to π is given by

$$D(x) = \sum_{i=1}^{n} \pi(p_i) d(x, p_i).$$

If a planning procedure is used, such as minimization of the function $D(x)$, the optimal solution is a *median* (or a *Weber point*); (see Bandelt, 1985; Bandelt and Barthelemy, 1984; Durier, 1989; Hansen and Labbé, 1988; and Hansen and Thisse, 1981). Let $Med(P)$ be the set of all median points of polygon P with respect to the weight function π.

In a similar way, we can define the median and Condorcet points for an arbitrary metric space (X,d).

3. PROPERTIES OF SIMPLE RECTILINEAR POLYGONS

Recall that the *interval* $I(u,v)$ between two points u, v of a metric space (X,d) consists of all points z between u and v, that is

$$I(u,v) = \{z \in X : d(u,v) = d(u,z) + d(z,v)\}.$$

A metric space (X,d) is a *median space* if every triple of points u, v, $w \in X$ admits a unique "median" point $z = \mathrm{m}(u,v,w)$, such that

$$d(u,v) = d(u,z) + d(z,v),$$
$$d(u,w) = d(u,z) + d(z,w),$$
$$d(v,w) = d(v,z) + d(z,w),$$

i.e. $z = I(u,v) \cap I(v,w) \cap I(w,u)$ (for an illustration, see Figs 1 and 2).

The median spaces represent a common generalization of different mathematical structures such as median semilattices and median algebras (Bandelt and Hedlikova, 1983), median graphs (including trees and hypercubes) (Mulder, 1980), median networks (Bandelt, 1985) and linear spaces with L_1-metric. For classical results on median spaces, the reader is referred to Bandelt and Hedlikova (1983) and van de Vel (1993).

A set M of metric space (X,d) is *convex* if for any points $x, y \in M$ the interval $I(x,y)$ belongs to M. For a subset $S \subset X$ by $conv(S)$ we denote the convex hull of S, i.e. the intersection of

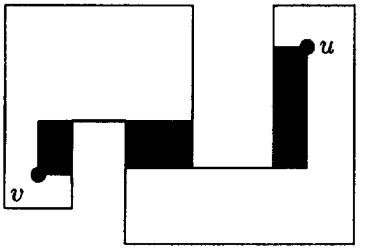

Fig. 1. The interval $I(u,v)$.

all convex sets containing S. A subset H of X is a *half-space* provided both H and $X\setminus H$ are convex. Recall also that the subset M is called *gated* (Dress and Scharlau, 1987), provided every point $x \in X$ admits a *gate* in M, i.e. a point $x' \in M$ such that $x' \in I(x,y)$ for all $y \in M$ (see Fig. 3). Any gated subset of a metric space is convex. The converse holds for median spaces:

Lemma 1. Any convex compact subset of a median space is gated.

For a proof of this result, see van de Vel (1993). The next result is proved in Chepoi and Dragan (1994).

Lemma 2. A simple rectilinear polygon P equipped with L_1-metric is a median space.

An axis-parallel segment c is called a *cut segment* of a polygon P if it connects two parallel

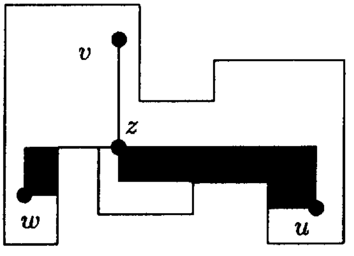

Fig. 2. z is the median of u, v and w.

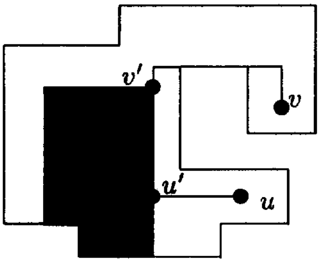

Fig. 3. u' and v' are the gates of u and v.

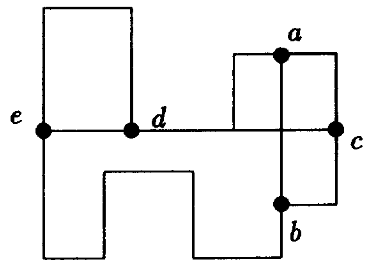

Fig. 4. $[a,b]$, $[c,e]$ and $[d,e]$ are cut segments.

edges of P and is fully contained in P. Note that any cut segment of a polygon P is a gated subset of P; for an illustration, see Fig. 4.

Lemma 3. If M is a compact convex subset of a simple rectilinear polygon P and $x \in P \setminus M$ then there exists a cut c of P which separates x and M, i.e. $M \cap c = \emptyset$, $x \notin c$, and M and x belong to different subpolygons defined by c.

