GRAPHS WITH CONNECTED MEDIANS

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Abstract. The median set of a graph G with weighted vertices comprises the vertices minimizing the average weighted distance to the vertices of G. We characterize the graphs in which, with respect to any non–negative vertex weights, median sets always induce connected subgraphs. The characteristic conditions can be tested in polynomial time (by employing linear programming) and are immediately verified for a number of specific graph classes.

Key words: graphs, medians, local medians, majority rule, LP duality.

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1. Introduction. Given a (finite, connected) graph G one is sometimes interested in finding the vertices minimizing the total distance

$$F(x) = \sum_u d(u,x)$$

to the vertices u of G, where the distance $d(u,x)$ between u and x is the length of a shortest path connecting u and x. The subgraph induced by all minima of F need not be connected: actually every (possibly disconnected) graph can be realized as such a "median" subgraph of another graph (Slater [23]); see Hendry [18] for further information and pertinent references. Here we will focus on the weighted version of the median problem, which arises with one of the basic models in discrete facility location [25] and with majority consensus in classification and data analysis [3, 9, 10, 11, 22]. A *weight function* is any mapping π from the vertex set to the non–negative real numbers, which is not the zero function (in order to avoid trivialities). The total weighted distance of a vertex x in G is given by

$$F_\pi(x) = \sum_u \pi(u) d(u,x).$$

A vertex x minimizing this expression is a *median* (vertex) of G with respect to π, and the set of all medians is the *median set* $\mathrm{Med}(\pi)$. By a *local median* one means a vertex x such that $F_\pi(x)$ does not exceed $F_\pi(y)$ for any neighbour y of x. Denote by $\mathrm{Med}_{\mathrm{loc}}(\pi)$ the set of all local medians. We will consider the following questions:

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- When are all median sets $\mathrm{Med}(\pi)$ of a graph connected?

- When does $\mathrm{Med}(\pi) = \mathrm{Med}_{\mathrm{loc}}(\pi)$ hold for all weight functions π ?

If we allow only 0–1 weight functions π, then recognition of graphs with $\mathrm{Med}(\pi) = \mathrm{Med}_{\mathrm{loc}}(\pi)$ is an NP–complete problem [2]. We will show that connectivity of median sets with respect to arbitrary weight functions turns out to be equivalent to the condition that for each function F_π all local minima are global. This property can also be formulated as a certain convexity condition.

In the next section we investigate weakly convex functions on graphs, and in Section 3 we then obtain the basic characterizations of graphs with connected medians. One of those equivalent conditions, weak convexity of F_π for each weight function π, can be formulated as a linear programming problem, thus allowing to recognize graphs with connected medians in polynomial time. This LP condition, however, entails a lot of redundancy and cannot be read off from other graph properties, which are known to imply median connectedness. Striving for less redundancy in the requirements, we establish the main result in Section 4 by employing LP duality to a particular instance of the original LP formulation. This theorem can conveniently be used to derive median properties in several specific classes of graphs, as is demonstrated in the final section.

2. Weakly convex functions. A real–valued function f defined on the vertex set V of a graph G is said to be *weakly convex* if for any two vertices u, v, and a real number λ between 0 and 1 such that $\lambda d(u,v)$ and $(1-\lambda)d(u,v)$ are integers, there exists a vertex x such that

$$d(u,x) = \lambda d(u,v), \qquad d(v,x) = (1-\lambda)d(u,v),$$
$$f(x) \leq (1-\lambda)f(u) + \lambda f(v).$$

Weakly convex functions were introduced by Arkhipova and Sergienko [1] under the name "r–convex functions"; see also Lebedeva, Sergienko, and Soltan [19].

The *interval* $I(u,v)$ between two vertices u and v consists of all vertices on shortest u,v–paths, that is:

$$I(u,v) = \{x : d(u,x) + d(x,v) = d(u,v)\}.$$

For convenience we will use the short–hand

$$I^\circ(u,v) = I(u,v) - \{u,v\}$$

to denote the "interior" of the interval between u and v.

LEMMA 1. *For a real–valued function f defined on the vertex set of a graph G the following conditions are equivalent:*

(i) *f is weakly convex;*

(ii) *for any two non–adjacent vertices u and v there exists $w \in I^\circ(u,v)$ such that*

$$d(u,v) \cdot f(w) \leq d(v,w) \cdot f(u) + d(u,w) \cdot f(v);$$

(iii) *any two vertices u and v at distance 2 have a common neighbour w with*

$$2f(w) \leq f(u) + f(v).$$

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Consider two vertices u and v at distance $d(u,v) = n$. Among all shortest paths connecting u and v select a path $P = (u = w_0, w_1, \ldots, w_{n-1}, w_n = v)$ such that $\sum_{i=0}^{n} f(w_i)$ is as small as possible. Condition (iii) implies that $2f(w_i) \leq f(w_{i-1}) + f(w_{i+1})$ for each $1 < i < n$. Regard

$$(0, f(w_0)), (1, f(w_1)), \ldots, (n-1, f(w_{n-1})), (n, f(w_n))$$

as points in the plane $\mathbb{R}^2$. Connecting the consecutive points by segments, we will get a graph of a piecewise–linear function. This function is necessarily convex (in the usual sense), because it coincides on P with the function f. From this we conclude that

$$f(w_i) \leq \lambda f(u) + (1-\lambda) f(v),$$

where $\lambda = (n-i)/n$ and $1 - \lambda = i/n$. $\square$

Recall that a real–valued function f defined on a path $P = (w_0, w_1, \ldots, w_p)$ is *peakless* [13, p.109] if $0 \leq i < j < k \leq p$ implies $f(w_j) \leq \max\{f(w_i), f(w_k)\}$ and equality holds only if $f(w_i) = f(w_k)$. Now, we will say that a function f defined on the vertex set of a graph G is *pseudopeakless* if any two vertices of G can be joined by a shortest path along which f is peakless [15]. Equivalently, f is pseudopeakless if for any two non–adjacent vertices u, v there is a vertex $w \in I^\circ(u,v)$ such that $f(w) \leq \max\{f(u), f(v)\}$ and equality holds only if $f(u) = f(v)$. The key property of pseudopeakless functions is their unimodality, that is, every local minimum of f is global. The proof is simple: let u be a global minimum of f and v a local minimum of G. Consider a shortest path P between u and v along which f is peakless. Then for any neighbour w of v we have $f(w) \geq \max\{f(u), f(v)\}$, whence $f(u) = f(v) = f(w)$, as required. Note that every weakly convex function is pseudopeakless. The following result constitutes a partial converse.

