# Distance approximating trees for chordal and dually chordal graphs*

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**Abstract.** In this note we show that, for each chordal graph $G$, there is a tree $T$ such that $T$ is a spanning tree of the square $G^2$ of $G$ and, for every two vertices, the distance between them in $T$ is not larger than the distance in $G$ plus two. Moreover, we prove that, if $G$ is a chordal graph without 3– and 4–suns, or a dually chordal graph, then there exists a spanning tree $T$ of $G$ which is an additive 3–spanner as well as a multiplicative 4–spanner of $G$. Both classes comprise the well-known strongly chordal graphs. In all cases the tree $T$ can be computed in linear time.

## 1. Introduction

Many combinatorial and algorithmic problems concern the distance $d_G$ on the vertices of a possibly weighted graph $G = (V, E)$. Approximating $d_G$ by a simpler distance (in particular, by tree–distance) is useful in many areas such as communication networks, data analysis, motion planning, image processing, network design, and phylogenetic analysis. The goal is, for a given graph $G$, to find a sparse graph $H = (V, E')$ with the same vertex–set, such that the distance $d_H(u, v)$ in $H$ between two vertices $u, v \in V$ is reasonably close to the corresponding distance $d_G(u, v)$ in the original graph $G$. There are several ways to measure the quality of this approximation, two of them leading to the notion of a spanner. For $t \geq 1$ a spanning subgraph $H$ of $G$ is called a *multiplicative $t$–spanner* of $G$ [23, 7, 22] if $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$. If $r \geq 0$ and $d_H(u, v) \leq d_G(u, v) + r$ for all $u, v \in V$, then $H$ is called an *additive $r$–spanner* [17].

For many applications the condition that $H$ must be a spanning subgraph of $G$ can be dropped. In this case there is a striking way to measure how sharp $d_H$ approximates $d_G$, based on the notion of a pseudoisometry between two metric spaces. This idea is borrowed from the geometry of hyperbolic groups [10, 13]. For graphs and finite metric spaces a related

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notion of a near–isometry has been already used by Linial et al [18]. For our purposes we present a simplified version of this notion (the interested reader can consult [10, pp.71–72] and [13] for the general definition and related material).

Let $t \geq 1$ and $r \geq 0$ be real numbers. Two graphs $G = (V, E)$ and $H = (V, E')$ are called $(t, r)$–*pseudoisometric* if

$$d_H(u, v) \leq t \cdot d_G(u, v) + r$$

and

$$d_G(u, v) \leq t \cdot d_H(u, v) + r$$

for all $u, v \in V$. In this case we will say that $H$ is a *distance* $(t, r)$–*approximating graph* for $G$ (and conversely, $G$ will be a distance $(t, r)$–approximating graph for $H$). The graphs $G$ and $H$ are $(t, 0)$–pseudoisometric iff

$$\frac{1}{t} \cdot d_G(u, v) \leq d_H(u, v) \leq t \cdot d_G(u, v)$$

for $u, v \in V$. If, in addition, $H$ is a spanning subgraph of $G$, then we obtain the notion of the multiplicative $t$–spanner. Now $G$ and $H$ are $(1, r)$–pseudoisometric iff $|d_G(u, v) - d_H(u, v)| \leq r$ for $u, v \in V$. Again, if $H$ is a spanning subgraph of $G$, this is the usual notion of the additive $r$–spanner.

Recently Cai and Corneil [5] have considered multiplicative tree spanners in graphs. They showed that for a given graph $G$ and integer $t$, the problem to decide whether $G$ has a tree $t$–spanner is $NP$–complete for $t \geq 4$ and is linearly solvable for $t = 1, 2$. The status of the case $t = 3$ is still open. Tree 3–spanners exist for interval and permutation graphs and they can be found in linear time [19]. Similar results are known for the additive tree $r$–spanner problem. [24] proposes a simple approach to construct additive tree 2–spanners in interval and distance–hereditary graphs and such 4–spanners in cocomparability graphs. Both papers [5, 24] ask which important graph classes have tree $t$–spanners and $r$–spanners with small $t$ and $r$. As it is mentioned in [24], McKee showed that for every fixed integer $t$ there is a chordal graph without tree $t$–spanners (additive as well as multiplicative). Nevertheless, from the metric point of view chordal graphs look like trees. In this note we prove that for every chordal graph $G$ there exists a tree $T$ (actually, $T$ is a spanning tree of the square $G^2$) such that

