On distance–preserving and domination elimination orderings

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Abstract. A *distance–preserving elimination ordering* of a graph G is a linear ordering $v_1, v_2, \ldots, v_n$ of the vertices such that each subgraph $G_i = G(v_1, \ldots, v_i)$, $i < n$, is an isometric subgraph of G. We prove that the ordering of the vertices of a pseudo–modular or a house–free weakly modular graph G produced by the Breadth–First Search is distance–preserving. We specify this result by showing that if in addition G does not contain the cycles $C_n, n \geq 5$, and the bipyramids $bipyr(C_m), m \geq 6$, as an isometric subgraph, then any ordering produced by the Lexicographic Breadth–First Search is a *domination elimination ordering* (i.e. every vertex v_i is dominated by some vertex v_j, $j < i$, or, in other words, every vertex v_k, $k < i$, adjacent to v_i is adjacent to v_j, too).

Key words: distance–preserving ordering, domination elimination ordering, Breadth–First Search, Lexicographic Breadth–First Search, weakly modular graph.

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1. Introduction

Various authors considered specific elimination orderings to characterize certain graph classes or discrete structures. The theory of elimination orderings is used in designing efficient algorithms for solving a number of combinatorial optimization, facility location and scheduling problems, as well as in Gaussian elimination on sparse systems of linear equations. One of the first classes of graphs to be recognized by a specific ordering of vertices was the class of chordal graphs. They are the graphs having *perfect elimination orderings,* i.e. the greater neighbours of any vertex form a complete subgraph. Linear time algorithms to recognize chordal graphs and to compute a perfect elimination ordering are the *Lexicographic Breadth–First Search (LBFS)* of Rose, Tarjan and Lueker [35] and the *Maximum Cardinality Search (MCS)* of Tarjan and Yannakakis [39] (Farber and Jamison [22] and Shier [37] prove similar results while studying the notion of induced–path convexity in chordal graphs). Jamison and Olariu [24] introduced the notion of *semisimplicial elimination ordering,* which relaxes that of

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perfect elimination ordering. They stated that any ordering produced by the procedure LBFS is a semiperfect elimination ordering if and only if the graph does not contain the house, the domino or any cycle C_n, with $n \geq 5$, as an induced subgraph; such graphs are called HHD–*free* (for a new proof of this result see Dragan et al. [19]). Dahlhaus et al. [21] established that if the graph does not contain the house and any cycle C_n, $n \geq 5$ as an induced subgraph (in [21] such graphs are called *HC–free graphs*) then any ordering produced by MCS or LBFS is a domination elimination ordering (a new proof was given by Dragan [20]). Finally, note that Scharlau [36] used a variant of Breadth–First Search to prove shellability of a class of numbered simplicial complexes.

In recent years several classes of graphs have been investigated from a metric point of view. These are modular graphs (which generalize the median graphs, cf. Bandelt [2], Bandelt and Hedlíková [4], Mulder [30]), distance–hereditary graphs (cf. Bandelt and Mulder [5], Hammer and Maffray [27], Howorka [29]), bridged graphs (which generalize chordal graphs, cf. Anstee and Farber [1], Farber and Jamison [23], Soltan and Chepoi [38]), pseudo–modular and pseudo–median graphs (Bandelt and Mulder [6] and Bandelt and Mulder [7]), the absolute retracts of reflexive graphs alias Helly graphs and their bipartite variants (cf. Bandelt and Pesch [10], Nowakowski and Rival [32], Quilliot [33]) and their numerous subclasses. These classes have their distinctive features, but anyway their members share two metric properties, namely, the triangle and the quadrangle conditions [3, 16, 8]. In [3] such graphs were dubbed *weakly modular graphs.* Again, the recognition problem for a majority of these classes of graphs is solvable in polynomial time by applying special vertex elimination schemes. Usually, the members of these classes admit a *domination elimination ordering,* i.e. an ordering $v_1, \ldots, v_n$ of the vertices such that each v_i is dominated by some $v_j, j < i$, i.e. all vertices $v_k, k < i$, adjacent to v_i are adjacent to v_j, too. If, in addition, v_j is a neighbour of v_i, then the graphs admitting such an ordering are called *dismantlable* or *cop–win graphs* in view of the results of Nowakowski and Winkler [31] and Quilliot [34]. That bridged graphs are cop–win graphs has been established by Anstee and Farber [1]. Recently, we proved [18] that a dismantling scheme of a bridged graph can be computed in linear time just by applying the *Breadth–First Search (BFS).* In this note we continue this line of research, by investigating the metric properties of the orderings produced by BFS and LBFS. We prove that for a large class of graphs (house–free weakly modular graphs and all pseudo–modular graphs) every ordering $v_1, \ldots, v_n$ of a graph G generated by BFS is a *distance–preserving ordering,* i.e. every subgraph G_i induced by the vertices $v_1, \ldots, v_i$ is a distance–preserving alias isometric subgraph of G. For this purpose we introduce and investigate the class of pseudo–peakless functions (the lower sets of such functions induce isometric subgraphs). To find a distance–preserving ordering of a graph with n vertices is equivalent to construct a pseudo–peakless function which takes n distinct values. As for domination elimination orderings we characterize the house–free graphs such that any ordering produced by BFS or by LBFS of each isometric subgraph is a domination elimination ordering.

2. Weakly modular graphs

All graphs occuring in this note are finite, connected, and without loops or multiple edges. The *distance* $d(u, v)$ between two vertices u and v of a graph G is the length of a shortest path between u and v. The set of all vertices w on shortest paths between u and v is called the *interval* $I(u,v)$ between u and v, that is,

$$I(u,v) = \{x : d(u,v) = d(u,x) + d(x,v)\}.$$

An induced subgraph H of a graph G is *isometric* if the distance between any pair of vertices in H is the same as that in G. An induced subgraph (or a subset of vertices) H is called *convex* if H includes every interval $I(u,v)$ with u,v in H.

The *disk* with centre u and radius k is the set

$$D_k(u) = \{x : d(u,x) \leq k\}.$$

We will also use the notation $N[u]$ for the disk $D_1(u)$. In particular, $N[u] = N(u) \cup \{u\}$, where $N(u) = \{x : d(u,x) = 1\}$ is the *(open) neighbourhood* of u. More generally, for a subset S let $N(S)$ denote the *neighbourhood* of S, i.e. $N(S) = \cup_{v\in S} N(v)$. Finally, $G(S)$ denotes the subgraph induced by S.

A graph G is *weakly modular* [3, 8, 16] if its shortest–path metric $d = d_G$ satisfies the following two conditions:

Triangle condition: for any three vertices u,v,w with

$$1 = d(v,w) < d(u,v) = d(u,w)$$

there exists a common neighbour x of v and w such that $d(u,x) = d(u,v) - 1$;

Quadrangle condition: for any four vertices u,v,w,z with

$$d(v,z) = d(w,z) = 1 \quad \text{and} \quad d(u,v) = d(u,w) = d(u,z) - 1,$$

there exists a common neighbour x of v and w such that $d(u,x) = d(u,v) - 1$.

We can define weakly modular graphs using the concept of metric triangle. Recall that three vertices u,v and w form a *metric triangle* uvw if the intervals $I(u,v), I(v,w)$ and $I(w,u)$ pairwise intersect only in the common end vertices. (If uvw is a metric triangle, then any permutation of the letters u,v, and w defines the same metric triangle.) According to [16], G is weakly modular if and only if for every metric triangle uvw all vertices of the interval $I(v,w)$ are at the same distance $k = d(u,v)$ from u. The number k is called the *size* of the metric triangle uvw. A metric triangle uvw is a *pseudo–median* of the triple x,y,z if the following metric equalities are satisfied:

$$d(x,y) = d(x,u) + d(u,v) + d(v,y),$$

$$d(y,z) = d(y,v) + d(v,w) + d(w,z),$$

$$d(z,x) = d(z,w) + d(w,u) + d(u,x).$$

A graph in which every metric triangle is degenerate, that is, has size 0 is called *modular* [2]. In other words, a graph is modular if $I(x,y) \cap I(y,z) \cap I(z,x)$ is nonempty for every triple x,y,z. Now, a *pseudo–modular graph* is a graph in which each metric triangle has the size at most 1. According to [6], pseudo–modular graphs can be characterized in the following way:

if $1 \leq d(u,w) \leq 2$ and $k = d(u,v) = d(v,w) \geq 2$ for vertices u,v,w, then there exists a common neighbour x of u and w such that $d(v,x) = k - 1$.

A graph is called *hereditary weakly modular* if every isometric subgraph is weakly modular. In a similar way we can define *hereditary modular* [2] and *hereditary pseudo–modular graphs* [6]. Recall from [2], that hereditary modular graphs are graphs in which all isometric cycles have length four, while hereditary pseudo–modular graphs are the graphs which do not contain the house, the 3–sun, or any cycle $C_n, n \geq 5$, as an isometric subgraph (the forbidden isometric subgraphs listed in Figure 1). Chordal bipartite and distance–hereditary graphs represent two important instances of hereditary modular and hereditary pseudo–modular graphs, respectively. Finally, hereditary weakly modular graphs are the graphs which do not contain the house or any cycle $C_n, n \geq 5$, as an isometric subgraph [16]. Besides these two classes of graphs, the class of hereditary weakly modular graphs comprises the important classes of *bridged graphs* (where all isometric cycles have length three [23, 38]), HHD-*free graphs* ([24]), *chordal graphs*, HC–free graphs [21] alias *house–free weakly triangulated graphs* ([28]).

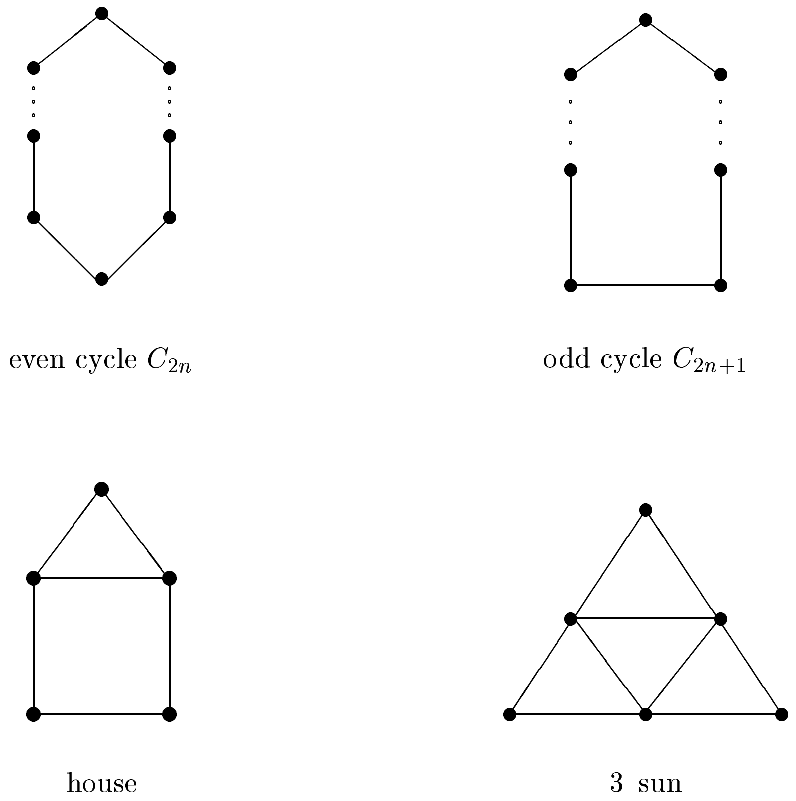


FIGURE 1. Forbidden graphs.