Proof. Let c' and c'' be the maximal cuts of P which pass through the point x. If both these segments intersect the set M, say $y' \in c' \cap M$ and $y'' \in c'' \cap M$, then necessarily $x \in I(y', y'')$, which is impossible. Let $c' \cap M = \emptyset$. Consider a shortest path Q between x and the gate x_M of x in M. Moving the cut c' a little such that c' intersects Q we obtain the required cut of P. □

The next property is a particular instance of a general result of Bandelt *et al.* (1993) on convexity structures; see also van de Vel (1993). A direct proof is given in Schuierer (1993).

Lemma 4. For any finite subset S of a simple rectilinear polygon P

$$conv(S) = \bigcup_{u, v \in S} I(u, v).$$

4. BASIC RESULTS

In this section, we investigate the structural properties of sets of Condorcet points and median points of a simple rectilinear polygon P. We use some results for the median problem in median graphs and discrete median spaces, established in Bandelt and Barthelemy, 1984; Chepoi, 1995; and Soltan and Chepoi, 1987.

Consider all horizontal and all vertical cuts that pass through the vertices of the polygon P or the points of P where the users are located. These cuts together with the edges of P generate a *rectilinear grid*; see Fig. 5 for an illustration. Denote by V the vertices (intersection points) of this grid and by G its graph. Recall that a graph G is *median* (Mulder, 1980) if G is a median space with respect to the standard graph distance.

Lemma 5. (Chepoi, 1995). G is a median graph.

Denote by $Med(G)$ the set of median vertices of the graph G.

Lemma 6. (Bandelt and Barthelemy, 1984; Soltan and Chepoi, 1987). $Med(G)$ is convex in G. Moreover, $Med(G)$ is an interval of G.

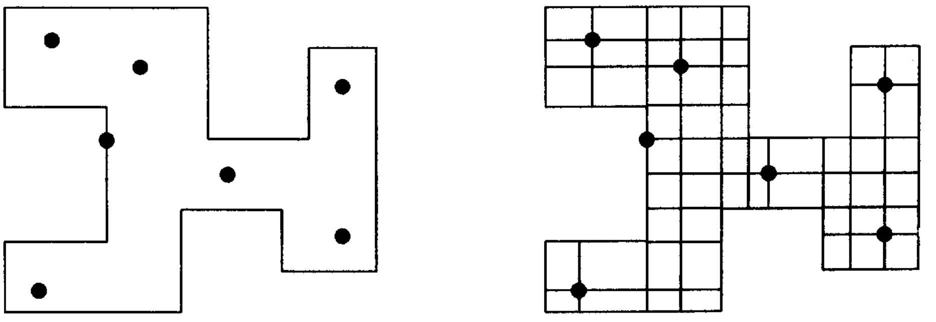

Fig. 5. A polygon P with users and its rectilinear grid.

Let c be a cut of P. Then for the subpolygons P' and P'' defined by this cut we have

$$\pi(P')+\pi(P'')=\pi(P)+\pi(c).$$

Lemma 7. (Bandelt and Barthelemy, 1984; Soltan and Chepoi, 1987). If $\pi(P')>\pi(P'')$ then $Med(P)\subset P'$, otherwise if $\pi(P')=\pi(P'')$ then $Med(P)\cap c\neq\emptyset$.
The converse is true for the set $Med(G)$.

Lemma 8. (Soltan and Chepoi, 1987). If $Med(G)$ belongs to the half-space H of G then $\pi(H)>\pi(V)/2=\pi(P)/2$.

Lemma 9. Let x and y be points on a cut c of P. If x and y belong to a common rectangle of the grid then

$$D(x)-D(y)=d(x,y)(\pi[y\succ x]-\pi[x\succ y]).$$

Proof. From the choice of points x and y, we conclude that for every vertex v of V either $x\in I(v,y)$ or $y\in I(v,x)$. In particular, $\pi[y\succ x]+\pi[x\succ y]=\pi(P)$. Then the proof that the required equality is true is straightforward. □

Lemma 10. $Med(P)=conv(Med(G))$.

Proof. By Lemma 6 the set $Med(G)$ is convex in G. Applying Lemma 4 we deduce that $conv(Med(G))$ in P coincides with the union of all rectangles (including degenerated ones) of the grid whose four corners belong to $Med(G)$. Let $R=conv(a,b,c,d)$ be such a rectangle and let x be an arbitrary point of R. First suppose that x belongs to the boundary of R, say $x\in[a,b]$. By Lemma 9 we have

$$D(x)-D(a)=d(x,a)(\pi[a\succ x]-\pi[x\succ a]).$$

Observe that $[a\succ x]=[a\succ b]$ and $[x\succ a]=[b\succ a]$. Applying this fact and Lemma 9 to the vertices a and b we get

$$D(a)-D(b)=d(a,b)(\pi[b\succ a]-\pi[a\succ b])=d(a,b)(\pi[x\succ a]-\pi[a\succ x]).$$

Since $D(a)=D(b)$ in both G and P (Soltan and Chepoi, 1987) we conclude that $\pi[b\succ a]=\pi[a\succ b]$). Therefore $D(x)=D(a)$.