$$d_T(u, v) \leq 3 \cdot d_G(u, v) \qquad \text{and} \qquad d_T(u, v) \leq d_G(u, v) + 2$$

for all vertices $u, v$ of $G$. In other words, $T$ is a $(3, 0)-$ and $(1, 2)-$approximating tree for $G$. Moreover, if $G$ is a dually chordal graph, or a chordal graph without 3– and 4–suns, then there exists a spanning tree $T$ of $G$ which is an additive 3–spanner and a multiplicative 4–spanner. Since both classes contain strongly chordal graphs, this answers the question posed in [24]. In all three cases the tree $T$ can be computed in linear time.

## 2. Preliminaries

All graphs occurring in this note are connected and simple, i.e. finite, undirected, loopless and without multiple edges. A graph $G = (V, E)$ is *chordal* [4, 8] if it does not contain any induced (chordless) cycle of length at least four. In a graph $G$ the *length* of a path from a vertex $v$ to a vertex $u$ is the number of edges in the path. The *distance* $d_G(u, v)$ between the vertices $u$ and $v$ is the length of a shortest $(u,v)$–path and the *interval* $I(u,v)$ between these vertices is the set

$$I(u,v) = \{w \in V : d_G(u,v) = d_G(u,w) + d_G(w,v)\}.$$

Let $k$ be a positive integer. The *kth power* $G^k$ of $G$ has the same vertices as $G$ and two vertices are joined by an edge in $G^k$ if and only if the distance in $G$ is at most $k$. The *disk* of radius $k$ centered at $v$ is the set of all vertices with distance at most $k$ to $v$,

$$D_k(v) = \{w \in V : d_G(v,w) \le k\},$$

and the *kth neighbourhood* $N_k(v)$ of $v$ is defined as the set of all vertices at distance $k$ to $v$, that is

$$N_k(v) = \{w \in V : d_G(v,w) = k\}.$$

By $N(v)$ we denote the *neighbourhood* of $v$, i.e., $N(v) := N_1(v)$. More generally, for a subset $S \subseteq V$ let $N(S) = \cup_{v\in S} N(v)$ denote the *neighbourhood* of $S$.

A subset $S \subseteq V$ of a graph $G$ is called *$m$–convex* [12] if for any pair of vertices $u,v \in S$ each induced path connecting $u$ and $v$ is contained in $S$. If two vertices $x$ and $y$ of a $m$–convex set $S$ can be joined outside $S$ by a path, then by definition of $m$–convexity $x$ and $y$ must be adjacent.

PROPOSITION 1. [12] *Any disk $D_k(v)$ of a chordal graph $G$ is $m$–convex.*

This basic metric property of chordal graphs has some immediate but important consequences (some of them being already used by different authors). In the subsequent results the graph $G$ is assumed to be chordal and $u$ is an arbitrary but fixed vertex of $G$.

LEMMA 1. *For every vertex $v$ of $G$ and every $k$, $0 < k < d_G(u,v)$, the set $N_k(u) \cap I(u,v)$ induces a complete subgraph of $G$.*

PROOF. Pick vertices $x,y \in N_k(u) \cap I(u,v)$. Concatenating arbitrary shortest paths which connect $v$ with $x$ and $y$, we will obtain a path outside the disk $D_k(u)$. By the $m$–convexity of $D_k(u)$ we conclude that $x$ and $y$ are adjacent. □

LEMMA 2. *If two vertices $v,w \in N_k(u)$ are adjacent, then they have a common neighbour in the set $N_{k-1}(u)$.*

PROOF. Indeed, take $x,y \in N_{k-1}(u)$ such that $v$ is adjacent to $x$ and $w$ is adjacent to $y$. By Lemma 1 $x$ and $y$ are adjacent or coincide. If $x = y$ we are done. Otherwise, from the chordality of $G$ we will deduce that either $vy \in E$ or $wx \in E$, concluding the proof. □

LEMMA 3. *For any connected component $S$ of the subgraph of $G$ induced by the kth neighbourhood $N_k(u)$ of $u$, the set $N(S) \cap N_{k-1}(u)$ induces a complete subgraph. Moreover, there exists a vertex $w \in S$ such that $N(S) \cap N_{k-1}(u) = N(w) \cap N_{k-1}(u)$.*

PROOF. Let $x,y \in N(S) \cap N_{k-1}(u)$. Then we can find two vertices $v,w \in S$ such that $x$ is adjacent to $v$ and $y$ is adjacent to $w$. Consider a path $P$ of $S$ connecting the vertices $v$ and $w$. Then $P$ together with the edges $xv$ and $yw$ will give a path which intersects the disk $D_{k-1}(u)$

only in the vertices $x$ and $y$. Since $D_{k-1}(u)$ is $m$–convex, we deduce that the vertices $x$ and $y$ are adjacent.