3. Distance–preserving orderings and pseudo–peakless functions

For a given linear ordering $v_1, v_2, \ldots, v_n$ of the vertices of a graph G let G_i denote the subgraph induced by $\{v_1, v_2, \ldots, v_i\}$, and let $N_i(v_j)$ denote $N(v_j) \cap \{v_1, \ldots, v_i\}$, and accordingly $N_i[v_j] = N_i(v_j) \cup \{v_j\}$.

An ordering $v_1, v_2, \ldots, v_n$ of G is *distance–preserving* if each $G_i, i = 1, 2, \ldots, n$, is an isometric subgraph of the graph G. It is quite evident that perfect elimination orderings, semiperfect elimination orderings and domination elimination orderings are distance–preserving. In Figure 2 we present an example of a distance–preserving ordering of a 4–cube.

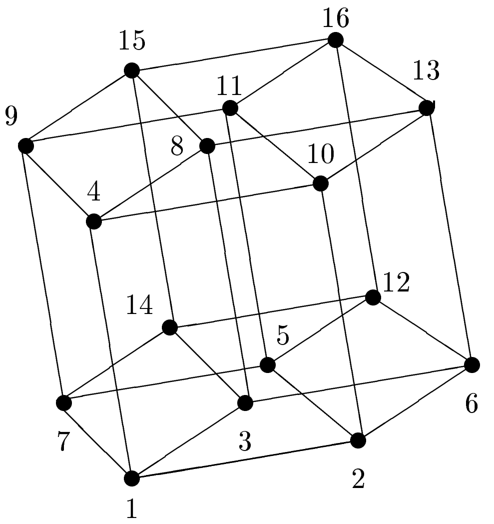


FIGURE 2. A distance–preserving ordering of a 4–cube.

There are close relationships between distance–preserving orderings and a certain class of functions, which we shall call pseudo–peakless. Let $P = (x_0, x_1, \ldots, x_p)$ be a path of a graph G. A real valued function f defined on P is *peakless* if $0 \leq j < i < k \leq p$ implies $f(x_i) \leq \max\{f(x_j), f(x_k)\}$ and equality holds only if $f(x_j) = f(x_k)$. A function f defined on the vertices of a graph G is *peakless* if the restriction of f on any shortest path of G is peakless. Peakless functions were introduced and studied by Busemann [12] and Busemann and Phadke [13] in the geometry of geodesics; see [14] for a recent survey. In graphs peakless functions were considered in [17]. Peakless functions inherit and generalize the properties of usual convex functions. Now, a function f defined on a graph G is called *pseudo–peakless* if any two vertices of G can be joined by a shortest path along which f is peakless. Equivalently, f is pseudo–peakless if for any two non–adjacent vertices u, v there is a vertex $w \in I(u,v) - \{u,v\}$ such that $f(w) \leq \max\{f(u), f(v)\}$ and equality holds only if $f(u) = f(v)$. The most useful property of pseudo–peakless functions is their unimodality, that is, any local minimum of f is global. The proof is simple: let u be a global minimum of f and v an arbitrary vertex of G. Consider the shortest path P between u and v along which f is peakless. Then either f is constant on this path, or for the neighbour w of v in P we have $f(w) < max\{f(u), f(v)\} = f(v)$. In the either case v is a global minimum too, otherwise v is not a local minimum.

Finally, a function f is *strictly pseudo–peakless,* if any two non–adjacent vertices u, v of G can be connected by a shortest path P along which f is strictly peakless, i.e. $f(w) < max\{f(u), f(v)\}$ for any $w \in P - \{u, v\}$.

For a subgraph H of a graph G by $f|_H$ we denote the *restriction* of a function f to H. For a real number a let

$$[f \leq a] = \{v \in V : f(v) \leq a\}$$

be the *lower set* of a function f. Let G_a denote the subgraph of G induced by $[f \leq a]$. The following evident lemmas capture some basic properties of pseudo–peakless functions.

LEMMA 1. *If f is a pseudo–peakless function defined on the vertices of a graph G with n vertices, then every G_a is an isometric subgraph of G. Conversely, if f has $|V|$ distinct values and G_a is an isometric subgraph for any a, then f is pseudo–peakless.*

LEMMA 2. *If $v_1, \ldots, v_n$ is a distance–preserving ordering of G, then the function $\alpha(v_i) = i$ is pseudo–peakless. Moreover, G admits a distance–preserving ordering if and only if there is a pseudo–peakless function which has n distinct values.*

For similar relationships between perfect elimination orderings and peakless functions see [37, 17] (more exactly, Shier [37] used quasiconcave and quasiconvex functions, but all quasiconvex functions which have n distinct values are peakless).

LEMMA 3. *If H is a convex subgraph of G and f is a pseudo–peakless function on G then $f|_H$ is pseudo–peakless on H.*

We shall say that a function g *refines* a function f whenever every lower set of f is a lower set of g. Two functions are *equivalent* if they have identical lower sets.

LEMMA 4. *If f is a strictly pseudo–peakless function on G then any refinement g is strictly pseudo–peakless, too.*

PROOF. Let v and w be two non–adjacent vertices of G and $x \in I(v, w) - \{v, w\}$ such that $a = f(x) < max\{f(v), f(w)\} = f(w)$. Since $[f \leq a]$ is a lower set of the function g and $w \notin [f \leq a]$, we conclude that $g(x) < g(w)$. □

Let f be a function defined on G. By a *f–minimal path* connecting two vertices u and v is meant a shortest path $P_f(u, v) = (u = w_0, w_1, \ldots, w_p = v)$ with the minimum sum $\sum_{i=0}^{p} f(w_i)$.

LEMMA 5. *(1) A function f on G is pseudo–peakless if and only if it is locally pseudo–peakless, i.e., for any two vertices u and v at distance two there is a common neighbour w of u and v such that $f(w) \leq max\{f(u), f(v)\}$ and equality holds only if $f(u) = f(v)$.*

(2) A function f on G is strictly pseudo–peakless if and only if for any two vertices u, v, with $d(u, v) = 2$, there exists a common neighbour w of u and v such that $f(w) < max\{f(u), f(v)\}$.

PROOF. Pick two non–adjacent vertices u and v, and let $P_f(u, v)$ be a f–minimal path connecting u and v. We assert that f is peakless along this path. Since every subpath of $P_f(u, v)$ again is a f–minimal path, it suffices to verify the peaklessness condition only for the vertices $w_0 = u, w_p = v$ and a vertex w_i where f attains its maximum on $P_f(u, v) - \{u, v\}$. Assume that w_i is as close as possible to w_0. Applying an induction argument on the distance

$d(u,v)$ departing from $d(u,v) = 2$, we can assume that f is peakless along each of the f–minimal paths $(w_0, \ldots, w_i)$ and $(w_i, \ldots, w_p)$. Consider the vertices w_{i-1} and w_{i+1}. Since f is locally pseudo–peakless , from the choice of the path $P_f(u,v)$ we conclude that $f(w_i) \leq max\{f(w_{i-1}), f(w_{i+1})\}$. From the choice of w_i we obtain that $w_i = w_1$. If $f(w_1) < f(w_0)$ we are done. Otherwise, $f(w_0) = f(w_1) = f(w_2)$. Now, letting w_2 play the role of w_1 and w_1 and w_3 the roles of w_0 and w_2, respectively, we obtain the equality $f(w_3) = f(w_2)$. And so on, until we arrive at the vertex w_p. Hence, $f(w_0) = f(w_1) = \ldots = f(w_p)$. □

Similar Tietze–type results were established for convexity in graphs in [9, 16] and for some kind of isometricity of disks of Cayley graphs in [15].

For a graph G a function f is called *locally unimodal* if the restriction of f on every interval $I(u,v)$ with $d(u,v) = 2$ is a unimodal function on $I(u,v)$.

LEMMA 6. *Any locally unimodal function f on G is pseudo–peakless. Conversely, if G does not contain the graphs F_1 and F_2 (Figure 3) as induced subgraphs then any pseudo–peakless is locally unimodal.*

PROOF. Let u and v be two arbitrary vertices of G with $d(u,v) = 2$. If f attains its minimum on $I(u,v)$ in some vertex $w \neq u,v$, then either $f(u) = f(w) = f(v)$ or $f(w) < max\{f(u), f(v)\}$. So, assume that u is the unique minimum of f on $I(u,v)$. Since v cannot be a local minimum of f, necessarily $f(v) > f(x)$ for some common neighbour x of u and v. Thus, $f(x) < max\{f(u), f(v)\}$. By Lemma 5 f is pseudo–peakless.

If G does not contain the graphs F_1 and F_2 as induced subgraphs, then the interval $I(u,v)$ between any two vertices u, v at distance 2 is convex. By Lemma 3 the restriction of any pseudo–peakless function on $I(u,v)$ is pseudo–peakless, and therefore, unimodal. □

The numberings of vertices of the graphs F_1 and F_2 presented in Figure 3 provide examples of pseudo–peakless but non–locally unimodal functions.

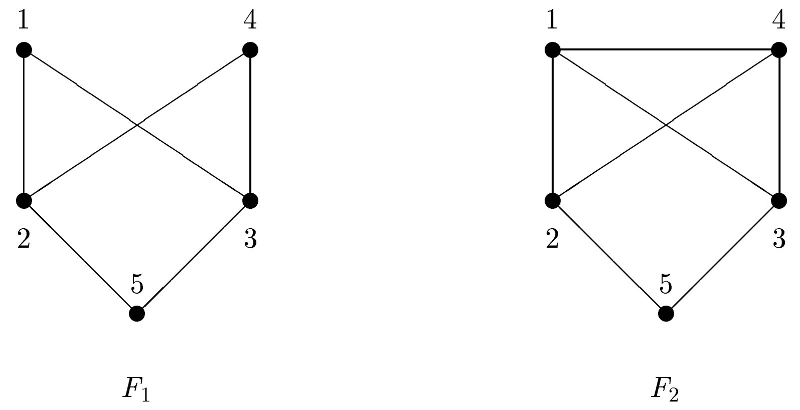


FIGURE 3.