REMARK 1. *A real–valued function f defined on the vertex set of a graph G is pseudopeakless if and only if the composition $\alpha \circ f$ is weakly convex for some strictly isotone transformation α of the reals.*

Proof. The property of being pseudopeakless is clearly invariant under strictly isotone transformations of the range. Conversely, let f be a pseudopeakless function taking n distinct values $a_1 < a_2 < \ldots < a_n$. Let α be a strictly isotone map which assigns to each a_i the integer 2^i. We assert that the composition $\alpha \circ f$ is weakly convex. For given vertices u and v at distance 2, let w be their common neighbour such that f is peakless along the path (u, w, v). Say, $a_i = f(u) \leq f(v) = a_k$ and $a_j = f(w) \leq a_k$. If $i = k$, then $i = j = k$ and hence $\alpha \circ f(w) = \alpha \circ f(u) = \alpha \circ f(v)$. Else, we have $i < k$ and thus $j < k$, and consequently

$$2\alpha \circ f(w) = 2^{j+1} \leq 2^k < 2^i + 2^k = \alpha \circ f(u) + \alpha \circ f(v),$$

as required. $\square$

In what follows we will apply Lemma 1 to the functions F_π of G. The simplest instance is given by the weight function π assigning 1 to a distinguished vertex w and 0 otherwise: then $F_\pi = d(\cdot, w)$ just measures the distance in G to that vertex. We say that G is *meshed* [6] if the function $d(\cdot, w)$ is weakly convex for every choice of w. A somewhat weaker property is used in subsequent results. Three vertices u, v, w of a G are said to form a *metric triangle* uvw if the intervals $I(u,v), I(v,w)$, and $I(w,u)$ pairwise intersect only in the common end vertices. If $d(u,v) = d(v,w) = d(w,u) = k$, then this metric triangle is called *equilateral* of *size* k.

REMARK 2. *Every metric triangle in a meshed graph G is equilateral.*

To see this, suppose the contrary: let uvw be a metric triangle in G with $d(u,w) < d(v,w)$. The above definition of weak convexity applied to $f = d(\cdot, w)$, the pair u, v, and the number $\lambda = 1 - 1/d(v,w)$ then provides us with a neighbour x of v which necessarily belongs to $I(u,w) \cap I(v,w)$, a contradiction.

3. Basic characterizations. We commence by giving first answers to the questions raised in the introduction.

PROPOSITION 1. *For a graph G the following conditions are equivalent:*

(i) $\mathrm{Med}_{\mathrm{loc}}(\pi) = \mathrm{Med}(\pi)$ *for all weight functions* π;

(ii) F_π *is weakly convex for all* π;

(iii) F_π *is pseudopeakless for all* π;

(iv) *all level sets* $\{x : F_\pi(x) \leq \lambda\}$ *induce isometric subgraphs;*

(v) *all median sets* $\mathrm{Med}(\pi)$ *induce isometric subgraphs;*

(vi) *all median sets* $\mathrm{Med}(\pi)$ *are connected.*

Any of the conditions (i) *to* (vi) *is equivalent to the analogous condition with the additional requirement that π be positive.*

The following observation is basic for the proof of Proposition 1.

LEMMA 2. *If the function F_π is not weakly convex on the vertex set V of a graph G for some weight function π, then there exist a positive weight function π^+ and vertices u, v at distance 2 such that* $\mathrm{Med}(\pi^+) = \{u, v\}$.

Proof. If F_π is not weakly convex, then by Lemma 1 there exist two vertices u and v at distance 2 such that $2F_\pi(w) > F_\pi(u) + F_\pi(v)$ for all (common neighbours) $w \in I^\circ(u,v)$. This inequality can be maintained under sufficiently small positive perturbations of π : viz., add any δ satisfying

$$0 < \delta \cdot 2 \cdot \#V < \min_{w \in I^\circ(u,v)}(2F_\pi(w) - F_\pi(u) - F_\pi(v))$$

to all weights, yielding the new weights $\pi'(x) = \pi(x) + \delta$. Then

$$\begin{aligned}
&2F_{\pi'}(w) - F_{\pi'}(u) - F_{\pi'}(v) \\
&= 2F_\pi(w) - F_\pi(u) - F_\pi(v) - \delta \cdot \sum_{x \in V} (d(u,x) + d(v,x) - 2d(w,x)) \\
&\geq 2F_\pi(w) - F_\pi(u) - F_\pi(v) - \delta \cdot 2 \cdot \#V > 0
\end{aligned}$$

for all $w \in I^\circ(u,v)$, that is, the initial inequality remains valid with respect to the thus perturbed weight function. We may therefore assume that π is actually positive.