Now consider a vertex $w \in S$ with maximal number of neighbours in $N_{k-1}(u)$ and assume that $wy \notin E$ for some $y \in N(S) \cap N_{k-1}(u)$. Among neighbours of $y$ in $S$ we choose a vertex $v$ for which the distance $d_S(w, v)$ is minimal. Let $P$ be a shortest path in $S$ connecting the vertices $w$ and $v$. For every neighbour $x$ of $w$ in $N_{k-1}(u)$ consider the cycle formed by path $P$ and the edges $wx$, $xy$ and $yv$. Since $G$ is chordal and $d_S(w, v)$ is minimal, the vertices $x$ and $v$ must be adjacent. So, every neighbour of $w$ in $N_{k-1}(u)$ is a neighbour of $v$ as well. But then from $wy \notin E$ and $vy \in E$ we get a contradiction with the choice of the vertex $w$. □

Let $q = \max \{d_G(u, v) : v \in V\}$. For a given $k$, $0 \leq k \leq q$, let $S_1^k, \ldots, S_{p_k}^k$ be the connected components of the subgraph of $G$ induced by the $k$th neighbourhood of $u$. Define a new graph $\Gamma$ whose vertices are the connected components $S_i^k$, $k = 0, \ldots, q$ and $i = 1, \ldots, p_k$. Two vertices $S_i^k$ and $S_j^{k-1}$ are adjacent if and only if there is an edge of $G$ with one end in $S_i^k$ and another end in $S_j^{k-1}$. Lemma 3 implies that every $S_i^k$, $k > 0$ is adjacent in $\Gamma$ to exactly one connected component of $N_{k-1}(u)$. This shows that the following holds.

LEMMA 4. *$\Gamma$ is a tree.*

In what follows we will consider $\Gamma$ rooted at the vertex $u = S_1^0$. As usually, the *nearest common ancestor* $nca(S_i^k, S_j^l)$ of two vertices $S_i^k$ and $S_j^l$ of $\Gamma$ is the root of the smallest subtree of $\Gamma$ that contains both the vertices $S_i^k$ and $S_j^l$.

## 3. Distance approximating trees for chordal graphs

In this section for a given chordal graph $G = (V, E)$ we construct a tree $T = (V, E')$ which is a distance $(3, 0)-$ and $(1, 2)-$approximating tree for $G$. As in the previous section we fix an arbitrary vertex $u$ of $G$. To construct $T$, for each connected component $S_j^k$ $(k \geq 1)$ we select a vertex $v \in N(S_j^k) \cap N_{k-1}(u)$ and make $v$ adjacent in $T$ to all vertices of $S_j^k$ (a formal description is given below).

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E' := ∅;
for k := q downto 1 do
    for j := 1 to p_k do
        pick an arbitrary vertex v in N(S_j^k) ∩ N_{k-1}(u);
        for all x ∈ S_j^k add xv to E'.
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By Lemma 4 it follows easily that $T$ is a tree. For any edge $xv$ of $T$ which is not an edge of $G$ we have $d_G(x, v) = 2$. Indeed, by Lemma 3 every neighbour of $x$ in $N_{k-1}(u)$ is adjacent to $v$. Therefore, $T$ is a spanning tree of $G^2$. Since for constructing $T$ we need only to find the connected components of the $k$th neighbourhoods of $u$, the complexity of this procedure is $O(|V| + |E|)$.