For a d–polytope $P \subset \mathbb{R}^d$ a numbering of its vertices is called *completely unimodal* [26, 41, 40] if every k–face $(2 \leq k \leq d)$ has a unique local minimum, that is, every face F has only one vertex such that all its neighbours on F get a larger number. In a similar way we can define *completely unimodal functions.* It is established in [26, 41] that if P is the d–dimensional cube, then a numbering is completely unimodal if and only if is unimodal on every 2–face. Let $G(P)$ denotes the graph (1–skeleton) of a polytope P. From previous results we obtain the following fact.

PROPOSITION 1. *Let P be a polytope such that all its faces induce convex subgraphs in $G(P)$ and let f be a function defined on the vertices of P.*

(1) If f is locally unimodal, then f is completely unimodal;

(2) If every 2–face of P is a 4–cycle and every $I(u,v)$ with $d(u,v)=2$ constitutes a 2–face of P, then the following conditions are equivalent:

(i) f is completely unimodal;

(ii) f is unimodal on each 2–face;

(iii) f is pseudo–peakless.

The proof of both assertions follows from Lemmas 6 and 3 and unimodality of pseudo–peakless functions.

For a vertex v and a subset of vertices M by $d(v,M)$ we denote the minimum over all distances $d(v,x), x \in M$. For a fixed $M \subseteq V$ let $d_M(v) = d(v,M)$. We shall now characterize the graphs for which the functions $d_u(v) = d(v,u)$ and $d_C(v) = d(v,C)$ are pseudo–peakless for all vertices $u \in V$ and all cliques C of G (by a clique is meant any complete subgraph of G). A graph G is called *meshed* [9], if for every triple u,v,w of vertices with $d(v,w)=2$ and

$$1 \leq d(u,v) \leq d(u,w) \leq d(u,v)+1,$$

there exists a common neighbour x of v and w with $d(u,x) \leq d(u,v)$.

PROPOSITION 2. *(1) For any vertex u of a graph G the distance function d_u is pseudo–peakless if and only if G is meshed. In particular, the distance function d_u of a weakly modular graph is pseudo–peakless.*

(2) for any vertex u of a graph G the distance function d_u is strictly pseudo–peakless if and only if G is pseudo–modular.

The proof of this result is immediate in view of Lemma 5. The characterization of pseudo–modular graphs as graphs with strictly pseudo–peakless functions follows from Lemma 5(2) and the characterization of pseudo–modular graphs of [6] presented in Section 2.

PROPOSITION 3. *For a graph G the following conditions are equivalent:*

(i) for any clique C of G the distance function d_C is pseudo–peakless;

(ii) for any edge e of G the distance function d_e is pseudo–peakless;

(iii) G is a weakly modular graph.

PROOF. $(i) \to (ii)$ is evident.

$(ii) \to (iii)$. It suffices to verify the quadrangle condition, because any meshed graph satisfies the triangle condition [9]. So, let $p, q \in I(x, y)$ be two neighbours of x, where x and y are selected as close as possible. By Proposition 2 there exists a common neighbour w of p and q such that $d(w, y) \leq d(p, y) = d(q, y)$. We can suppose that $d(w, y) = d(p, y)$, otherwise we are done. By the triangle condition there are two common neighbours y' and y'' of p, w and q, w, respectively, one step closer to y. By the induction assumption there exists a common neighbour $z \in I(y', y) \cap I(y'', y)$ of y' and y''. Consider the distance function d_e, where $e = xq$. Then $d_e(p) = 1$, while $d_e(z) = d_e(y') = 2$. Since d_e is pseudo–peakless, there is a common neighbour u of p and z with $d_e(u) = 1$. Since $d(x, z) = 3$, the vertex u must be adjacent to q. Therefore $u \in I(p, y) \cap I(q, y)$, as required.

$(iii) \to (i)$. According to Lemma 5 it suffices to establish that d_C is locally pseudo–peakless. Let $d(u, v) = 2$. Pick two vertices $x, y \in C$ such that $d(u, x) = d(u, C)$ and $d(v, y) = d(v, C)$. We can assume that $d(u, x) < d(u, y)$ and $d(v, y) < d(v, x)$, otherwise u and v will have a common closest vertex in C, and we are in a position to apply Proposition 2. Moreover, $|d(u, x) - d(v, y)| \leq 1$, otherwise, if, say $d(u, x) = d(v, y) + 2$, then $d(t, y) = d(v, y) + 1$ for any common neighbour t of u and v, and we are done. We proceed by induction on $d(u, x) + d(v, y)$. If $d(u, x) = d(v, y)$, then $d(v, x) = d(u, x) + 1$. By Proposition 2 there is a vertex $w \in I(u, v) - \{u, v\}$ such that $d(w, x) \leq d(u, x)$. So, assume that $d(v, y) = d(u, x) + 1$. Then $u, y \in I(v, x)$. Pick an arbitrary neighbour z of x in $I(v, x)$. By the quadrangle condition we would find a common neighbour t of z and y which is one step closer to v. Consider the clique $C' = \{z, t\}$. By the induction assumption there exists a vertex $w \in I(u, v) - \{u, v\}$ such that $d(w, C') \leq d(u, z)$. Since $d(u, z) = d(u, x) - 1$, this finishes the proof. □

The following example shows that in Proposition 2 we cannot replace cliques and edges by isometric subgraphs or shortest paths. In the following graph (which is pseudo–modular) consider the isometric subgraph induced by the set $M = \{a, b, c\}$; see Figure 4. The function d_M is not pseudo–peakless because $d_M(y) = d_M(x) = d_M(z) = 1$, however $d_M(c) = 0$ and $I(y, c) = \{y, x, z, c\}$.

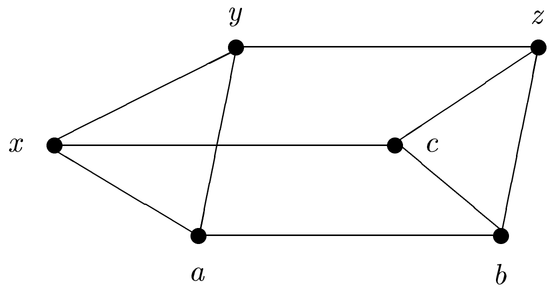


FIGURE 4.

The results of [17] for graphs as well as the results for G–spaces and G–surfaces (see [14] for references) show that the nonconstant peakless functions are quite rare. In contrast, at least in meshed or weakly modular graphs nonconstant and even locally nonconstant pseudo–peakless functions exist in abundence. Propositions 2 and 3 provide a method for constructing such kind of functions. Consider for example the distance function d_u. The lower sets of this function are the disks $D_k(u)$ centered at u. In particular, d_u takes the constant value k on the *sphere* $S_k(u) = \{v : d(u,v) = k\}$. One method to produce a distance–preserving ordering of a meshed or a weakly modular graph G is to extend the function d_u to a pseudo–peakless function which takes $|S_k(u)|$ distinct values on each sphere $S_k(u)$. The simplest way to realize this is to apply the *Breadth–First Search (BFS)* starting from the vertex u. In the Breadth–First Search the vertices of a graph G with n vertices are numbered from 1 to n in increasing order. We number with 1 the vertex u and put it on an initially empty queue of vertices. We repeatedly remove the vertex v at the head of the queue and consequently number and place onto the queue all still unnumbered neighbours of v. BFS constructs a spanning tree T_u of G with the vertex u as a root. Then a vertex v is the *father* in T_u of any of its neighbours w in G included in the queue when v is removed. The procedure is called once for each vertex, so the total complexity of its implementation is $O(|V|+|E|)$ (a detailed description of the BFS procedure is presented, for example, in the book of Golumbic [25]). Another method to order the vertices of a graph in linear time is the *Lexicographic Breadth–First Search* proposed by Rose, Tarjan, and Lueker [35]. According to LBFS the vertices of a graph G are numbered from n to 1 in decreasing order. The *label* $\mathsf{label}(w)$ of an unnumbered vertex w is the list of its numbered neighbours. As the next vertex to be numbered, select the vertex v with the (lexicographic) largest label, breaking ties arbitrarily [35]. Evidently, LBFS is a particular instance of BFS, i.e. every ordering produced by LBFS can be generated by BFS, too. Below we reproduce the details of LBFS.

procedure LBFS(G);

Input: the adjacency list of G;
Output: an ordering of the vertices of G.

```
begin
    for every vertex w in V do label(w) ← ∅;
    for i ← n downto 1 do begin
        pick an unnumbered vertex v with the largest label;
        number the vertex v with i;
        for each unnumbered w ∈ N[v] do
            add i to label(w)
    end
end;
```

Let $v_1, \ldots, v_n$ be the ordering of G produced by BFS. For each vertex v let $\alpha(v) = i$ if $v = v_i$. We close the section with some basic properties of the BFS–orderings, formulated in terms of the function α. Important Convention: in all subsequent results where BFS is used, $\alpha(u) = 1$, i.e., the procedure BFS starts from the vertex u.

LEMMA 7. *(1) Let x and y be the fathers in T_u of the vertices v and w, respectively. If $\alpha(v) < \alpha(w)$ then $\alpha(x) \leq \alpha(y)$. Conversely, if $\alpha(x) < \alpha(y)$ then $\alpha(v) < \alpha(w)$;*

(2) the function α refines the distance function d_u, i.e., if $d(u,v) < d(u,w)$ then $\alpha(v) < \alpha(w)$;

(3) the function α is monotone along any shortest path connecting the root u with any vertex v.

4. Distance–preserving orderings of weakly modular graphs

We are now in a position to state the main result of this paper. We start with a special case of pseudo–modular graphs. Already from Lemmas 7 and 4 we can deduce that α is strictly pseudo–peakless, because it refines the strongly pseudo–peakless distance function d_u. Alternatively, consider the base point order $<_u$ defined by $v <_u w$ if and only if $v \in I(u, w)$. Then a function f is called *strictly isotone* if $v <_u w$ implies $f(v) < f(w)$.

LEMMA 8. *Any strictly isotone function f on a pseudo–modular graph G is strictly pseudo–peakless.*

The proof is immediate in view of previous results and because the distance function of pseudo–modular graphs is strictly pseudo–peakless.