We stipulate that $\epsilon = \frac{1}{2}(F_\pi(v) - F_\pi(u)) \geq 0$. Let μ denote the maximum value of F_π. Define a new positive weight function π^+ by

$$\pi^+(u) = \pi(u) + 2\mu, \qquad \pi^+(v) = \pi(v) + 2\mu + \epsilon,$$

and $\pi^+(x) = \pi(x)$ otherwise. Then

$$F_{\pi^+}(u) = F_\pi(u) + 2(2\mu + \epsilon) = F_\pi(v) + 4\mu = F_{\pi^+}(v)$$

and $\mathrm{Med}(\pi^+) \subseteq I(u,v)$ because

$$F_{\pi^+}(x) \geq F_\pi(x) + 2\mu(d(u,x) + d(v,x)) \geq F_\pi(x) + 6\mu > F_{\pi^+}(u)$$

for every vertex x outside $I(u,v)$. For $w \in I^\circ(u,v)$ one obtains

$$F_{\pi^+}(w) = F_\pi(w) + 4\mu + \epsilon > \frac{1}{2}(F_\pi(u) + F_\pi(v)) + 4\mu + \epsilon = F_{\pi^+}(u).$$

Therefore $\mathrm{Med}(\pi^+)$ consists only of u and v. This concludes the proof. □

Proof of Proposition 1. The implications (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (vi) and (ii) ⇒ (i) are trivial, while (vi) ⇒ (ii) is covered by Lemma 2. It remains to verify that (i) implies (ii). Suppose by way of contradiction that some function F_π is not weakly convex. By Lemma 2 there exist a positive weight function π^+ and vertices u, v at distance 2 such that $\mathrm{Med}(\pi^+) = \{u, v\}$. Pick any ϵ satisfying

$$0 < \epsilon < \min_{x \in V - \{u,v\}} F_{\pi^+}(x) - F_{\pi^+}(v),$$

and define a new weight function π' by

$$\pi'(u) = \pi^+(u) + \epsilon$$

and $\pi'(x) = \pi^+(x)$ otherwise. Then

$$\begin{aligned}
&F_{\pi'}(u) = F_{\pi^+}(u) = F_{\pi^+}(v), \\
&F_{\pi'}(x) = F_{\pi^+}(x) + d(x,u)\epsilon \geq F_{\pi^+}(x) + \epsilon > F_{\pi^+}(v) + 2\epsilon = F_{\pi'}(v)
\end{aligned}$$

for all $x \in V - \{u, v\}$. Therefore both u and v are local minima of $F_{\pi'}$, but $v \notin \mathrm{Med}(\pi')$. This establishes the implication (i) ⇒ (ii).

The same arguments can be applied to prove that the analogous conditions (i$^+$) to (vi$^+$) additionally requiring the weight functions to be positive are all equivalent. Since (i)$\Rightarrow$ (i$^+$) is trivial and (vi$^+$)$\Rightarrow$ (i) is covered by Lemma 2, the proof is complete. □

In view of Lemma 1(iii) and Proposition 1, all median sets are connected if and only if for each pair u, v of vertices at distance 2 the following system of linear inequalities is unsolvable in π:

$$D^{uv}\pi < 0 \text{ and } \pi \geq 0 \text{ with matrix}$$

$$D^{uv} = (d(u,x) + d(v,x) - 2d(w,x))_{w \in I^\circ(u,v), x \in V}.$$

Since LP problems can be solved in polynomial time, we thus obtain the following result.

COROLLARY 1. *The problem to decide whether all median sets* $\mathrm{Med}(\pi)$ *of a graph* G *are connected is solvable in polynomial time.*

4. The main result. In order to obtain a convenient characterization of median connectedness, we will restrict the supports of the weight functions π to be tested and then take advantage of LP duality in the following form:

REMARK 3. *Let* u, v *be any vertices of a graph* G *with* $d(u, v) = 2$, *and let* W *be a nonempty subset of* $I^\circ(u, v)$ *and* X *be a nonempty subset of the vertex set of* G. *Then for every weight function* π *with support included in* X *there exists some* $w \in W$ *such that*

$$F_\pi(u) + F_\pi(v) - 2F_\pi(w) \geq 0$$

if and only if there exists a weight function η *with support included in* W *such that for every* $x \in X$

$$\sum_{w \in W} (d(u,x) + d(v,x) - 2d(w,x))\eta(w) \geq 0.$$

Proof. Let D denote the submatrix of D^{uv} (as just defined) with rows and columns corresponding to W and X, respectively. The asserted equivalence is then a particular instance of LP duality, as formulated by J.A.Ville (cf. [16, p.248]) in terms of systems of linear inequalities:

$$D\pi < 0, \pi \geq 0 \text{ is unsolvable } \iff \eta D \geq 0, \eta \geq 0, \eta \neq 0 \text{ is solvable.}\square$$

For vertices x and y define the following superset of $I(x, y)$:

$$J(x, y) = \{z \in V : I(z, x) \cap I(z, y) = \{z\}\}.$$

PROPOSITION 2. *The median sets* $\mathrm{Med}(\pi)$ *in a graph* G *are connected for all weight functions* π *if and only if* (1) *all metric triangles are equilateral and* (2) *for any vertices* u, v *with* $d(u, v) = 2$ *and every weight function* π *whose support* $\{x \in V : \pi(x) > 0\}$ *is included in the set* $J(u, v)$ *there exists a vertex* $w \in I^\circ(u, v)$ *such that* $2F_\pi(w) \leq F_\pi(u) + F_\pi(v)$.

Proof. Necessity immediately follows from Remark 2 and Proposition 1. To prove sufficiency assume that (1) and (2) hold but there exists a weight function π for which F_π is not weakly convex. Then by Lemma 1 there are vertices u and v at distance 2 such that

$$2F_\pi(w) > F_\pi(u) + F_\pi(v) \text{ for all } w \in I^\circ(u,v).$$

Now, fixing u and v, we may assume that π was chosen so that this inequality still holds and the number of vertices from the support of π outside $J(u,v)$ is as small as possible. By (2) we can find a vertex x with $\pi(x) > 0$ and $x \notin J(u,v)$. To arrive at a contradiction, we will slightly modify π by setting the weight of x to zero (and possibly transferring its old weight to some vertex in $J(u,v)$). We distinguish three cases.