Next assume that x is an interior point of R. Let x' and x'' be the boundary points of R which lie on a common horizontal cut with x. Then as we already proved

$$D(x')=D(a)=D(b)=D(x'').$$

Notice that $\pi[a \succ b]=\pi[x' \succ x]$ and $\pi[b \succ a]=\pi[x \succ x']$. Applying Lemma 9 to the points x and x' we obtain that $D(x)=D(x')$. Thus $D(\cdot)$ is constant on the set $conv(Med(G))$.

In order to prove the required equality, it is sufficient to establish that $z \notin Med(P)$ for an arbitrary point $z \notin conv(Med(G))$. By Lemma 3 there is a cut c that separates the sets $\{z\}$ and $conv(Med(G))$, i.e.

$$z \in P' \backslash c, \quad conv(Med(G)) \subseteq P'',$$

where P' and P'' are the subpolygons defined by c. Then $H'=P' \cap V$ and $H''=P'' \cap V$ represent complementary half-spaces of the graph G. Since $Med(G) \subset H''$ by Lemma 8, we conclude that $\pi(P'')=\pi(H'')>\pi(P)/2$. Let z^* be gate for z in the subpolygon P''. A straightforward verification shows that

$$D(z^*)-D(z)<\pi(P')-\pi(P'')<0,$$

and thus $z \notin Med(P)$. Hence $Med(P) \subseteq conv(Med(G))$. Since $D(\cdot)$ is constant on the set $conv(Med(G))$, we conclude that $Med(P)=conv(Med(G))$. □

Theorem 1. $Cond(P)=Med(P)$.

Proof. First we prove that $Med(P) \subseteq Cond(P)$. Assume the contrary, i.e. for some median point x there exists a point y such that $\pi[y \succ x]>\pi(P)/2$. If $x \in conv([y \succ x])$ then by Lemma 4 we have $x \in I(z',z'')$ for two points $z',z'' \in [y \succ x]$. Since $d(y,z')<d(x,z')$ and $d(y,z'')<d(x,z'')$ we obtain a contradiction with the choice of z' and z''. So, assume that $x \notin conv[(y \succ x])$. Let x^* be the gate for x in the gated set $conv([y \succ x])$. In the interval $I(x,x^*)$ pick a close neighbor z of x, such that x and z belong to a common rectangle of the grid and to a common cut of P. By Lemma 9.

$$D(z)-D(x)=d(z,x)(\pi[x \succ z]-\pi[z \succ x]).$$

Since $[z \succ x] \supseteq [x^* \succ x] \supseteq [y \succ x]$ and $\pi[y \succ x]>\pi(P)/2$ we get $D(z)<D(x)$, in contradiction with the assumption that $x \in Med(P)$. Therefore, any median point of P is a Condorcet point.

Conversely, assume that some Condorcet point x is not a median point. As $Med(P)$ is convex by Lemma 3, there exists a cut c which separates the set $Med(P)$ and the point x. Let $x \in P' \backslash c$, $Med(P) \subseteq P''$, where P' and P'' are subpolygons defined by c. Since $Med(G) \subseteq P'' \cap V$ by Lemma 8 necessarily $\pi(P'')=\pi(P'' \cap V)>\pi(P)/2$. But then for the gate x^* of x in P'' we have $P'' \subset [x^* \succ x]$ and thus $\pi[x^* \succ x]>\pi(P)/2$, a contradiction. □

5. COMPUTING CONDORCET AND MEDIAN POINTS

Using the results of the previous section, we describe how to compute the set of Condorcet alias median points of a simple rectilinear polygon P. We start with an outline of the algorithm. Instead of constructing the rectilinear grid G (which in the worst case can contain $O((N+n)^2)$ vertices), we divide the polygon P into horizontal strips by using the horizontal cuts that pass through the vertices of P. The dual graph of this subdivision is a tree $T(P)$. Assign to each vertex of $T(P)$ a weight equal to the number of users located in the corresponding rectangle. Let $Med(T(P))$ be the set of median vertices with respect to this weight function. It is well-known that $Med(T(P))$ induces a subpath of $T(P)$. There is a close relationship between the sets $Med(P)$ and $Med(T(P))$. Namely, $Med(P)$ intersects a rectangle of the subdivision if and only if it belongs to $Med(T(P))$. In order to derive $Med(P)$ as a union of its intersections with the rectangles of $Med(T(P))$, we have to compute $M(x)=Med(P)\cap R(x)$ and $M(y)=Med(P)\cap R(y)$, where $R(x)$ and $R(y)$ are the rectangles that correspond to the end-vertices x and y of the path $Med(T(P))$. This is done by solving two median problems on the rectangles $R(x)$ and $R(y)$, where each user is replaced by its gate in $R(x)$ and $R(y)$, respectively. The problem is now reduced to joining the sets $M(x)$ and $M(y)$ along the path $Med(T(P))$ in order to compute the whole set $Med(P)$. For this purpose, it suffices to find the intersection of $Med(P)$ with each horizontal cut c separating the rectangles $M(x)$ and $M(y)$. We show that $Med(P)\cap c$ is the smallest subsegment of c containing the gates of the corners of $M(x)$ and $M(y)$ in the cut c. These gates can be computed in constant time pro cut by processing twice the edges (they correspond to horizontal cuts of P) of the path $Med(T(P))$.