THEOREM 1. *Let G be a pseudo–modular or a house–free weakly modular graph. Then any ordering $v_1, \ldots, v_n$ of the vertices of G produced by BFS is a distance–preserving ordering.*

PROOF. The case of pseudo–modular graphs is covered by Lemma 8. Henceforth, let G be a house–free weakly modular graph. It is necessary to verify that every G_i, $i = 1, \ldots, n$, is an isometric subgraph of G, or equivalently, that the function α is pseudo–peakless. Suppose the contrary, and let G_i be the first non–isometric subgraph among the subgraphs induced by the lower sets of the function α. Pick a vertex w of G_i as close as possible to $v = v_i$ such that v and w cannot be connected inside G_i with a shortest path. Then v and w are the only common vertices of the interval $I(v, w)$ and G_i. Denote by x and y the fathers of the vertices v and w, respectively. First we establish some metric relationships between the vertices v, wx, and y. Let $k = d(u, v)$ and $t = d(v, w)$.

CLAIM 1. *All vertices of the interval $I(v, w)$ are at distance $k = d(u, v)$ from u.*

PROOF OF CLAIM 1. Indeed observe that $I(v, w) \cap I(v, u) = \{v\}$ and $I(v, w) \cap I(w, u) = \{w\}$. For, otherwise we will get a vertex $p \in I(v, w) - \{v, w\}$ which is closer to u than v or w. By BFS, $\alpha(p) < max\{\alpha(v), \alpha(w)\} = i$ and $p \in G_i$, contrary to the choice of the vertices v and w. Hence, any pseudo–median of the triple u, v, w has the form u', v, w. From the properties of weakly modular graphs presented in Section 2 we know that $d(u', v) = d(u', q) = d(u', w)$ for any vertex $q \in I(v, w)$. Since $u' \in I(u, v) \cap I(u, w)$, we obtain that $d(u, v) = d(u, w) = k \geq d(u, q)$. The choice of the vertices v and w implies that $d(u, q) = k$. This settles Claim 1. $\square$

Claim 1 implies that $x \neq y$. Since $\alpha(w) < \alpha(v)$, we obtain $\alpha(y) < \alpha(x)$ by BFS. From the choice of v and w we conclude that the function α is peakless along some shortest path $P_\alpha(x, y)$ connecting x and y.

CLAIM 2. $d(y, v) = d(y, x) = d(v, w) + 1 = d(p, v)$, *where p is the neighbour of y in the path $P_\alpha(x, y)$.*

PROOF OF CLAIM 2. First we will prove that $d(x, y) \leq d(v, w) + 1$. Consider a shortest path $P = (v = v_0, v_1, \ldots, v_{t-1} v_t = w)$ between v and w. We infer from Claim 1 that all vertices of P are at distance k from u. Applying the triangle condition to u and the consecutive vertices of the path P we can find vertices $z_1, \ldots, z_t$ at distance $k - 1$ from u such that every z_j is a common neighbour of the vertices v_{j-1} and v_j. Notice that z_j cannot be adjacent to any other vertex of the path P, otherwise $z_j \in I(v,w)$, which is impossible by Claim 1. If x or y are adjacent to at least one vertex v_j, $0 < j < t$, then $d(x,y) \leq d(v,w)+1$, and we are done. The same inequality holds when the vertices $x = z_0, z_1, \ldots, z_t, z_{t+1} = y$ induce a path. So, we can assume that two consecutive vertices z_{j-1} and z_j are non–adjacent. Since $z_{j-1}, z_j \in I(v_j,u)$, by the quadrangle condition there exists a common neighbour u' of z_{j-1}, z_j at distance $k-2$ to u. Then either the vertices $v_{j-1}, v_j, z_{j-1}, z_j, u'$ or the vertices $v_{j+1}, v_j, z_{j-1}, z_j, u'$ induce a house. This shows that $d(x,y) \leq d(v,w)+1$, in particular, $d(y,v) \geq d(v,w)$.

If w and y are equidistant from v, then by the triangle condition there is a common neighbour z of w and y one step closer to v. Since $\alpha(y) < \alpha(x)$, by BFS we get $\alpha(z) < \alpha(v)$. Then we obtain a contradiction to the choice of the vertices v and w, because $z \in I(v,w)$. Hence, $w \in I(y,v)$. Next, suppose that the vertex p also belongs to the interval $I(y,v)$. By the quadrangle condition there is a common neighbour $q \in I(w,v) \cap I(p,v)$ of w and p. And again, since $\alpha(p) < max\{\alpha(y), \alpha(x)\} = \alpha(x)$, we conclude that $\alpha(q) < \alpha(v) = i$, contrary to our assumption. By Claim 2 $d(x,y) \leq d(v,w)+1$. If $d(x,y) = d(v,w)$, then $x \in I(v,y)$ and $p \in I(x,y) \subset I(v,y)$, which is impossible. Therefore, $d(x,y) = d(v,p) = d(v,w)+1$, thus establishing Claim 2. □

We are now prepared to prove the theorem. Let $P_\alpha(y,x) = (y, p = p_1, p_2, \ldots, p_t, x)$. Since $\alpha(p_j) < \alpha(x)$, from the choice of v and w and the procedure BFS we conclude that p_j cannot be adjacent to any vertex of the interval $I(v,w)$. As $d(v,y) = d(v,p_1)$, by the triangle condition there exists a common neighbour q_1 of p_1 and y which is one step closer to v. We assert that $q_1 \neq w$. Indeed, otherwise $p_2, w \in I(p_1,v)$ and by the quadrangle condition there is a common neighbour $w' \in I(w,v) \cap I(p_2,v)$ of w and p_2, contrary to our assumption. So, let $q_1 \neq w$. Then $w, q_1 \in I(y,v)$ and $q_1, p_2 \in I(p_1,v)$. Again applying the quadrangle condition we can find the vertices $w_1 \in I(w,v) \cap I(q_1,v)$ and $q_2 \in I(q_1,v) \cap I(p_1,v)$, adjacent to w, q_1 and q_1, p_1, respectively; see Figure 5.

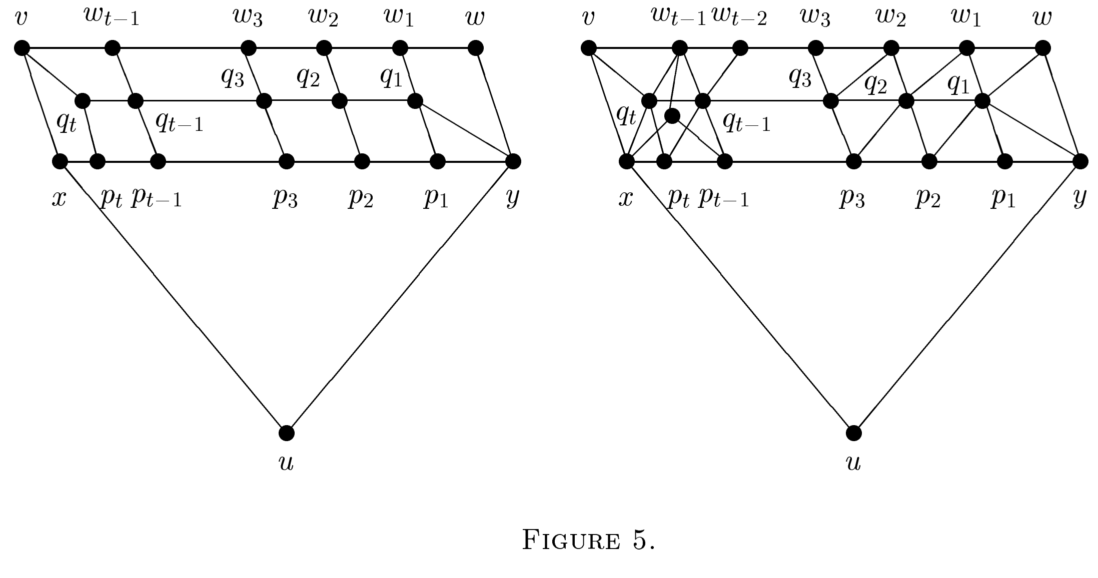


FIGURE 5.

Recall that the vertices p_2 and w_1 cannot be adjacent. Therefore, $q_2 \neq w_1$. If $q_2 = p_3$, then $p_3, w_1 \in I(q_1,v)$. Applying the quadrangle condition we would get a common neighbour

$w' \in I(w_1, v) \cap I(p_3, v)$ of w_1 and p_3. Since $\alpha(p_3) < \alpha(x)$, necessarily $\alpha(w') < \alpha(v)$, contrary to the fact that $w' \in I(v, w)$. So, let $q_2 \neq p_3$. Since $q_2, p_3 \in I(p_2, v)$ and $q_2, w_1 \in I(q_1, v)$, we could find the vertices q_3 and w_2 one step closer to v and adjacent to q_2, p_3 and q_2, w_1, respectively. Walking this way along the path $P_\alpha(x, y)$, we will find distinct vertices $w_3, \ldots, w_{t-1}$ and $q_3, \ldots, q_{t-1}, q_t$ such that $w, w_1, w_2, \ldots, w_{t-1}, v$ induce a shortest path between w and v, and $y, q_1, q_2, \ldots, q_{t-1}, q_t, v$ induce a shortest path between y and v. In addition, every q_j, $1 \leq j < t$, is adjacent to w_j and p_j and $q_j \in I(p_j, v), w_j \in I(q_j, v)$, whereas the last vertex q_t is adjacent to p_t and v.

We claim that every q_j, $1 < j < t$, is adjacent to w_{j-1} and p_{j+1}, while q_1 is adjacent to w, p_2 and q_t is adjacent to x and w_{t-1}. Consider the subgraphs induced by the vertices w, w_1, q_1, y, p_1 and y, q_1, p_1, p_2, q_2. As we already established w_1 and p_1 cannot be adjacent. Since G is house–free, from the metric relations between the involved vertices and the vertex v we deduce that q_1 must be adjacent to p_2 and w. Again, in order to avoid induced houses, q_2 must be adjacent to w_1 and p_3. Continuing this way, we eventually obtain that q_{t-1} would be adjacent to p_t and w_{t-2}, while q_t would be adjacent to x and w_{t-1}; see Figure 5. The vertices v and p_{t-1} are at distance three. Since $x, w_{t-1} \in I(v, p_{t-1})$, we will find a common neighbour z of the vertices x, w_{t-1} and p_{t-1}. We assert that $z \neq q_{t-1}$. Indeed, if x and q_{t-1} were adjacent, we would get a path $y, q_1, q_2, \ldots, q_{t-1}, x$ of length $t = d(v, w)$ between x and y, contrary to Claim 2. Consider the subgraph induced by the vertices $w_{t-1}, w_{t-2}, z, q_{t-1}$, and p_{t-1}. Since the graph G is house–free and the vertex p_{t-1} cannot be adjacent to w_{t-2} or w_{t-1}, we deduce that z is adjacent either to w_{t-2} or to q_{t-1}. In both cases we will get that x and w_{t-1} must be adjacent, otherwise we obtain a forbidden house induced by the vertices $v, x, z, w_{t-1}, w_{t-2}$ or $v, x, z, w_{t-1}, q_{t-1}$. Then, however, the vertices x, w_{t-1}, q_{t-1}, p_t and p_{t-1} induce a house. This final contradiction shows that the function α is pseudo–peakless. By Lemma 2 the ordering produced by BFS is distance–preserving. □

The assertion of Theorem 1 is not true for all weakly modular graphs. We take the 3–cube Q_3 and its dual graph (the 3–octahedron O_3) and join any vertex of O_3 to all vertices of Q_3 which belong to the respective face of Q_3; see Figure 6. A straightforward analysis shows that the resulting graph H is weakly modular. In Figure 6 we present a BFS ordering of H which is not distance–preserving. Namely, if we consider the vertices with numbers 10 and 11, then all their common neighbours get a larger number.