Case 1: $|d(u,x) - d(v,x)| = 2$.

Then $2d(w,x) = d(u,x) + d(v,x)$ for every $w \in I^\circ(u,v)$. Hence the weight function π' defined by

$$\pi'(y) = \begin{cases} \pi(y) & \text{if } y \neq x, \\ 0 & \text{if } y = x \end{cases}$$

satisfies $2F_{\pi'}(w) > F_{\pi'}(u) + F_{\pi'}(v)$, contrary to the minimality assumption on π.

Case 2: $|d(u,x) - d(v,x)| = 1$.

Without loss of generality assume that $d(v,x) = d(u,x)+1$. Let x' be a vertex in $I(u,x) \cap I(v,x)$ at maximal distance to x. Then $x' \in J(u,v)$ and $d(v,x') = d(u,x') + 1$. From the latter equality and $d(u,v) = 2$ we infer that $I(u,v) \cap I(u,x') = \{u\}$. Let t be a vertex in $I(u,v) \cap I(v,x')$ at maximal distance to v. Since $u \notin I(v,x)$, the vertices u, t, x' must constitute an equilateral metric triangle. Therefore $d(u,x') = d(t,u) = 1$ and $d(v,x') = 2$. Define a new weight function π', where

$$\pi'(y) = \begin{cases} \pi(y) & \text{if } y \neq x, x', \\ 0 & \text{if } y = x, \\ \pi(x) + \pi(x') & \text{if } y = x'. \end{cases}$$

Let $k = d(u,x) = d(v,x) - 1$. Then

$$F_{\pi'}(u) + F_{\pi'}(v) = F_\pi(u) + F_\pi(v) - (2k-2)\pi(x).$$

For every $w \in I^\circ(u,v)$ we have either

$$d(w,x) = k \quad \text{and } 1 \leq d(w,x') \leq 2,$$

or

$$d(w,x) = k+1 \quad \text{and } d(w,x') = 2.$$

Hence

$$F_{\pi'}(w) = F_\pi(w) - (d(w,x) - d(w,x'))\pi(x) \geq F_\pi(w) - (k-1)\pi(x),$$

and further

$$2F_{\pi'}(w) - F_{\pi'}(u) - F_{\pi'}(v)$$

$$\geq 2F_\pi(w) - 2(k-1)\pi(x) - F_\pi(u) - F_\pi(v) + 2(k-1)\pi(x)$$
$$= 2F_\pi(w) - F_\pi(u) - F_\pi(v) > 0,$$

again a contradiction to the choice of π.

Case 3: $d(u,x) = d(v,x)$.

Let x' be a vertex in $I(u,x) \cap I(v,x)$ at maximal distance to x. Since all metric triangles of G are equilateral and $d(u,v) = 2$, we obtain that

$$1 \leq d(x',u) = d(x',v) \leq 2.$$

Define the new weight function π' as in Case 2. Then

$$F_{\pi'}(u) = F_\pi(u) - d(x,x')\pi(x),$$

$$F_{\pi'}(v) = F_\pi(v) - d(x,x')\pi(x),$$

and for every $w \in I^\circ(u,v)$

$$F_{\pi'}(w) = F_\pi(w) - (d(x,w) - d(w,x'))\pi(x)$$
$$\geq F_\pi(w) - d(x,x')\pi(x),$$

a final contradiction. □

In view of Remark 2 we may require in Proposition 2 (as well as the Theorem below) that G be meshed instead that all metric triangles in G be equilateral.

Let $M(u,v)$ denote the set of those vertices of $J(u,v)$ which are equidistant to u and v :

$$M(u,v) = \{x \in V : I(x,u) \cap I(x,v) = \{x\} \text{ and } d(u,x) = d(v,x)\}.$$

The *neighbourhood* $N(x)$ of a vertex x consists of all vertices adjacent to x. Note that if $d(u,v) = 2$ then $N(u) \cap N(v) = I^\circ(u,v) \subseteq M(u,v)$.

LEMMA 3. *If all median sets of a graph G are connected, then for any vertices u and v at distance 2 there exist (not necessarily distinct) vertices $s,t \in I^\circ(u,v)$ such that*

$$d(s,y) + d(t,y) \leq d(u,y) + d(v,y)$$

for all vertices $y \in M(u,v)$.

Proof. We define a weight function π for which the hypothesis that F_π be pseudopeakless implies the asserted inequality: let

$$\pi(x) = \begin{cases} 0 & \text{if } x \notin M(u,v), \\ 4 & \text{if } x \in M(u,v) - I^\circ(u,v), \\ 2 & \text{if } x \in I^\circ(u,v) \ \text{ with } \ \#(I^\circ(u,v) \cap N(x)) \geq \#I^\circ(u,v) - 2 \\ & \quad \text{and } d(x,y) = 2 \text{ for all } y \in M(u,v) - I^\circ(u,v), \\ 3 & \text{otherwise.} \end{cases}$$

Since weights are zero outside $M(u,v)$, we have $F_\pi(u) = F_\pi(v)$. Select $s \in I^\circ(u,v)$ such that F_π is peakless along the (shortest) path (u,s,v), that is, $F_\pi(s) \le F_\pi(u) = F_\pi(v)$. Note that

$$d(u,x) + 1 \ge d(s,x) \ge d(u,x) = d(v,x) \text{ for all } x \in M(u,v) - \{s\}.$$