In the following pages, we perform a rather detailed description of the algorithm. The algorithm is based on the Chazelle algorithm for computing all vertex-edge visible pairs (Chazelle, 1991) and on the Goldman algorithm for finding the median set of a tree (Goldman, 1971). By the first algorithm, we obtain a decomposition of a polygon P into $O(N)$ rectangles, using only maximal horizontal cuts. The dual graph of this decomposition is a tree $T(P)$: vertices of this tree are the rectangles and two vertices are adjacent in $T(P)$ if the corresponding rectangles in the decomposition share a common cut (see Figs 6 and 7). Denote by $R(v)$ the rectangle that corresponds to a vertex v of the tree $T(P)$. Assign to each vertex of $T(P)$ the weight of its rectangle. (The weight of a rectangle R is the sum of weights of its points (users) minus one half of the total weight of points that belong to the horizontal sides of R and do not belong to the boundary of P.) In order to compute these weights, we first have to compute which rectangles of the decomposition of P contain each of the users. Using one of the optimal point location methods (Edelsbrunner *et al.*, 1985 or Kirkpatrick, 1983) this can be done in time $O(n\log N)$ with a structure that uses $O(N)$ storage. (Here N is the number of vertices of polygon P, while n is the number of users.) Observe that the induced subdivision is monotone and hence, the point location structure can be built in linear time. Therefore the weights of vertices of $T(P)$ can be defined in total time $O(n\log N+N)$. When all users are located only on vertices of P then this assignment takes $O(N+n)$ time.

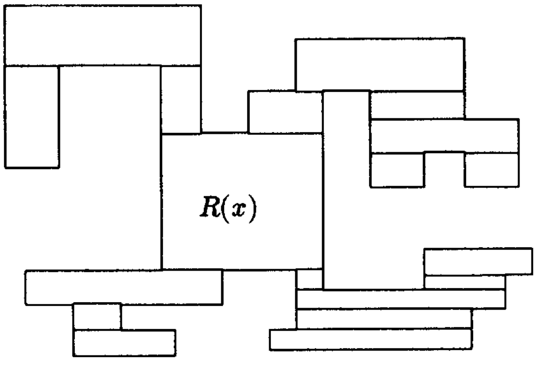

Fig. 6. Subdivision of P into horizontal strips.

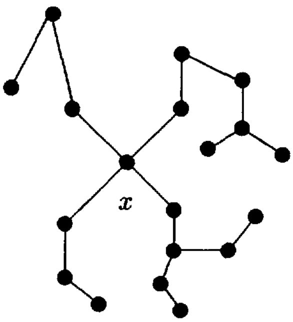

Fig. 7. Dual graph $T(P)$.

Now using the Goldman algorithm (Goldman, 1971) we compute the set $Med(T(P))$ of median vertices of the tree $T(P)$. It is well-known that $Med(T(P))$ induces a path of $T(P)$; see for example Tansel *et al.* (1983).

Lemma 11. $Med(P) \subseteq \bigcup\{R(v)\colon v \in Med(T(P))\}$. Moreover, $Med(P) \cap R(v) \neq \emptyset$ for every vertex $v \in Med(T(P))$.

Proof. By the majority rule for trees $v \in Med(T(P))$ if and only if $\pi(T_v) > \pi(T_{v'})$ for any neighbor v' of v; Goldman (1971). (By T_v and $T_{v'}$ we denote the subtrees obtained by deleting the edge (v, v')). In the polygon P, the rectangles $R(v)$ and $R(v')$ are separated by the horizontal cut c which coincides with $R(v) \cap R(v')$. Let P_v and $P_{v'}$ be the subpolygons defined by c and let $R(v) \subset P_v$ and $R(v') \subset P_{v'}$. All rectangles that correspond to vertices from T_v lie in the subpolygon P_v. If $v \in Med(T(P))$ then $\pi(P_v) - \pi(c)/2 = \pi(T_v) > \pi(T_{v'}) = \pi(P_{v'}) - \pi(c)/2$ for all neighbors of v. Since $R(v)$ coincides with the intersection of the subpolygons of the type P_v by Lemma 7, we conclude that $Med(P) \cap R(v) \neq \emptyset$. Conversely, if $v \notin Med(T(P))$ then $\pi(P_v) - \pi(c)/2 = \pi(T_v) < \pi(T_{v'}) = \pi(P_{v'}) - \pi(c)/2$ for some vertex v' adjacent to v. By Lemma 7, we obtain that $Med(P) \cap R(v) = \emptyset$. □