Corollary 1. *For a graph G the following conditions are equivalent:*

(i) for each isometric subgraph of G, the ordering of its vertices produced by BFS is distance–preserving;

(ii) G is a hereditary weakly modular graph.

Proof. Any cycle C_n, $n \geq 5$, does not have a distance–preserving ordering. Figure 8 provides a BFS–ordering of the vertices of a house which is not distance–preserving. Thus $(i) \to (ii)$. The converse follows from Theorem 1. □

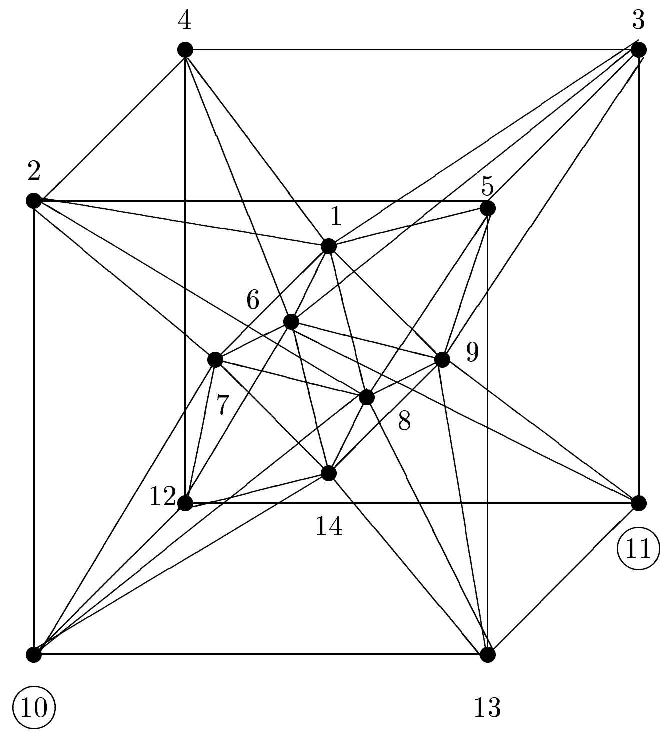


FIGURE 6.

5. Domination elimination orderings of hereditary weakly modular graphs

Next we will show that in hereditary weakly modular graphs and their subclasses the BFS– and LBFS–orderings have additional properties. First we need a lemma.

LEMMA 9. *Let v and w be two adjacent vertices of a hereditary weakly modular graph G which are equidistant to u. If x and y are the fathers of v and w, then either x and y are adjacent or coincide. In addition, if $\alpha(v) < \alpha(w)$, then y is adjacent to v.*

PROOF. We proceed by induction on $k = d(u, v)$. If $d(u, v) = 1$, then $x = u = y$ and we are done. So, let $k > 1$. Suppose by way of contradiction that $d(x, y) > 1$. Since $\alpha(v) < \alpha(w)$, according to BFS $\alpha(x) < \alpha(y)$. Therefore, the vertices x and w must be non–adjacent. We distinguish two cases.

Case 1. $d(x, y) = 3$.

Then any pseudo–median of the triple u, x and y has either size 3 or size 1. In the either case such a pseudo–median consists of x, y and a vertex u' at distance 3 from x and y. From the properties of weakly modular graphs presented in Section 2 we deduce that $d(v, u') = d(w, u') = 3$. Since $d(u', u) = k - 4$, we get a contradiction with $d(u, v) = d(u, w) = k$. So, assume that the pseudo–median x', y' and u' of the vertices x, y and u has size 1; see Figure 7.

Then $d(x',u) = d(y',u) = k-2$ and $d(u',u) = k-3$. Since $v, x' \in I(x,y)$, by the quadrangle condition there is a common neighbour p of v, x' and y. As G is house–free, the vertices p and y' must be adjacent. But then the vertices x, v, x', p and y' induce a house, because $d(x,y) = 3$ and $d(x',u) = d(y',u) = k-3$.

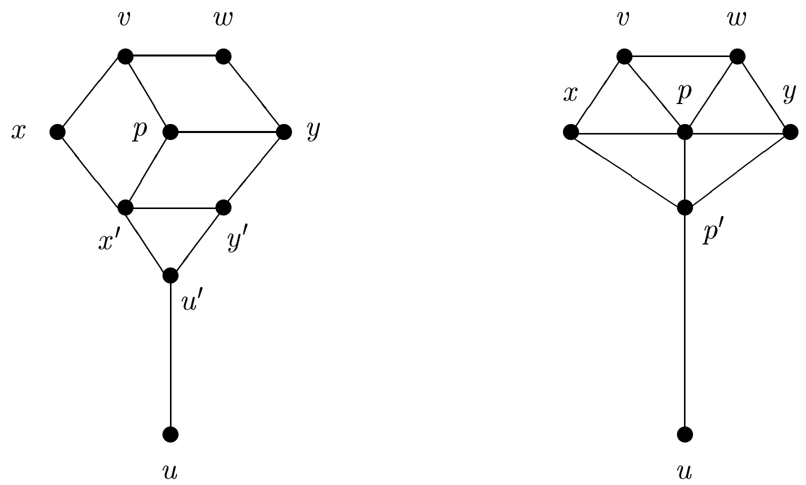


FIGURE 7.

Case 2. $d(x,y) = 2$.

Applying the triangle condition to the vertices x, y, w, we can find their common neighbour p. In order to avoid an induced house, the vertices p and v must be adjacent, too; see Figure 7. According to BFS $\alpha(p) > \alpha(y)$. We assert that $d(p,u) = k-1$. Indeed, otherwise, $d(p,u) = k$ and $x, y \in I(p,y)$. By the quadrangle condition there is a common neighbour q of x and y at distance $k-2$ from u. Then we get two houses, induced by the vertices v, w, x, y, p, and q. So, let $d(p,u) = k-1$. Consider the fathers x', p', and y' of the vertices x, p, and y, respectively. By the induction assumption $d(x',p') \leq 1$ and $d(p',y') \leq 1$. In addition, p' must be adjacent to both x and y, because $\alpha(x) < \alpha(p) > \alpha(y)$. Then, however, the vertices p', x, v, w, y induce a 5–cycle, which is impossible. So, $d(x,y) \leq 1$.

Now, assume that y and v were non–adjacent. Since x and y are at distance $k-1$ to u, by the triangle condition there is a common neighbour u' of x and y at distance $k-2$ to u. As a result we will get a house, induced by v, w, x, y and u'. □

COROLLARY 2 [1, 18]. *Bridged graphs are cop–win graphs. Moreover, any ordering of a bridged graph G produced by BFS is a cop–win ordering.*

PROOF. As in [18], by induction on i we will show that the vertex $w = v_i$ is dominated in G_i by its father y. Pick an arbitrary neighbour v of w in G_i. If $d(v,u) = d(w,u)$, then y and v would be adjacent in view of Lemma 9. Otherwise, $y, v \in I(w,u)$, and, by the quadrangle condition there is a vertex u' adjacent to y and v and at distance $d(w,u) - 2$ to u. Since G does not contain induced 4–cycles, the vertices y and v must be adjacent. □

LEMMA 10. *If Γ is an induced 6–cycle of a hereditary weakly modular graph G, there is a vertex adjacent to all vertices of Γ. In particular, $d(u,v) \leq 2$ for any $u, v \in \Gamma$.*

Proof. As Γ cannot be isometric, necessarily two opposite vertices of Γ, say u and v, are at distance two. Let w be a neighbour of v in Γ. By weak modularity there is a common neighbour x of u, v and w. In order to avoid the forbidden induced house or 5–cycle, x must be adjacent to the remaining vertices of Γ. $\square$

For a vertex v with $\alpha(v) = i$ denote

$$N'(v) = \{w \in N_i(v) : d(w,u) < d(v,u)\},$$

$$N''(v) = N_i(v) - N'(v).$$

Lemma 11. *Let v be a vertex of a hereditary weakly modular graph G at distance k from u. Then all neighbours of v in $I(v,u)$ are adjacent to some vertex v^* at distance $k-2$ to u.*

Proof. Let v^* be a vertex of $S_{k-2}(u)$ which is adjacent to a maximum number of vertices of $N'(v)$. Assume by way of contradiction that there is a neighbour w of v in $I(v,u)$ that is not adjacent to v^*. Pick an arbitrary vertex $z \in N(v) \cap N(v^*)$. As $z, w \in I(v,u)$, by the quadrangle condition there is a common neighbour $y \neq v^*$ of z and w at distance $k-2$ to u. And again, applying the quadrangle condition to the vertices $v^*, y \in I(z,u)$ we can find a common neighbour u' of v^* and y which is one step closer to u. From the choice of the vertex v^* we conclude that there exists a vertex $q \in N(v^*) \cap N(v)$ which is not adjacent to y. As a result we will get an induced 6–cycle (v, q, v^*, u', y, w). By Lemma 10 $d(v,u') = 2$, a contradiction. $\square$

Corollary 3. *Any ordering of the vertices of a hereditary modular graph G produced by BFS is a domination elimination ordering.*

Proof. Since G is bipartite, for any vertex v with $\alpha(v) = i$ we have $N_i(v) = N'(v)$. By Lemma 11 we are done. $\square$

Therefore, the simplest instances of hereditary weakly modular graphs have domination elimination orderings which can be computed by BFS. As the following example shows, this is not more true for all hereditary weakly modular graphs. Consider a graph $G = L(G_1, G_2)$ which consists of copies of the graphs G_1 and G_2 and all edges xy, where $x \in G_1$ and $y \in G_2$. We shall say that G is a *join* of the graphs G_1 and G_2. Notice that every such G is hereditary weakly modular, provides G_1 and G_2 do not contain 5–cycles and houses as induced subgraphs. Moreover, if both G_1 and G_2 are HC–free graphs, then G is HC–free, too. The graph $L(C_n, C_m)$, $n \geq 6, m \geq 6$, is hereditary weakly modular, however, it does not have a domination elimination ordering. We can dismantle all hereditary weakly modular graphs by relaxing the domination condition. Namely, we shall say that an edge xy *dominates* a vertex w whenever $N[w] \subseteq N[x] \cup N[y]$. An ordering $v_1, v_2, \ldots, v_n$ of the vertices of G is called an *edge–dominating elimination ordering* if for every $v_i, i < n$, there exists an edge $v_j v_k, j < i, k < i$, which dominates the vertex v_i in G_i.