THEOREM 1. *$T$ is a distance $(3, 0)-$ and $(1, 2)-$approximating tree of $G$.*

PROOF. First we will show that for any edge $xy$ of $G$ we have $d_T(x, y) \leq 3$. If $d_G(u, x) = d_G(u, y) = k$, then $x$ and $y$ belong to a common connected component of $N_k(u)$. Therefore, in $T$ both $x$ and $y$ are adjacent to the same vertex $v$. In this case $d_T(x,y) = 2$. Now suppose that $d_G(u,x) - 1 = d_G(u,y) = k$ and $xy$ is not an edge of $T$. Let $v$ be the neighbour of $x$ on the path connecting $u$ and $x$ in the tree $T$. By Lemma 3 the vertices $v$ and $y$ are adjacent. Since $d_G(u,v) = k$, from the previous case we obtain that $d_T(v,y) = 2$. Hence $d_T(x,y) = 3$. On the other hand, as we mentioned above, for every edge $xy$ of $T$, $d_G(x,y) \leq 2$.

Now consider two arbitrary vertices $x$ and $y$ of $G$ and a shortest $(x,y)$–path. Applying to every edge of this path the obtained inequalities, we will get

$$\frac{1}{2} \cdot d_G(x,y) \leq d_T(x,y) \leq 3 \cdot d_G(x,y).$$

To prove that $|d_G(x,y) - d_T(x,y)| \leq 2$ we proceed as follows. Suppose that $x \in S_i^k$ and $y \in S_j^l$. Let $S := nca(S_i^k, S_j^l)$ and assume that $S$ is a connected component of the $s$th neighbourhood of $u$. Denote by $S'$ and $S''$ the neighbours of $S$ in $\Gamma$ on the paths between $S, S_i^k$, and $S, S_j^l$, respectively. From the definition of the trees $\Gamma$ and $T$ we obtain that in $T$ and $G$ the distances between $u$ and any other vertex of $V$ are the same. Any shortest path in $G$ between $u$ and each of the vertices $x$ and $y$ shares a common vertex with $S$. Therefore, one can select two vertices $x', y' \in S$ such that $d_G(x,x') = k - s$ and $d_G(y,y') = l - s$ (note that $x'$ and $y'$ are vertices of $S$ closest to $x$ and $y$, respectively). Since $\Gamma$ is a tree, one can easily show that in $G$ any shortest $(x,y)$–path $P(x,y)$ between $x$ and $y$ passes through the set $S$. Let $x'', y''$ denote the first and the last vertices of $P(x,y)$ in $S$. Since $x', x'' \in N(S')$ and $y', y'' \in N(S'')$, by Lemma 3 the vertices $x', x''$ and $y', y''$ either coincide or are adjacent. From the choice of these vertices we conclude that $d_G(x,x'') = k - s$ and $d_G(y,y'') = l - s$. Therefore, $d_G(x,y) = k + l - 2 \cdot s + d_G(x'', y'')$. From Lemma 3 we have $d_G(x'',y'') \leq 3$. On the other hand, from the algorithm we obtain that $d_T(x,y) = k + l - 2 \cdot s + \alpha$, where $\alpha = 0$ if in $T$ the nearest common ancestor of $x$ and $y$ is a vertex of $S$, and $\alpha = 2$ if this ancestor belongs to the father of $S$ in $\Gamma$. In the first case necessarily $N(S')$ and $N(S'')$ share a vertex in $S$. Since by Lemma 3 $N(S') \cap S$ and $N(S'') \cap S$ are complete subgraphs, one can easily show that $d_G(x'', y'') \leq 2$. Therefore, in this case $|d_G(x,y) - d_T(x,y)| \leq 2$. The same inequality is evidently true if $\alpha = 2$. This completes the proof of the theorem. □

It is an open problem whether the distance matrix $D$ of a chordal graph $G = (V,E)$ can be computed in optimal $O(|V|^2)$ time. From the second assertion of Theorem 1 we obtain that within these time bounds we can compute the elements of $D$ with an error at most 2. Even more, given a pair of vertices $x, y \in V$ by the algorithm of Harel and Tarjan [15] the nearest common ancestor $nca(x,y)$ of $x$ and $y$ in $T$ can be computed in $O(1)$ time after a linear time preprocessing of $T$. Since $d_T(x,y) = d_T(x,u) + d_T(y,u) - 2 \cdot d_T(u, nca(x,y))$, the distance $d_T(x,y)$ can be found using a constant number of operations. Therefore, after a linear time preprocessing, in only $O(1)$ time we can compute $d_G(x,y)$ with an error at most 2.