Denote by x and y the end-vertices of the path $Med(T(P))$. Next we concentrate on finding the median points of rectangles $R(x)$ and $R(y)$. For this purpose, we use the method developed in Chepoi and Dragan (1994). Suppose that $R(x)$ is bounded by the horizontal cuts c' and c'' of the decomposition of P. Then P can be represented as a union of $R(x)$ with two subpolygons P' and P'' of P, where $P' \cap R(x) = c'$ and $P'' \cap R(x) = c''$. For any user z_i let g_i be the gate for z_i in the rectangle $R(x)$. Evidently $g_i \in c'$ if $z_i \in P'$, $g_i \in c''$ if $z_i \in P''$ and $g_i = z_i$ if $z_i \in R(x)$. In order to find these gates, we define the maximal histograms H' and H'' inside P' and P'' with c' and c'' as their bases, respectively. (A *histogram* is a rectilinear polygon that has one distinguished edge, its *base*, whose length is equal to the sum of the lengths of the other edges that are parallel to it); see for example de Berg (1991). The vertical edges of these histograms divide the polyons P' and P'' into subpolygons, called *pockets*. (In Fig. 8 P' has two pockets, while P'' has three pockets.) Consider for example the pockets from P'. Note that all points from the same pocket Q have one and the same gate. This is a point of a cut c' which has the same x-coordinate with the vertical cut of P' that separates the pocket Q and the histogram H'. (For example, in Fig. 8) a' and b' are the gates in $R(x)$ of the points a and b, respectively.) Hence it is sufficient to find the location of users in the pockets. This can be done by using the subdivisions of P' and P'' into rectangles by vertical vertex-

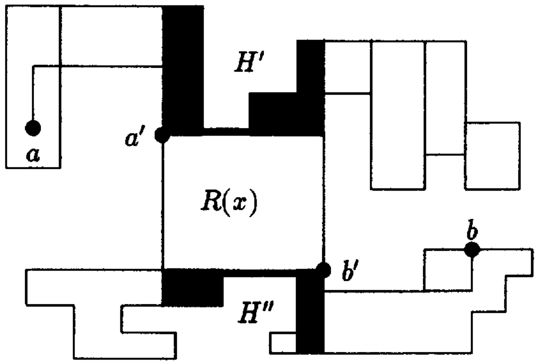

Fig. 8. Vertical subdivision of pockets of P'.

edge visible pairs. Let $T_v(P')$ be the dual graph of such a subdivision of P'. The rectangles from any pocket of P' induce a subtree in $T_v(P')$. Therefore $T_v(P')$ is a disjoint union of the subtrees corresponding to pockets of P' and of the subtree generated by the histogram H' (see Fig. 9). Such a partition can be computed in time proportional to the size of $T_v(P')$, just processing the boundary of each pocket separately. So, it suffices to establish which rectangles of the subdivision of P' contain each of the users. Again it is necessary to use the Chazelle algorithm (Chazelle, 1991) and the optimal point location methods (Edelsbrunner *et al.*, 1985; and Kirkpatrick, 1983). For every user $z_i \in P' \cup P''$ assign the weight $\pi(z_i)$ to its gate g_i, the weights of users from $R(x)$ remain unchanged. As a result, we obtain a median problem in the rectangle $R(x)$. Note that any solution of this problem belongs to $Med(P) \cap R(x)$. To see this, observe that for any two points $z', z'' \in R(x)$ it holds

$$D(z')-D(z'')=\sum_{i=1}^{n} \pi(z_i)(d(z',z_i)-d(z'',z_i))=\sum_{i=1}^{n} \pi(z_i)(d(z',g_i)-d(z'',g_i)).$$

The new median problem on $R(x)$ may be solved by decomposing it into two one-dimensional median problems and applying to each of them a modification of the selection algorithm from Blum *et al.*, 1972; see also Cormen *et al.*, 1990.

Let $M(x)=Med(P) \cap R(x)$. In a similar way, we find the set $M(y)=Med(P) \cap R(y)$. Both $M(x)$ and $M(y)$ are rectangles (possible degenerate) whose corners are vertices of the grid G introduced in Section 2. Denote the corners of $M(x)$ by a_1, a_2, a_3, a_4 and the corners of $M(y)$ by b_1, b_2, b_3, b_4, in the assumption that the segments $[a_3,a_4]$ and $[b_3,b_4]$ belong to the horizontal sides s_x and s_y of $R(x)$ and $R(y)$ which separate the rectangles $R(x)$ and $R(y)$. By

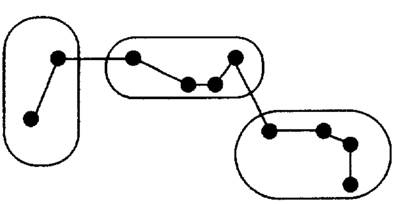

Fig. 9. The partition of $T_v(P')$ into subtrees of pockets.