Theorem 2. *For a house–free graph G the following conditions are equivalent:*

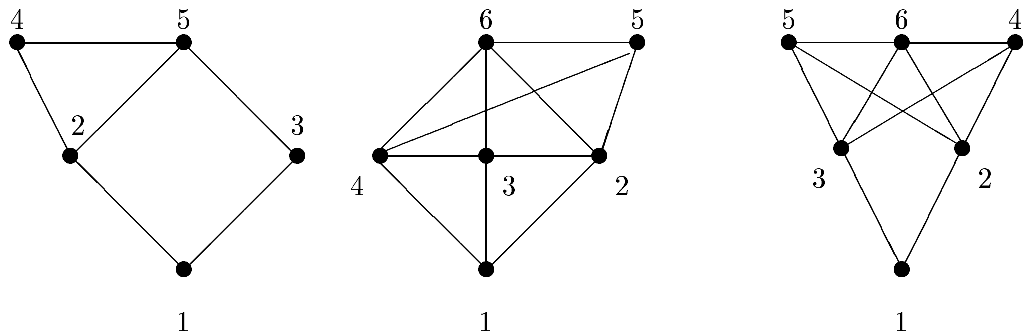
(i) G and each isometric subgraph admit an edge–domination elimination ordering;

(ii) G is a hereditary weakly modular graph;

(iii) for each isometric subgraph of G, the ordering of its vertices produced by BFS is an edge–dominating ordering.

PROOF. $(i) \rightarrow (ii)$ and $(iii) \rightarrow (i)$ are evident.

$(ii) \rightarrow (iii)$. The hereditary property of G infers that it is sufficient to prove the result only for the whole graph G. In the ordering produced by BFS consider the current vertex v, where $\alpha(v) = i$. If $N_i(v) = N'(v)$, according to Lemma 11 v is dominated by any vertex v^*. Then v^* together with any $p \in N_i(v)$ forms an edge which dominates the vertex v. So, suppose that the set $N''(v)$ is nonempty. By Lemma 9 the father x of v will be adjacent to any vertex $w \in N''(v)$. We assert that the edge xw dominates the vertex v. Suppose the contrary: then there exists a vertex $z \in N'(v)$ non–adjacent to both x and w. Let y be a common neighbour of z and x one step closer to u. Then the vertices v, w, x, z, y induce a house, a contradiction. □



L_1 L_2

FIGURE 8.

Next, we are interested in graphs with the property that each isometric subgraph has a domination elimination ordering and this ordering can be computed by BFS or LBFS.

THEOREM 3. *For a graph G the following conditions are equivalent:*

(i) for each isometric subgraph of G, the ordering of its vertices produced by BFS is a domination elimination ordering;

(ii) G does not contain the house, the graphs L_1, L_2 (Figure 8), and any cycle $C_n, n \geq 5$, as an isometric subgraph;

(iii) G is a hereditary weakly modular graph which does not contain the graphs L_1 and L_2 as induced subgraphs.

PROOF. $(i) \rightarrow (ii)$ follows from Corollary 1, while $(ii) \rightarrow (iii)$ follows from the characterization of hereditary weakly modular graphs presented in Section 2.

$(iii) \rightarrow (i)$. As in the preceding theorem, it suffices to establish the result only for the whole graph G. Consider a vertex v with $\alpha(v) = i$. Let $d(v,u) = k$. Again, if $N_i(v) = N'(v)$, by Lemma 11 v is dominated in G_i by some vertex v^* at distance $k-2$ from u. So, we can suppose that the set $N''(v) = N_i(v) - N'(v)$ is nonempty. According to the proof of Theorem 2, any edge xw, where $w \in N''(v)$ and x is the father of v in the tree T_u, dominates the vertex v. Assume that separately the vertices x and w do not dominate the vertex v. Therefore, we can find two vertices $p, q \in N_i(v)$ such that p and x on the one hand and q and w on the other hand are non–adjacent. We assert that $p \neq q$. Indeed, otherwise the vertices v, w, x, v^*, p induce a house. As x is adjacent to all vertices of $N''(v)$, necessarily $p \in I(v,u)$. So, $p \neq q, p \in I(v,u)$ and the vertices p and w are adjacent. If $q \in N'(v)$, the vertex q must be adjacent to both x and p, otherwise we get a house induced by the vertices v, q, x, v^*, w or by the vertices v, x, p, v^*, w. However, in this case the vertices v, w, x, p, q, v^* induce a forbidden subgraph L_1. Now, let $q \in N''(v)$. Again, the vertices p and q must be adjacent, otherwise q, v, x, p and v^* induce a house. In this case we will get the second forbidden subgraph L_2. □

Dahlhaus et al. [21] introduced the concept of a *domination graph* as a graph with the property that each induced subgraph has a domination elimination ordering. As follows from [21], HC–free graphs are exactly the house–free domination graphs. Moreover, a domination elimination ordering of HC–graphs can be computed by MCS or LBFS [21]. Further, we characterize house–free graphs with the property that any LBFS–ordering of each isometric subgraph is a domination elimination ordering. In this case a graph can contain induced cycles of any length $\neq 5$. However, two restrictions on induced cycles are imposed. First, any induced cycle $C_n, n \geq 6$ cannot be isometric. Second, it cannot occur as a cycle in a bipyramid. A *bipyramid* $bipyr(C_m)$ is the graph $L(C_m, \overline{K_2})$, i.e. it consists of a cycle of length m and two non–adjacent vertices which are adjacent to all vertices of this cycle. Figure 9 presents a LBFS–ordering of the bipyramid $bipyr(C_6)$, which is not a domination elimination ordering.

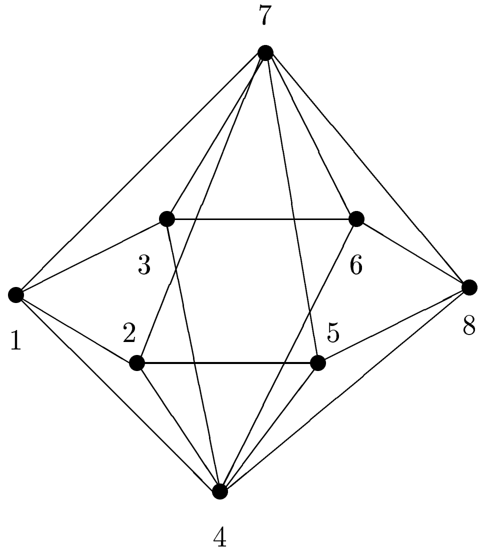


FIGURE 9.

THEOREM 4. *Let G be a house–free graph. The following conditions are equivalent:*

(i) for each isometric subgraph of G, the ordering $v_k, v_{k-1}, \dots, v_1$ of its vertices produced by LBFS is a domination elimination ordering;

(ii) G does not contain any cycle $C_m, m \geq 5$, and any bipyramid $bipyr(C_m), m \geq 6$, as an isometric subgraph;

(iii) G is a hereditary weakly modular graph which does not contain any bipyramid $bipyr(C_m), m \geq 6$, as an induced subgraph.

PROOF. It suffices to establish that $(iii) \to (i)$, namely, that any LBFS–ordering of a hereditary weakly modular graph G without bipyramids $bipyr(C_m), m \geq 6$, as an induced subgraph is a domination elimination ordering. Let the notations remain as before, only $N_i(v_j) = N(v_j) \cap \{v_i, v_{i+1}, \dots, v_n\}$ and $G_i = G(v_i, \dots, v_n)$. Again, we suppose that the procedure LBFS has the vertex u as the starting point, i.e., $\alpha(u) = n$. We proceed by induction on the number n of vertices of G. By Theorem 1 every G_i is an isometric subgraph of G. Therefore, it suffices to prove that the vertex v with $\alpha(v) = 1$ is dominated by some vertex of G. Let $d(v,u) = k$. We can suppose that $k \geq 2$, otherwise v is dominated by u, and we are done. Throughout in the proof for a vertex p by p^* we will denote a vertex at distance $d(p,u) - 2$ to u which is adjacent to all vertices of $N'(p)$ (see Lemma 11).

Assume by way of contradiction that no vertex of G dominates the vertex v. Then Lemma 11 implies that the set $N''(v) = N(v) - N'(v)$ is nonempty. By Lemma 9 the father f of v is adjacent to all vertices of $N''(v)$. Therefore, we could find a vertex $f^+ \in N'(v)$ which is non–adjacent to f. Then any $w \in N''(v)$ must be adjacent to f^+, otherwise, the vertices w, v, f, f^+, v^* induce a house. Our proof requires a number of auxiliary results and notations that we present next.

CLAIM 1. *Let $w, z \in S_k(u)$ be two adjacent vertices with $\alpha(z) > \alpha(w)$, and let $N = N'(w) \cap N'(z)$. If N contains two non–adjacent vertices, then $N'(w) \subseteq N'(z)$ and any vertex of $N'(z) - N'(w)$ is adjacent to all vertices of $N'(w)$. Otherwise, if N is a clique, then any vertex of N is adjacent to all vertices of $N'(w) \cup N'(z)$.*

PROOF OF CLAIM 1. Since $d(w, u) = d(z, u) = k$, by the triangle condition there is a common neighbour of w and z at distance $k-1$ to u, i.e., the set N is nonempty. If N is a clique, but two vertices $t \in N$ and $s \in N'(w) - N'(z)$ are non–adjacent, then z, w, t, s and w^* induce a house. So, suppose that N contains two non–adjacent vertices x and y, however, the sets $N'(w)$ and $N'(z)$ are incomparable. Then we can find vertices $p \in N'(w) - N'(z)$ and $q \in N'(z) - N'(w)$. By Lemma 11 there exists vertices w^* and z^* at distance $k-2$ to u, which are adjacent to the vertices p, x, y and q, x, y, respectively. Necessarily, p and q must be adjacent to both x and y, otherwise we would get an induced house. First suppose that $w^* \neq z^*$, i.e., the vertices w^*, q and z^*, p are non–adjacent. As $w^*, z^* \in I(x, u)$, by the quadrangle condition there is a common neighbour $u^+ \in I(w^*, u) \cap I(z^*, u)$ of w^* and z^*. In order to avoid an induced house, the vertices w^* and z^* must be adjacent. If p and q were non–adjacent, we will get a bipyramid *bipyr*(C_6) induced by p, q and the 6–cycle $(w, z, y, z^*, w^*, x, w)$. Otherwise, if p and q are adjacent, the vertices p, q, z^*, w^*, and u^+ induce a house. So, let $w^* = z^*$. But then we get either an induced 5–cycle, or, if p and q are adjacent, an induced house. The obtained contradiction implies that $N'(w)$ and $N'(z)$ must be comparable. Since $\alpha(z) > \alpha(w)$, by LBFS we deduce that $N'(w) \subseteq N'(z)$. Now, pick two arbitrary vertices $s \in N'(z) - N'(w)$ and $t \in N'(w)$. If s and t were non–adjacent, then the vertices z, w, s, t, and z^+ induce a house, which is impossible. □