We claim that $\pi(s) = 2$. Indeed, otherwise $\pi(s) = 3$ holds and either there are two distinct vertices $x, y \in I^\circ(u,v)$ with $d(s,x) = d(s,y) = 2$ or there exists some vertex $z \in M(u,v) - I^\circ(u,v)$ with $d(s,z) = 3$. In the first case we would get

$$F_\pi(s) - F_\pi(u) \ge \pi(x) + \pi(y) - \pi(s) \ge 2 + 2 - 3 > 0,$$

and in the second case

$$F_\pi(s) - F_\pi(u) \ge \pi(z) - \pi(s) = 4 - 3 > 0,$$

both giving a contradiction. Now, if all vertices in $M(u,v) - \{s\}$ are equidistant to s and u (as well as v), then

$$2d(s,y) \le d(u,y) + d(v,y) \text{ for all } y \in M(u,v),$$

and thus $t = s$ is the required solution to the asserted inequality. Else, there exists some $t \in M(u,v)$ with $d(s,t) = d(u,t) + 1$, yielding

$$0 \le F_\pi(u) - F_\pi(s) \le \pi(s) - \pi(t),$$

whence $\pi(t) = 2$ and $d(s,t) = 2$. Consequently,

$$d(s,y) + d(t,y) = d(u,y) + d(v,y) \text{ for all } y \in M(u,v),$$

and therefore s, t constitutes the required vertex pair in this case. □

LEMMA 4. *Let u and v be vertices at distance 2 in a graph G, for which all median sets are connected. Select a maximal subset S of $I^\circ(u,v)$ with the property that for each vertex $s \in S$ there exists a vertex $t \in S$ (not necessarily distinct from s) such that $d(s,y) + d(t,y) \le d(u,y) + d(v,y)$ for all $y \in M(u,v)$. Then for any weight function π with support included in $J(u,v)$ there exists some vertex $w \in S$ satisfying $2F_\pi(w) \le F_\pi(u) + F_\pi(v)$.*

Proof. By Lemma 3, the set S is nonempty. Now, assume the contrary and among all weight functions violating our assertion, choose a function π for which the set

$$\{x \in I^\circ(u,v) - S : 2F_\pi(x) - F_\pi(u) - F_\pi(v) \le 0\}$$

is as small as possible. This set contains some vertex x because F_π is weakly convex. Pick any $\delta > \frac{1}{2}(F_\pi(u) + F_\pi(v) - 2F_\pi(x)) \ge 0$.

First suppose that $d(x,y) = d(x,z) = 2$ for some distinct vertices $y, z \in I^\circ(u,v)$. Define a new weight function π' by

$$\pi'(w) = \begin{cases} \pi(w) + \delta & \text{if } w \in \{x,y,z\}, \\ \pi(w) & \text{otherwise.} \end{cases}$$

Then

$$F_{\pi'}(u) = F_\pi(u) + 3\delta, \qquad F_{\pi'}(v) = F_\pi(v) + 3\delta, \qquad F_{\pi'}(x) = F_\pi(x) + 4\delta,$$
$$F_{\pi'}(w) \geq F_\pi(w) + 3\delta \text{ for all } w \in I^\circ(u,v), w \neq x.$$

Hence

$$2F_{\pi'}(x) - F_{\pi'}(u) - F_{\pi'}(v) = 2F_\pi(x) - F_\pi(u) - F_\pi(v) + 2\delta > 0$$

and

$$2F_{\pi'}(w) - F_{\pi'}(u) - F_{\pi'}(v) \geq 2F_\pi(w) - F_\pi(u) - F_\pi(v)$$

for all $w \in I^\circ(u,v)$ distinct from x, contrary to the choice of π. Therefore, x is adjacent to all vertices $w \in I^\circ(u,v)$, $w \neq x$, except possibly one vertex.

Now, suppose that $d(x,z) = 3$ for some vertex $z \in M(u,v)$. Then for the modified weight function π' defined by

$$\pi'(w) = \begin{cases} \pi(z) + \delta & \text{if } w = z, \\ \pi(w) & \text{otherwise} \end{cases}$$

we have

$$F_{\pi'}(u) = F_\pi(u) + 2\delta, \quad F_{\pi'}(v) = F_\pi(v) + 2\delta, \quad F_{\pi'}(x) = F_\pi(x) + 3\delta,$$
$$F_{\pi'}(w) \geq F_\pi(w) + 2\delta \text{ for all } w \in I^\circ(u,v), w \neq x.$$

Then we obtain the same inequalities as in the preceding case and thus again arrive at a contradiction. We conclude that $d(x,z) = d(x,u) = d(x,v) = 2$ for all $z \in M(u,v) - I^\circ(u,v)$.

If x is adjacent to all other vertices of $I^\circ(u,v)$, then $2d(x,z) = d(x,u) + d(x,v)$ for all $z \in M(u,v)$, $z \neq x$, and hence we could adjoin x to S, contrary to the maximality of S. Therefore, by what has been shown, $I^\circ(u,v)$ contains exactly one vertex y distinct from x which is not adjacent to x. Since $x \notin S$, there exists a vertex $z \in M(u,v)$ such that $d(x,z) + d(y,z) > d(u,z) + d(v,z)$. Then necessarily $d(x,z) = d(u,z) = d(y,z) - 1 = k$ for $k \in \{1,2\}$. Define a new weight function π' by

$$\pi'(w) = \begin{cases} \pi(w) + \delta & \text{if } w \in \{y,z\}, \\ \pi(w) & \text{otherwise.} \end{cases}$$

Then

$$F_{\pi'}(u) = F_\pi(u) + (k+1)\delta, \quad F_{\pi'}(v) = F_\pi(v) + (k+1)\delta,$$
$$F_{\pi'}(x) = F_\pi(x) + (k+2)\delta, \quad F_{\pi'}(y) = F_\pi(y) + (k+1)\delta,$$
$$F_{\pi'}(w) \geq F_\pi(w) + (k+1)\delta \text{ for all } w \in I^\circ(u,v), w \neq x, y,$$

giving the same contradiction as before. This concludes the proof. □

Now we are in a position to formulate the principal result.