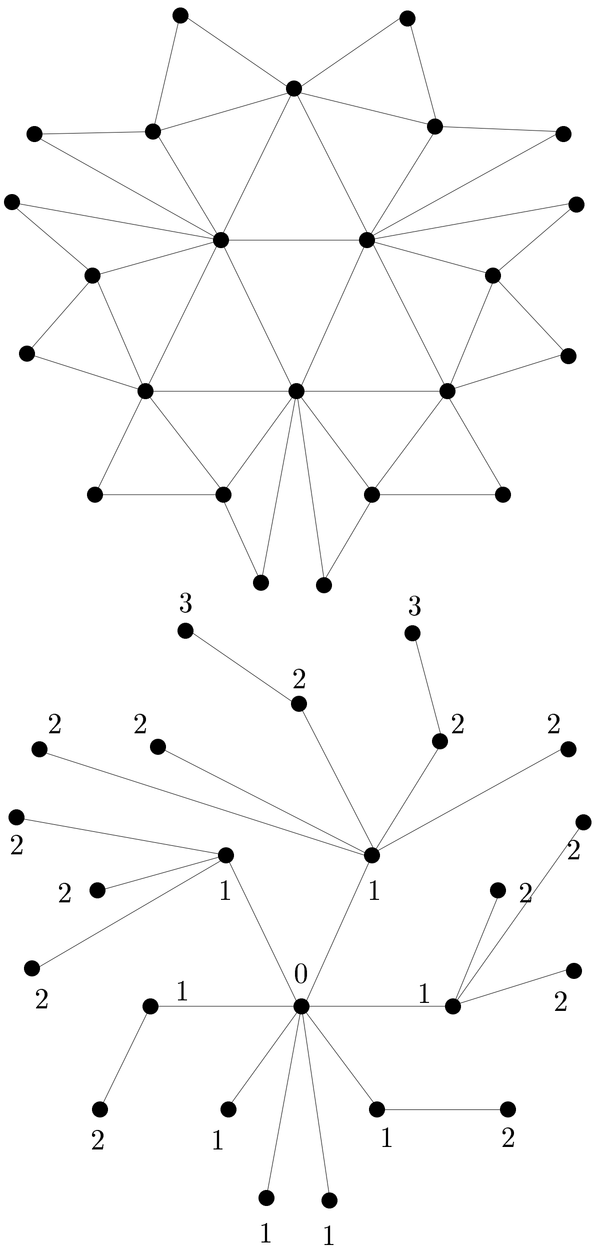


FIGURE 1. A chordal graph and a distance (1,2)–approximating tree of it.

A chordal graph together with some distance $(1,2)$–approximating tree produced by our procedure is given in Figure 1. Note that this chordal graph has no additive tree $r$–spanner for $r \leq 3$ (see [24]). It remains an open question whether the result of the Theorem 1 is sharp.

## 4. Spanners of chordal graphs without small suns

As we already mentioned in the introduction, chordal graphs do not have multiplicative or additive $t$–spanners with a fixed $t$. However, we will show that chordal graphs which do not contain 3–suns and 4–suns as induced subgraphs, have multiplicative 4–spanners and additive 3–spanners. We present an $O(|V| + |E|)$ time algorithm for computing such spanners. This class of graphs comprises the well–known strongly chordal graphs.

A *k–sun* $S_k$ ($k \geq 3$) is a graph consisting of a complete subgraph $U = \{u_1, \ldots, u_k\}$ and an independent set $W = \{w_1, \ldots, w_k\}$, such that every $w_i$ is adjacent only to $u_i$ and $u_{i+1} (\bmod k)$ ($S_3$ and $S_4$ are presented in Figure 2). A chordal graph $G$ is called *strongly chordal* if it does not contain any sun $S_k$ as an induced subgraph [6, 11]. For equivalent definitions and properties of strongly chordal graphs see [11] and for a recognition see [21, 25].

By a *two–set* of a graph $G$ we will mean a subset $M \subseteq V$ such that $d_G(x, y) \leq 2$ for any $x, y \in M$.