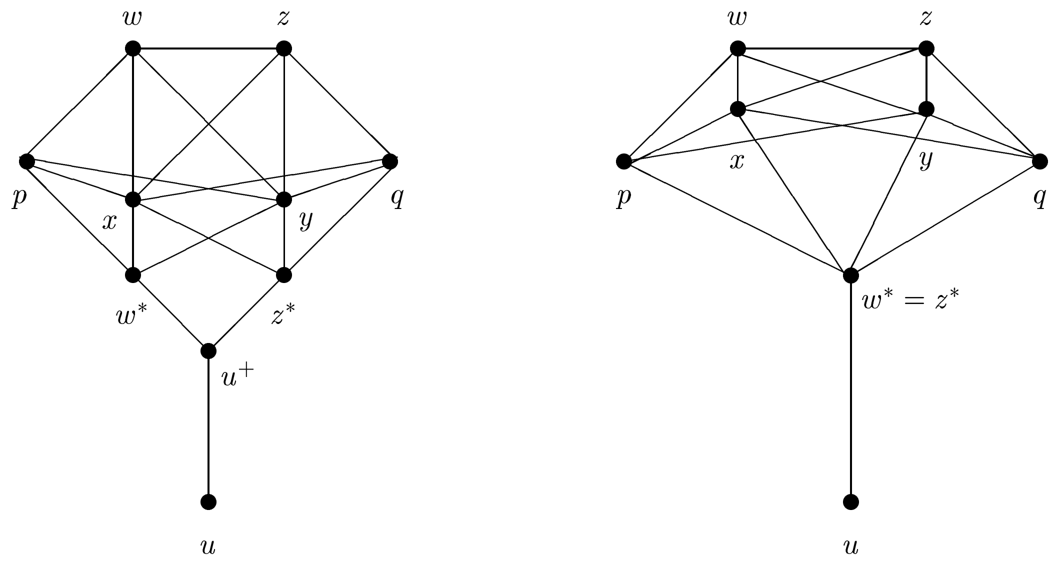


FIGURE 10.

According to Claim 1 and LBFS, $N'(v) \subseteq N'(w)$ for any $w \in N''(v)$ (recall that w is adjacent to non–adjacent vertices f and f^+ of $N'(v)$). Hence for any vertex $p \in N'(v)$ there is a vertex $p^+ \in N'(v)$ that is not adjacent to p. Denote by S the vertices $t \notin N[v]$ of G which are adjacent to all vertices of $N'(v)$. We let T be the union of S and $N[v]$.

Claim 2. *Let w and z be two adjacent vertices of $S_k(u)$. If $\alpha(z) > \alpha(w)$ and $N'(w) = N'(v)$, then $z \in T$.*

Proof of Claim 2. By the triangle condition there is a common neighbour p of w and z at distance $k-1$ to u. Since $p \in N'(v)$, we can find a vertex $p^+ \in N'(v)$ that is not adjacent to p. Then z and p^+ must be adjacent, otherwise we will obtain an induced house. By Claim 1 and because $\alpha(z) > \alpha(w)$ we conclude that $N'(v) = N'(w) \subseteq N'(z)$. Thus $z \in T$. □

Claim 3. *Every vertex $t \in N''(v)$ is adjacent to some vertex z of S with $\alpha(z) > \alpha(t)$.*

Proof of Claim 3. Since t does not dominate the vertex v, there is a vertex $t^+ \in N''(v)$ that is not adjacent to t. First, suppose that $\alpha(t^+) > \alpha(t)$. Then according to LBFS in step $\alpha(t)$ the label of t must be larger than that of v. Therefore we can find a vertex $z \in N(t) - N(v)$ with $\alpha(z) > \alpha(t)$. We assert that $z \in S$. If $z \in N'(t)$, then z would be adjacent to any vertex $p \in N'(v) \subseteq N'(t)$, otherwise t, v, z, p and t^* induce a house. So, let $z \in N''(t)$ and $N'(t) = N'(v)$. By Claim 2 we conclude that $z \in S$.

Now, assume that $\alpha(t) > \alpha(t^+)$. According to LBFS in step $\alpha(t)$ the labels of the vertices t and v either coincide or t has a larger label. In the second case t will be adjacent to a vertex $z \in N(t) - N(v)$ with $\alpha(z) > \alpha(t)$. By Claims 1 and 2 z has the required property. So, consider the either case. According to LBFS in step $\alpha(t)$ the labels of all vertices q with $\alpha(t) > \alpha(q) \geq \alpha(v)$ coincide with that of the vertex t. Moreover, all such q must be adjacent to t, because v and t are adjacent. Then we get a contradiction, since $\alpha(t) > \alpha(t^+) > \alpha(v)$, but t and t^+ are not adjacent. □

Claim 4. *The subgraph $G(T)$ induced by T is a HC–free graph.*

Proof of Claim 4. First, note that the subgraphs induced by $N'(v)$ and $T - N'(v)$ are HC–free. Indeed, otherwise we will get a forbidden bipyramid induced either by v, v^* and a cycle of length ≥ 6 of $G(N'(v))$, or by a similar cycle of $G(T - N'(v))$ and the vertices f and f^+. Now we are done, because $G(T)$ is the join of the graphs $G(N'(v))$ and $G(T - N'(v))$. □

From Claim 4 and Lemma 6 of [21] we obtain the following property of the subgraph $G(T)$.

Claim 5. *For each connected component C of $G(S) = G(T - N[v])$ there is a vertex p_C in C, such that*

$$N(C) \cap N(v) = N(p_C) \cap N(v).$$

Let w be the vertex with the greatest index $i = \alpha(w)$ over all $\alpha(p), p \in T_0$, where T_0 is the connected component of $T \cap S_k(u)$ containing the vertex v.

Claim 6. *The vertex w is adjacent to a vertex $q \in S \cap (N'(w) - N'(v))$.*

Proof of Claim 6. First we will show that $N'(w) - N'(v) \neq \emptyset$. Suppose the contrary, i.e., $N'(w) = N'(v)$. Then in step i of LBFS the labels of the vertices w and v must be equal to $N'(v)$. Indeed, otherwise we could find a vertex $z \in N(w) - N(v)$ with $\alpha(z) > i$. Since $d(z, u) = k$, by Claim 2 we obtain that $z \in S$. Consequently, $z \in T_0$, contrary to the choice of the vertex w. Hence, in step i the set $N'(v)$ represents the label of each vertex of T_0. It may be observed that the restriction of the function α on the set T_0 can be derived by applying LBFS to the graph $G(T_0)$. More exactly, there is a numbering of the vertices of T_0 produced by LBFS that is equivalent to $\alpha|_{T_0}$. The graph $G(T_0)$ is hereditary weakly modular, because it is an induced subgraph of the HC–free graph $G(T)$. By the induction assumption, the vertex v is dominated in $G(T_0)$ by some vertex. But then the same vertex still dominates v in the whole graph G, contrary to our initial assumption.

Pick an arbitrary vertex $q \in N'(w) - N'(v)$. We assert that $q \in S$. Let P be an induced path connecting the vertices w and v inside T_0. Recall that every vertex of P is adjacent to all vertices of $N'(v)$. By Claim 1 we could find two adjacent vertices t and s of P such that $q \in N'(t) - N'(s)$. Again, Claim 1 implies that q must be adjacent to all vertices of $N'(s) \supseteq N'(v)$, i.e., $q \in S$. □

Let the notations remain as before, and consider the connected component C^* of $G(S)$ containing the vertex q defined in Claim 6. Since by Claim 5 all vertices of $N(v) \cap N(C)$ are adjacent to a vertex $p_{C^*} \in C$, our initial assumption infers that $N(v) - N(C^*) \neq \emptyset$, otherwise p_{C^*} will dominate the vertex v. By Claim 3 any vertex $t \in N(v) - N(C^*)$ has a neighbour in some connected component C_t of the graph $G(S)$.

First suppose that still exists a component C_t that contains a vertex p at distance $k-1$ to u. Evidently, p can be selected to be adjacent to a vertex $z \in T_0$. Namely, either z is a vertex of C_t, or, if $C_t \cap T_0 = \emptyset$, we can let $z = t$. Consider the vertices w^* and z^*. They must be distinct and non–adjacent, because $w^* \in C^*$ and $z^* \in C_t$. Since $w^*, z^* \in I(f,u)$, there is a vertex $u^+ \in I(w^*,u) \cap I(z^*,u)$ that is adjacent to w^* and z^*. Then, however, we get two houses induced by the vertices q, f, w^*, z^*, u^+, and p; see Figure 11.

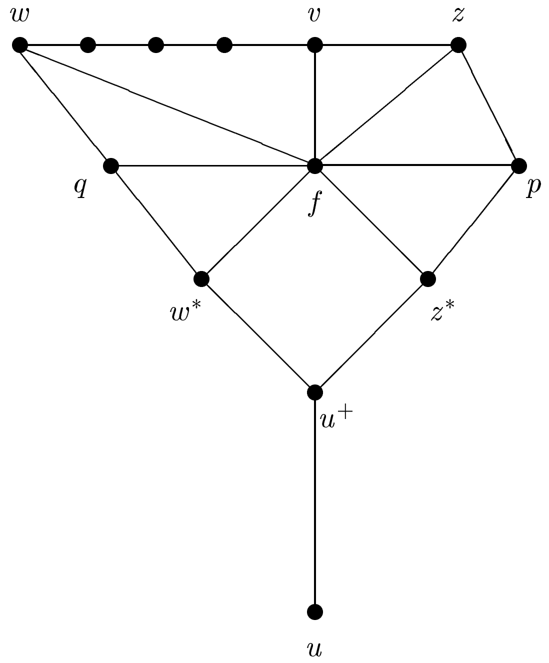


FIGURE 11.

Thus we can assume that each connected component $C_t, t \in N(v) - N(C^*)$, is entirely contained in the sphere $S_k(u)$. In particular, $N'(z) = N'(v)$ for any $z \in \cup\{C_t : t \in N(v) - N(C)\}$. Among such vertices z take the vertex z^+ with the greatest index $j = \alpha(z^+)$. We assert that in step j of LBFS the vertices z^+ and v have equal labels. Otherwise, there is a vertex $s \in N(z^+) - N(v)$ with $\alpha(s) > j$. As $s \in S_k(u)$, by Claim 2 we conclude that $s \in S$. Since $s \notin N[v]$, necessarily s and z^+ belong to a common connected component C_t, contrary to the choice of the vertex z^+.