Lemmas 6 and 10, the set $Med(P)$ coincides with the interval $I(v',v'')$ between two vertices v', v'' of the grid. As the cuts s_x and s_y separate the rectangles $R(x)$ and $R(y)$ from the rest of the set $Med(P)$, we deduce that $v' \in \{a_1,a_2\}$ while $v'' \in \{b_1,b_2\}$. Therefore $Med(P)=\bigcup_{i,j\in\{1,2\}} I(a_i,b_j)$, i.e. it is enough to compute this union of intervals (in fact, $Med(P)$ equals one of the intervals $I(a_i,b_j)$). Moreover, it is sufficient to find its intersection with each horizontal cut that separates the rectangles $R(x)$ and $R(y)$. Indeed, let $R(v)$ be a rectangle for $v \in Med(T(P))$. Assume that $R(v)$ is bounded by the horizontal cuts c_1 and c_2 and let

$$Med(P)\cap c_1=I' \quad \text{and} \quad Med(P)\cap c_2=I''.$$

Then $Med(P)\cap R(v)$ is a rectangle, whose corners can be computed in constant time by finding the intersection of segments I' and I'' with the respective horizontal sides of the rectangle $R(v)$.

Let $x=v_0,v_1,\ldots,v_{k-1}$, $v_k=y$ be the vertices of $Med(T(P))$ and let $c_1=c',\ldots,c_k$ be the horizontal cuts of P that correspond to edges of the path induced by $Med(T(P))$. First we find that gates $g^1(a_1)$, $g^1(a_2)$, $g^1(b_1)$, $g^1(b_2),\ldots,g^k(a_1)$, $g^k(a_2)$, $g^k(b_1)$, $g^k(b_2)$ for points a_1, a_2, b_1, b_2 in the cuts $c_1,\ldots,c_k$, respectively. In order to do this, we use the next evident remark: $g^{i+1}(a_1)$ and $g^{i+1}(a_2)$ are the gates for $g^i(a_1)$ and $g^i(a_2)$ in the cut c_{i+1}, while $g^i(b_1)$ and $g^i(b_2)$ are the gates for $g^{i+1}(b_1)$ and $g^{i+1}(b_2)$ in c_i. This follows from the fact that if c cuts P into subpolygons P' and P'' and $x \in P'$ then the gates for x in c and P'' coincide; see Chepoi and Dragan (1994) Lemma 3.

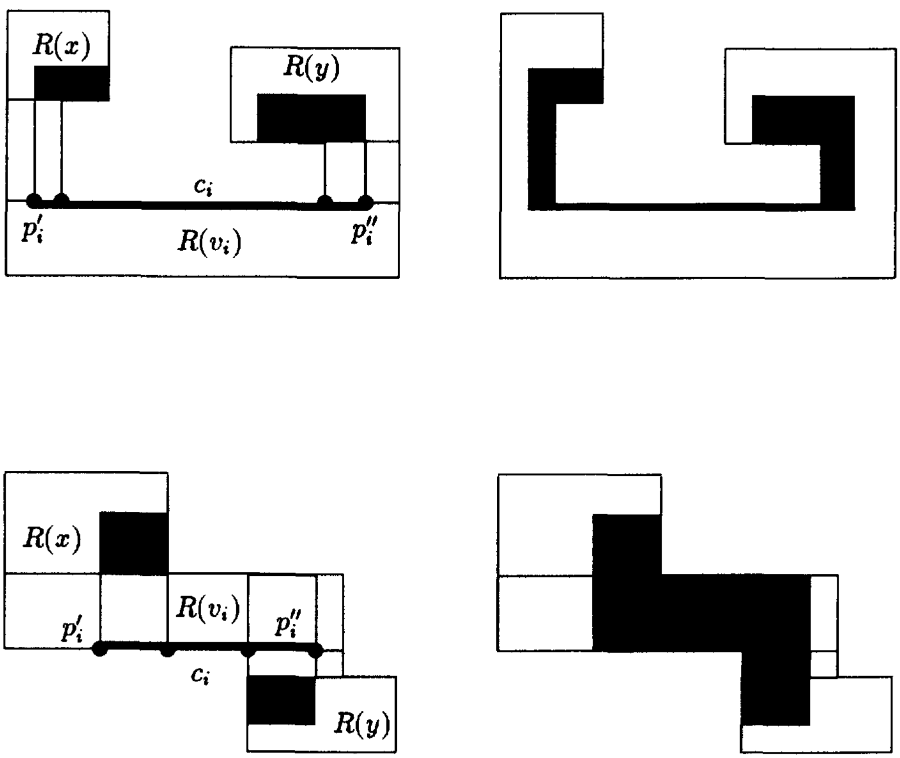

Fig. 10. Computation of $[p_i',p_i'']$ and $Med(P)\cap R(v_i)$.

Let $[p_i', p_i'']$ be the smallest segment of the cut c_i containing the points $g^i(a_1)$, $g^i(a_2)$, $g^i(b_1)$ and $g^i(b_2)$ (see Fig. 10 for an illustration).