THEOREM. *For a graph G all median sets are connected if and only if the following conditions are satisfied:*

(i) *all metric triangles of G are equilateral;*

(ii) *for any vertices u and v at distance 2 there exist a nonempty subset S of $I^{\circ}(u,v)$ and a weight function η with support included in S having the two properties:*

(α) *every vertex $s \in S$ has a companion $t \in S$ (not necessarily distinct from s) such that*

$$d(s,x) + d(t,x) \leq d(u,x) + d(v,x) \text{ for all } x \in M(u,v),$$

and $\eta(s) = \eta(t)$ whenever $d(s,t) = 2$;

(β) *the joint weight of the neighbours of any $x \in J(u,v) - M(u,v)$ from S is always at least half the total weight of S :*

$$\sum_{s \in S \cap N(x)} \eta(s) \geq \frac{1}{2} \sum_{s \in S} \eta(s) \text{ for all } x \in J(u,v) - M(u,v).$$

Proof. First assume that G is a graph with connected medians. By Proposition 2 all metric triangles of G are equilateral. Let u, v be a pair of vertices at distance 2. The nonempty subset S of $I^{\circ}(u,v)$ provided by Lemma 4 satisfies the inequality in (α) for each $s \in S$ and companion t. Moreover, Lemma 4 guarantees that for every weight function π with support included in $J(u,v)$ there exists a vertex $w \in S$ with $F_{\pi}(u) + F_{\pi}(v) - 2F_{\pi}(w) \geq 0$. Now, LP duality as formulated in Remark 3 for $W = S$ and $X = J(u,v)$ yields a weight function η with support included in S such that for every $x \in J(u,v)$ the weighted sum of all $d(u,x) + d(v,x) - 2d(w,x)$ ($w \in S$) is nonnegative. If $x \in J(u,v) - M(u,v)$, then

$$d(u,x) + d(v,x) - 2d(w,x) = \begin{cases} 1 & \text{if } \ w \in S \cap N(x), \\ -1 & \text{if } \ w \in S - N(x), \end{cases}$$

and therefore (β) holds. If $x \in M(u,v)$, then $d(s,x) \leq 2$ for all $s \in S$ and

$$d(u,x) + d(v,x) - 2d(w,x) = \begin{cases} 2 & \text{if } \ w = x \in S, \\ 0 & \text{if } \ w \in S \ \text{ with } \ d(w,x) = d(u,x), \\ -2 & \text{if } \ w \in S \ \text{ with } \ d(w,x) = 2 \ \text{ and } \ d(u,x) = 1. \end{cases}$$

In the latter case $t = x \in S$ is the companion of $s = w$. This yields the inequality $\eta(t) - \eta(s) \geq 0$, and by interchanging the role of s and t, we infer that $\eta(s) = \eta(t)$ whenever $d(s,t) = 2$.

Conversely, if conditions (i) and (ii) are satisfied, then by virtue of LP duality (Remark 3) conditions (1) and (2) of Proposition 2 are fulfilled, whence G has connected medians. □

COROLLARY 2. *All median sets in a meshed graph G are connected whenever the following condition is satisfied:*

($*$) *for any two vertices u and v with $\#I^\circ(u,v) \geq 2 = d(u,v)$ there exist (not necessarily distinct) vertices $s, t \in I^\circ(u,v)$ such that*

$$d(s,x) + d(t,x) \leq d(u,x) + d(v,x) \quad \textit{for all} \quad x \in J(u,v).$$

If, moreover, G satisfies the stronger condition requiring in addition that $d(s,t) \leq 1$, then $\mathrm{Med}(\pi)$ *induces a complete subgraph for every positive weight function π.*

Proof. First observe that the inequality of ($*$) also holds in the case that u and v are at distance 2 with a unique common neighbour s. Indeed, if $x \in J(u,v)$, then necessarily $2d(s,x) \leq d(u,x) + d(v,x)$ since G is meshed. To see that ($*$) implies condition (ii) of the Theorem, put $S = \{s,t\}$ and $\eta \equiv 1$. Then (α) is trivially satisfied. For $x \in J(u,v) - M(u,v)$ we have $d(u,x) + d(v,x) = 3$ because G is meshed, and therefore x is adjacent to at least one of s, t, thus establishing (β).

To prove the second statement, observe that the inequality in ($*$) actually holds for all vertices x of G. Indeed, for each vertex x select a vertex x' from $I(u,x) \cap I(v,x)$ at maximal distance to x. Then x' belongs to $J(u,v)$ and

$$\begin{aligned} d(s,x) + d(t,x) &\leq d(s,x') + d(t,x') + 2d(x,x') \\ &\leq d(u,x') + d(v,x') + 2d(x,x') \\ &= d(u,x) + d(v,x). \end{aligned}$$

Adding up all these inequalities each multiplied by the corresponding weight $\pi(x)$ then yields

$$F_\pi(s) + F_\pi(t) \leq F_\pi(u) + F_\pi(v),$$

where strict inequality holds exactly when one of the former inequalities is strict for x with $\pi(x) > 0$, for instance, when $\pi(s) > 0$. Hence, if π is positive, u and v cannot both belong to $\mathrm{Med}(\pi)$ under these circumstances. □

In general, we can neither dispense with weight functions nor impose any fixed upper bound on the cardinalities of the sets S occuring in the Theorem, as the following example shows.