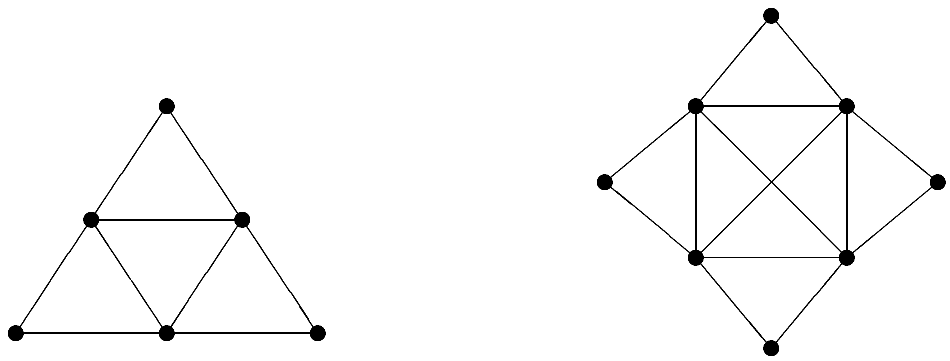

FIGURE 2. The suns $S_3$ and $S_4$.

LEMMA 5. *Let $G$ be a chordal graph which does not contain $S_3$ as an induced subgraph. For any two–set $M$ of the subgraph of $G$ induced by $N_k(u)$ there exists a vertex $v \in N_{k-1}(u)$ which is adjacent to all vertices of $M$.*

PROOF. Let $v$ be a vertex of $N_{k-1}(u)$ which is adjacent to the maximum number of vertices of $M$, and assume that $v$ is not adjacent to some $y \in M$. Take an arbitrary neighbour $w \in N_{k-1}(u)$ of $y$. From the choice of $v$ we conclude that $w$ is not adjacent to some neighbour $x$ of $v$ in $M$. Since $x$ and $y$ are connected in $N_k(u)$ by a path of length at most two, the vertices $v$ and $w$ are adjacent due to $m$–convexity of the disk $D_{k-1}(u)$. The vertices $x$ and $y$ cannot be adjacent, otherwise we will obtain an induced 4–cycle formed by $v, x, y$, and $w$. Let $z$ be a common neighbour of $x$ and $y$ in $N_k(u)$. From chordality of $G$ we obtain that $z$ must be adjacent to both $v$ and $w$. By Lemma 2 there exists a common neighbour $u' \in D_{k-2}(u)$ of $v$ and $w$. But then we have constructed a 3–sun, induced by the vertices $u', v, w, x, z, y$, contrary to our assumption. Hence the vertex $v$ is adjacent to all vertices of $M$. □

Pick an arbitrary vertex $u$ of $G$, and let $S_1^{k+1}, \ldots, S_{p_{k+1}}^{k+1}$ be the connected components of the subgraph of $G$ induced by $N_{k+1}(u)$, $k \geq 1$. Let $C_j = N(S_j^{k+1}) \cap N_k(u)$ ($j = 1, \ldots, p_{k+1}$) (due to Lemma 3 $C_j$ is a clique) and denote by $H_k$ the graph with $\cup_{j=1}^{p_{k+1}} C_j$ as the vertex set and two vertices $x, y$ are adjacent in $H_k$ if and only if they belong to a common clique $C_j$.

LEMMA 6. *Let $G$ be a chordal graph which does not contain $S_3$ and $S_4$ as induced subgraphs. For every $k$ the graph $H_k$ is a cograph (not necessarily connected), i.e., $H_k$ does not contain any induced path of length 3.*

PROOF. Assume by way of contradiction that $H_k$ contains an induced path $(x, a, b, y)$. Suppose that the edge $ab$ belongs to the clique $C_j$. By Lemma 3 the vertices $a$ and $b$ have a common neighbour in the connected component $S_j^{k+1}$, say $z$. Now consider the vertices $x, a, b$ and $a, b, y$. By Lemma 5 there exist vertices $v, w \in N_{k-1}(u)$ such that $v$ is adjacent to $x, a, b$, and $w$ is adjacent to $a, b, y$. From $m$–convexity of $D_{k-1}(u)$ it follows that either $v = w$ or $v$ and $w$ are adjacent. If $v = w$ we have constructed a 3–sun. Otherwise, if $vw \in E$, then by Lemma 2 there is a common neighbour $u'$ of $v$ and $w$ one step closer to $u$. Now the vertices $u', v, w, x, a, b, y$, and $z$ form a 4–sun. Since $G$ is chordal, one can easily show that the obtained suns are induced. □