Lemma 12. $[p_i', p_i''] = Med(\pi, P) \cap c_i$.

Proof. Assume for example that $Med(P) = I(a_1, b_1)$. Then necessarily a_2, $b_2 \in I(a_1, b_1)$. The points p_i', p_i'' being the gates for some of the points a_1, a_2, b_1 or b_2 necessarily belong to the interval $I(a_1, b_1)$. By the convexity of the set $I(a_1, b_1)$ we conclude that $[p_i', p_i''] \subset I(a_1, b_1)$. Next consider a point p outside the segment $[p_i', p_i'']$. Then

$$d(a_1, p) = d(a_1, g^i(a_1)) + d(g^i(a_1), p),$$

$$d(b_1, p) = d(b_1, g^i(b_1)) + d(g^i(b_1), p).$$

Since $d(g^i(a_1), p) + d(g^i(b_1), p) \geqslant d(g^i(a_1), g^i(b_1))$ we conclude that $d(a_1, p) + d(b_1, p) > d(a_1, b_1)$, i.e. $p \notin I(a_1, b_1)$. Thus $Med(P) \cap c_i = [p_i', p_i'']$. Another proof of this equality follows from the fact that gate functions in median spaces map intervals precisely to intervals; see van de Vel, 1993. □

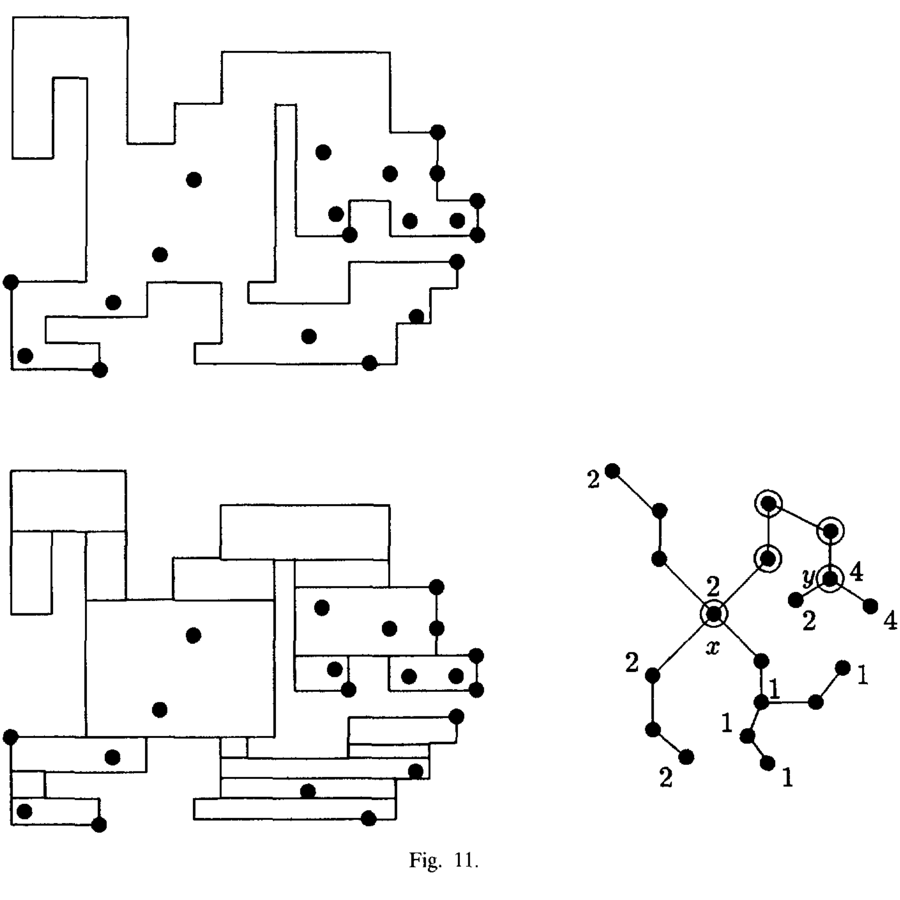

Fig. 11.

The computing of gates for points a_1, a_2, b_1 and b_2 in cuts $c_1,\ldots,c_k$ takes $O(N)$ time. The same number of operations is necessary to compute the segments $Med(P)\cap c_i$ and the sets $Med(P)\cap R(v_i)$, $i=1,\ldots,k$. Thus we obtain the set $Med(P)$ as a union of at most N rectangles.

Summarizing the results of this section and taking into account that $Cond(P)=Med(P)$ the next result is obtained.

Theorem 2. The sets of Condorcet and median points of a simple rectilinear polygon P can be found in time $O(nlogN+N)$. If all users are located on vertices of P then the time becomes $O(N+n)$.

6. THE ALGORITHM

We are now in position to describe the complete procedure for the calculation of $Cond(P)=Med(P)$. It consists of the following steps.