EXAMPLE. Let G be the chordal graph with $2\binom{2k}{2}+2k+6$ ($k \geq 2$) vertices comprising a set $S = \{x_1, x_2, y_1, \ldots, y_{2k}\}$ of mutually adjacent vertices, two vertices u and v adjacent to all of S, and vertices w_Z associated with certain subsets Z of S, namely the sets $\{x_1, x_2\}, \{y_1, \ldots, y_{2k}\}$, and $\{x_i\} \cup Y$ for all $i = 1, 2$ and k–subsets Y of $\{y_1, \ldots, y_{2k}\}$, such that each vertex w_Z is adjacent to u and all vertices from Z. Then a weight function η with support included in $I^\circ(u,v) = S$ satisfies (β) for the pair u, v if and only if for each of the above sets Z (which come in complementary pairs) one has

$$\sum_{z \in Z} \eta(z) = \sum_{y \in S - Z} \eta(y),$$

which is equivalent to

$$\eta(x_i) = k \cdot \eta(y_j) \quad \text{for all} \quad i = 1, 2 \quad \text{and} \quad j = 1, \dots, 2k.$$

Thus, in order to satisfy (β), we are forced to take a weight function η having a large support on which it is not constant. Note that (α) for u, v is trivially fulfilled with any choice of $s = t$ from S. Finally, every other pair of vertices at distance 2 meets condition $(*)$ of Corollary 2 when either replacing u by w_Z and selecting $s = t$ from Z or replacing u, v by $w_Z, w_{Z'}$ $(Z \neq Z')$ and setting $s = t = u$. This shows that G has connected medians.

5. Particular cases. A number of graph classes [12] consist of particular meshed graphs, for instance, the classes of chordal graphs (and more generally, bridged graphs), Helly graphs, and weakly median graphs (see below), respectively. In the case of chordal graphs the first statement of Corollary 2 is due to Wittenberg [26, Theorem 1.1]. Whenever a chordal graph satisfies condition $(*)$, then any pair s, t selected in $(*)$ necessarily satisfies $d(s,t) \leq 1$. Corollary 2 comes into full action for the class of Helly graphs. A *Helly graph* G (alias "absolute retract of reflexive graphs") has the characteristic "Helly property" that for any vertices $v_1, \dots, v_n$ and non–negative integers $r_1, \dots, r_n$ the system of inequalities $d(v_i, x) \leq r_i \quad (i = 1, \dots, n)$ admits a solution x whenever $d(v_i, v_j) \leq r_i + r_j \quad (i, j = 1, \dots, n)$ holds [8]. Since this Helly property for $n = 3$ implies weak convexity (as formulated in Lemma 1(iii)) of the function $d(\cdot, z)$, all Helly graphs are meshed. To verify condition $(*)$ (with $d(s,t) \leq 1$), first observe that the Helly property for $n = 3$ entails $J(u,v) \subseteq N(u) \cup N(v)$ and thus $M(u,v) = I^\circ(u,v)$ for every pair u, v with $d(u,v) = 2$. Now, all vertices in $(N(u) \cap J(u,v)) \cup \{v\}$ are pairwise at distance ≤ 2, whence the Helly property guarantees a common neighbour s. Similarly, the vertices from $(N(v) \cap J(u,v)) \cup \{u\}$ have a common neighbour t. Necessarily $s, t \in I^\circ(u,v)$ and $d(s,t) \leq 1$ hold, and we have therefore established the following observation, which implies the result of Lee and Chang [20] on strongly chordal graphs (being special Helly graphs):

COROLLARY 3. *All median sets* $\mathrm{Med}(\pi)$ *of a Helly graph* G *are connected, and moreover, if* π *is positive, then* $\mathrm{Med}(\pi)$ *induces a complete graph in* G.

In some particular classes of meshed graphs the pair s, t meeting $(*)$ can always be selected by the following trivial rule: given vertices u, v at distance 2, choose any pair s, t from $I^\circ(u,v)$ for which $d(s,t)$ is as large as possible. Evidently, a meshed graph G satisfies condition $(*)$ with this selection rule provided that the following two requirements are met:

(§) if $I^\circ(u,v)$ is a complete subgraph for vertices u and v at distance 2, then $d(s,x) + d(t,x) \leq d(u,x) + d(v,x)$ for all $s, t \in I^\circ(u,v)$ with $s \neq t$ and all $x \in J(u,v)$;

(§§) if s, t, u, v induce a 4–cycle where $d(s,t) = d(u,v) = 2$, then $d(s,x) + d(t,x) = d(u,x) + d(v,x)$ for all $x \in J(u,v)$.

Notice that in (§§) we could replace the last equality by an inequality because the inequality would imply $J(s,t) = J(u,v)$ in a meshed graph and thus the reverse inequality would follow by symmetry. We will now show that the so–called weakly median graphs satisfy (§) and (§§). A graph is *weakly median* [14] if (i) any three distinct common neighbours of any two distinct vertices u and v always induce a connected subgraph and (ii) all functions $d(\cdot, z)$ satisfy the following condition, stronger than weak convexity: if u, v are at distance 2 and $|d(u,z) - d(v,z)| \leq 1$, then $2d(w,z) < d(u,z) + d(v,z)$ for some $w \in I^{\circ}(u,v)$. Condition (ii) is also referred to *weak modularity*; see [4]. Trivially, weakly modular graphs are meshed.