Hence in step j all vertices $p \in T_0$ with $\alpha(p) \leq j$ have one and the same label. It consists of $N'(v)$ and all neighbours r of v in $S_k(u)$ with the property that $\alpha(r) > j$. Denote the collection of such vertices r by R. Assume that $R = \emptyset$. Since the connected component C^* contains the vertex $q \in S_{k-1}(u) - N'(u)$ with $\alpha(q) > j$, we conclude that LBFS numbers all vertices of C^* before z^+. Therefore in step j the label of any vertex from $N(C^*) \cap N(v)$ must be larger than that of z^+, i.e., any such vertex must be already numbered. This implies that R is nonempty. Moreover, since v is not dominated, for any $r \in R$, there exists a vertex $r^+ \in R$ that is not adjacent to r. On the other hand, at least one vertex of $N''(v)$ must be outside R, otherwise z^+ dominates v.

Finally, consider a new graph G' obtained from the subgraph of G induced by $R \cup \{p \in T_0 : \alpha(p) \leq j\}$ by adding a new vertex y that has R as its neighbourhood and by removing all edges between vertices of R. Evidently, G' is a HC–free graph, because it is the join of an edgeless graph R and a HC–graph $G(\{p \in T_0 : \alpha(p) \leq j\}) + y$. It may be observed that the ordering α of the vertices of the set $R \cup \{p \in T_0 : \alpha(p) \leq j\}$ may be obtained as a LBFS–ordering of the vertices of G'. Indeed, LBFS numbers the vertex y first, after that the vertices of R in accordance with α (recall, that in G' all vertices of R are pairwise non–adjacent). Finally, it numbers the vertices from $\{p \in T_0 : \alpha(p) \leq j\}$ exactly repeating the corresponding steps of the procedure LBFS in G. Since G' has less than n vertices, by the induction hypothesis the vertex v is dominated in G' by some vertex t. Evidently, $t \neq y$, because $N''(v) - R \neq \emptyset$. Since $G' - y$ is a partial subgraph of G and $N(v)$ is completely included in G', we deduce that t dominates the vertex v in the whole graph G. With this the proof of the theorem is complete. □

6. Lexicographic orderings of hypercubes

A *d–dimensional hypercube* Q_d (*d–cube* for short) is a graph whose vertices are all (0,1)–vectors of length d, two vertices being joined if they differ in exactly one coordinate. Clearly Q_d is a bipartite d–regular graph with $n = 2^d$ vertices and $m = d \cdot 2^{d-1}$ edges. In the sequel we assume that only the adjacency matrix of Q_d is available.

The usual shortest path distance between any two vertices u and v of Q_d is the number of positions in which u and v differ (this distance is called the *Hamming distance* between u and v). Any subcube of Q_d is called a *face* of Q_d. Any hypercube is a median graph [30].

Let T be a rooted tree. Recall that the *height* of T is the largest distance from the root to a leaf of T. The *height* of a vertex v is the height of the subtree of T with root v. Then the leaves of T are exactly the vertices of height 0, while the root of T is the unique vertex of maximum height.

THEOREM 5. *Let T be the spanning tree of the d–cube Q_d constructed by LBFS. Then the leaves of T induce in Q_d a $(d-1)$–cube. More generally, the vertices of height h of T induce a $(d-h-1)$–dimensional face of Q_d.*

PROOF. Let u be the root of the tree T, i.e., $\alpha(u) = n$. Denote by L and $I = T - L$ the leaves and the interior vertices of the tree T. First we will show that for any vertex $v \in I$ its neighbour v' in Q_d with the smallest number in the LBFS ordering is a leaf of T and v is the father of v'. Suppose the contrary, and let v' be the father of some other vertex w. Consider the second common neighbour v'' in Q_d of v and w. Since $\alpha(v') < \alpha(v'')$, we obtain a contradiction with the fact that v' is the father of w. Therefore v' must be a leaf of T. In addition, since v is the father of some vertex z and $\alpha(v') < \alpha(z)$, by LBFS we conclude that v is the father of v', too. We will say that the vertices v and v' form a *peripherical pair.*

Consider the peripherical pair (u, u'). We can split Q into two disjoint $(d-1)$–cubes Q and Q' with respect to the unique coordinate where u and u' differ. Let us assume that $u \in Q$ and $u' \in Q'$. Taking all edges with one end in Q and another one in Q' we will get a perfect matching of Q_d. We assert that every peripherical pair (v, v') defines an edge in this matching with $v \in Q$ and $v' \in Q'$. Pick an arbitrary vertex $v \in I$. We apply an induction argument on the distance $d(u, v)$ departing from $d(u, v) = 0$. The vertex v necessarily belongs to Q. Indeed, the father w of v being closer to u lies in Q, while the leaf w' is a vertex of Q'. Since $v \neq w'$ and w' is a unique neighbour of w in Q', we deduce that $v \in Q$. Suppose by way of contradiction that the vertex v' belongs to Q, too. Let x be the neighbour of v in Q'. According to LBFS, in the step when we number x the label of this vertex must be larger or equal than the labels of vertices numbered later. At this moment $\mathsf{label}(x)$ consists of v and all neighbours of x on shortest paths between x and u'. On the other hand, the label $\mathsf{label}(v')$ of v' comprises all neighbours of v' at distance $d(u, v)$ to u. Evidently, $\mathsf{label}(x) \subset Q' \cup \{v\}$, while $\mathsf{label}(v') \subset Q$. Pick an arbitrary vertex $y \in \mathsf{label}(x)$, $y \neq v$. Let z be the neighbour of y in Q. Notice that z cannot be a leaf of T, otherwise the peripherical pair consisting of z and its father would violate the induction assumption. Hence $z \in I$, whence by induction assumption (z, y) is a peripherical pair. Let t denote the second common neighbour of the vertices z and v' in Q_d. Evidently, $t \in \mathsf{label}(v')$, because $d(t, u) = d(v, u)$. Since y is the neighbour of z with the minimum number, necessarily $\alpha(y) < \alpha(t)$. Therefore the label of v' is larger than that of y, contrary to the fact that y is numbered before v'. This shows that $v' \in Q'$. Every interior vertex is a member of at least one peripherical pair, whence $I \subseteq Q$ and $Q' \subseteq L$. Since the father of any vertex of Q' will be its unique neighbour in Q, we obtain that $I = Q$ and $L = Q'$, concluding the proof of the first assertion.

Next, observe that applying LBFS to the $(d-1)$–cube Q we can construct the same spanning tree of Q as the subtree of T induced by its interior vertices. It remains to notice that all vertices of height $h > 0$ of T in the new tree have height $h - 1$. So, to establish the second assertion we can proceed by induction on the dimension d. □

In order to recognize whether a given connected graph G is a d–cube first we construct by LBFS a spanning tree T of G. After that we arrange the vertices of G according to their heights in T. Necessarily, G must have 2^{d-h-1} vertices of height h. Starting with the root of T, for a current h we have to check if the edges of T between the vertices of height h and their fathers define a perfect matching and an isomorphism between the subgraph of G induced by the vertices of height h and the subgraph induced by the vertices of height larger than h. The complexity of the given step is proportional to the number of edges of the second subgraph (it contains exactly $(d-h-1)\cdot 2^{d-h-1}$ edges). Therefore the total complexity is linear with respect to the size of G. The graph G is the d–cube if and only if G passes each test. For another linear test see [11] ; some characterizations of hypercubes are given in [30].

In Figure 12 we present an ordering and a spanning tree of the 4–cube Q_4 produced by the LBFS procedure. The leaves of this tree generate a 3–dimensional face of Q_4.

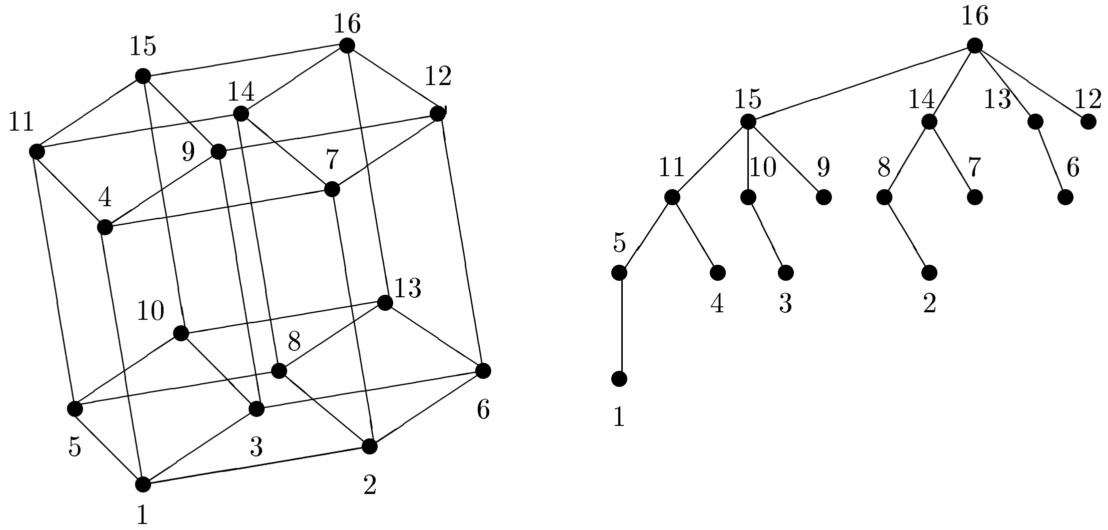


FIGURE 12. A LBFS ordering of Q_4.

We conclude with a BFS ordering of the 3–cube Q_3 so that the leaves of the tree T do not induce a face of Q_3.

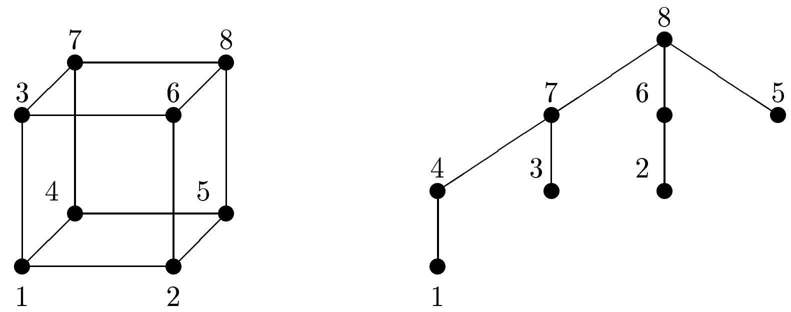


FIGURE 13.

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