Step 1: Divide P into horizontal strips. Let $T(P)$ be the tree of this subdivision. For each vertex v of $T(P)$ $R(v)$ denotes the corresponding strip alias rectangle.

Step 2: Assign to each vertex v of $T(P)$ the sum of weights of users located in $R(v)$ minus one half of the weight of users located on horizontal sides of $R(v)$ which do not

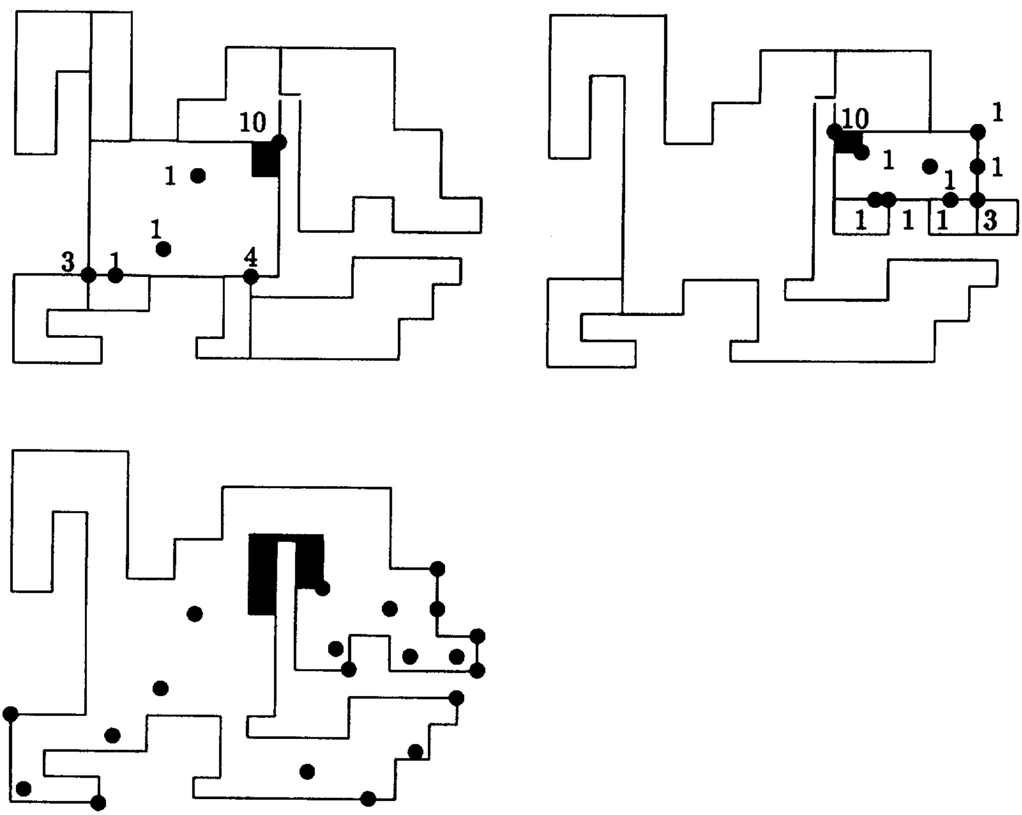

Fig. 12. The sets $M(x)$, $M(y)$ and $Med(P)$.

belong the boundary of P. To this end, we use the point location methods in planar subdivisions.

Step 3: Compute the median $Med(T(P))$ of the tree $T(P)$. It is known that $Med(T(P))$ is a subpath of $T(P)$, say $Med(T(P))=(x=v_0, v_1, \ldots, v_k=y)$.

Step 4: Compute the rectangles $M(x)=Med(P)\cap R(x)$ and $M(y)=Med(P)\cap R(y)$. To this end, we solve two median problems on $R(x)$ and $R(y)$, where each user is replaced by its gates in $R(x)$ and $R(y)$, respectively. These gates (say in $R(x)$) can be determined by dividing $P-R(x)$ into pockets and two histograms. All users from one pocket have a common gate in $R(x)$. In order to find all users located in each pocket, we divide the pockets into vertical strips and apply the point location methods to the obtained subdivision of P.

Step 5: Recursively compute the gates of the corners of $M(x)$ and $M(y)$ in each horizontal cut c_i separating the rectangles $R(x)$ and $R(y)$. Find the smallest segment $[p_i', p_i'']$ of c_i containing these gates. Using the segments $[p_i', p_i'']$ and $[p_{i+1}', p_{i+1}'']$ compute the intersection $M(v_i)$ of $Med(P)$ with the rectangle $R(v_i)$ ($R(v_i)$ shares its horizontal sides with the cuts c_i and c_{i+1}).

Step 6: Output $Med(P)=Cond(P)=\bigcup_{i=0}^{k} M(v_i)$.

7. AN EXAMPLE

In Figs 11 and 12 we present a concrete step-by-step example (all users are assumed to have weight 1).

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