REMARK 4. *A weakly modular graph G satisfies conditions (§) and (§§) if and only if G does not contain any of the graphs of Fig.1 as an induced subgraph. In particular, weakly median graphs have connected medians.*

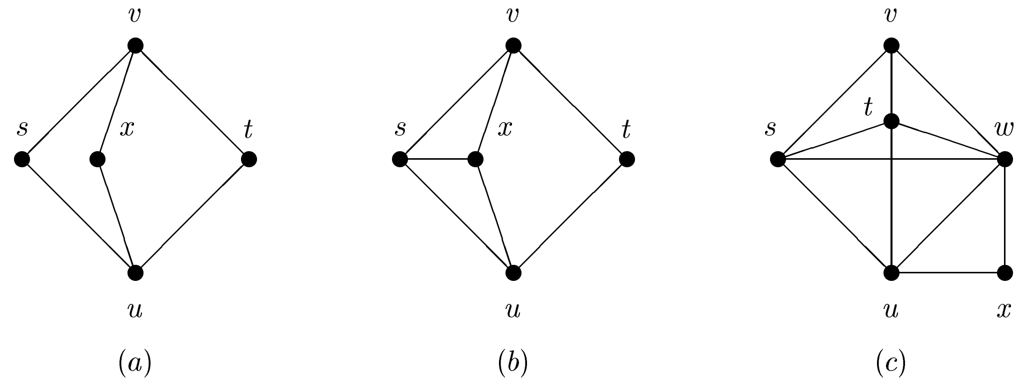

FIG. 1

Proof. If G includes an induced subgraph from Fig. 1, then s, t, u, v, x violate either (§§) or (§). Conversely, assume that G is weakly modular but violates (§) or (§§) for some vertices s, t, u, v, x. Then, as $x \in J(u,v)$, we have $d(u,x) + d(v,x) \leq 3$ by weak modularity. If $x \in I^{\circ}(u,v)$, then necessarily s, t, u, v, x induce one of the first two graphs in Fig. 1. Otherwise, say, $d(u,x) + 1 = 2 = d(v,x) = d(s,x) = d(t,x)$. As G is meshed, u, v, x have some common neighbour w. If w is not adjacent to both s and t, then we are back in the preceding case with w playing the role of x. Therefore we may assume that w is a common neighbour of s and t. If s and t are not adjacent, then s, t, u, v, w induce a subgraph isomorphic to Fig. 1b. Otherwise, s and t are adjacent, and we obtain the graph of Fig. 1c as an induced subgraph. Finally, note that the latter graph minus v as well as Fig. 1a,b are all forbidden in a weakly median graph. $\square$

Inasmuch as pseudo–median graphs and quasi–median graphs are weakly median, Remark 4 (in conjunction with Proposition 1) generalizes some results from [3, 7, 17, 24]. Another class of meshed graphs fulfilling (§) and (§§) is associated with matroids. A *matroid* $\mathcal{M}$ can be defined as a finite set E together with a collection $\mathcal{B}$ of subsets (referred to as the *bases* of $\mathcal{M}$) such that for $B, B' \in \mathcal{B}$ and $e \in B - B'$ there exists some $e' \in B' - B$ with $(B' - \{e'\}) \cup \{e\} \in \mathcal{B}$. The *basis graph* of $\mathcal{M}$ is the (necessarily connected) graph whose vertices are the bases of $\mathcal{M}$ and edges are the pairs B, B' of bases differing by a single exchange. We immediately infer from the characterization established by Maurer [21] that the basis graph of every matroid is meshed (but not weakly modular in general) and satisfies (§) and (§§).

COROLLARY 4. *The basis graph of every matroid has connected medians.*

In view of Corollary 4 and Proposition 1 one can solve the median problem in the basis graph of a matroid $\mathcal{M} = (E, \mathcal{B})$ with the greedy algorithm. Given a weight function π on

$\mathcal{B}$ assign a weight to each element $e \in E$ by $w(e) = \sum\{\pi(B) : e \in B\}$. Applying the greedy algorithm one finds a base B^* maximizing the function $B \mapsto w(B) = \sum\{w(e) : e \in B\}$. We assert that B^* is a minimum for the function F_π in the basis graph. Indeed, if $B' = (B^* - \{e\}) \cup \{e'\}$ is any neighbour of B^*, then

$$\begin{aligned} F_\pi(B^*) - F_\pi(B') &= \sum\{\pi(B) : e \notin B, e' \in B\} - \sum\{\pi(B) : e \in B, e' \notin B\} \\ &= w(e') - w(e) \\ &= w(B') - w(B) \leq 0. \end{aligned}$$

Hence $B^* \in \mathrm{Med}_{\mathrm{loc}}(\pi) = \mathrm{Med}(\pi)$, as required.

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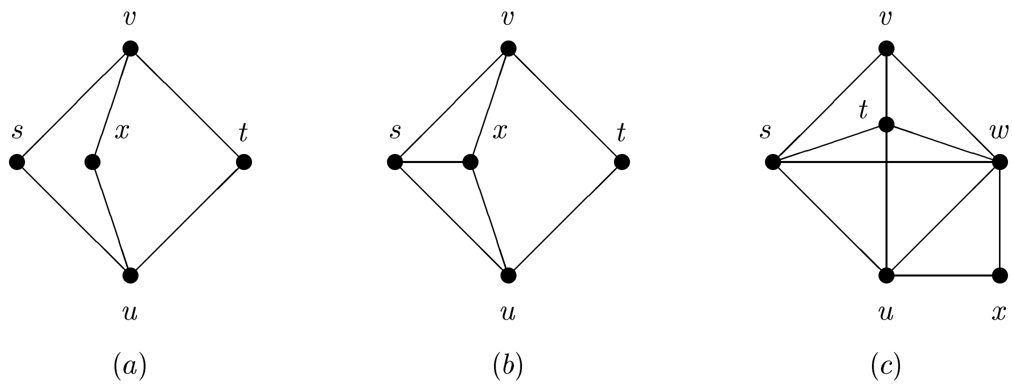

FIG. 1