Now we have all prerequisites to formulate the main result of this section. Let $G$ be a chordal graph which does not contain the suns $S_3$ and $S_4$ as induced subgraphs. Consider an arbitrary fixed vertex $u$ and a tree $\Gamma$ rooted at $u$ which is defined as in Section 2. Let $q := max\{d(u, v) : v \in V\}$ and $S_1^k, \ldots, S_{p_k}^k$ be the connected components of the $k$th neighbourhood $N_k(u)$ of $u$. We construct a distance approximating spanning tree $T = (V, E')$ of $G$ step by step, starting from the leaves of $\Gamma$ from $N_q(u)$. At first, for each vertex $v \in N_q(u)$ we add to $E'$ (initially $E'$ is empty) one edge of the form $vw$, where $w$ is a neighbour of $v$ in $N_{q-1}(u)$. Now consider an arbitrary $k$ which runs from $q-1$ to 1 and the cliques $C_j = N(S_j^{k+1}) \cap N_k(u)$ $(j = 1, \ldots, p_{k+1})$. By Lemma 6 every connected component $F$ of the graph $H_k$ is a two–set. Therefore all vertices of $F$ have a common neighbour $v_F$ in $N_{k-1}(u)$ (see Lemma 5). Now if $x$ is a vertex of $H_k$, say $x \in F$ for some connected component $F$ of $H_k$, then we add to the current $E'$ the edge $xv_F$. For every vertex $y$ of $N_k(u)$ which does not belong to $H_k$, we add to $E'$ an edge connecting $y$ with an arbitrary neighbour of it in $N_{k-1}(u)$.

> $E' := \emptyset$;
> **for every** $v \in N_q(u)$ pick a neighbour $w$ in $N_{q-1}(u)$ and add the edge $vw$ to $E'$;
> **for** $k := q-1$ **downto** 1 **do**
>   compute the connected components $S_1^{k+1}, \ldots, S_{p_{k+1}}^{k+1}$ of $N_{k+1}(u)$;
>   determine the graph $H_k$ and compute its connected components;
>   **for every** $y \in N_k(u) \setminus H_k$ pick a neighbour $w$ in $N_{k-1}(u)$ and add the edge $yw$ to $E'$;
>   **for every** connected component $F$ of the graph $H_k$ **do**
>     choose in $N_{k-1}(u)$ a common neighbour $v_F$ of all vertices of $F$;
>     **for every** $x \in F$ add the edge $xv_F$ to $E'$.

One can easily show that the graph $T = (V, E')$ constructed by this procedure is a spanning tree of $G$. Next we will show that the procedure can be implemented in linear time. In the preprocessing step we apply the breadth–first search to compute the $k$th neighbourhoods of the vertex $u$ in $O(|V| + |E|)$ time. Denote by $deg(v)$ the degree of a vertex $v$ in $G$. The second line of the procedure requires at most $\sum_{v \in N_q(u)} deg(v)$ operations. We spend $O(|N_{k+1}(u)| + \sum_{v \in N_{k+1}(u)} deg(v))$ time for computing the connected components in $N_{k+1}(u)$.

To determine the graph $H_k$ we need to find the cliques $C_j = N(S_j^{k+1}) \cap N_k(u)$, $(j = 1, \ldots, p_{k+1})$. To do this we proceed in the following way. By Lemma 3 a vertex $w_j$ of the connected component $S_j^{k+1}$ which has the maximum number of neighbours in $N_k(u)$ obeys the condition $N(w_j) \cap N_k(u) = N(S_j^{k+1}) \cap N_k(u)$. Therefore, we have to find the vertices $\{w_1, \ldots, w_{p_{k+1}}\}$ and put $C_j := N(w_j) \cap N_k(u)$. This can be done in $\sum_{v \in N_{k+1}(u)} deg(v)$ time. Having the cliques $C_1, \ldots, C_{p_{k+1}}$, the connected components of the graph $H_k$ can be computed by constructing a special bipartite graph $B_k = (W, K; U)$. In this graph $W = \{w_1, \ldots, w_{p_{k+1}}\}$, $K = \cup_{i=1}^{p_{k+1}} C_i$, and a vertex of $W$ and a vertex of $K$ are adjacent in $B_k$ if and only if they are adjacent in $G$. This graph $B_k$ can be constructed in $O(\sum_{j=1,\ldots,p_{k+1}} |C_j|) = O(\sum_{v \in N_{k+1}(u)} deg(v))$ time. One can easily see that the connected components of $H_k$ are exactly the intersections of the connected components of $B_k$ with the set $K$, and thus can be found within the same time bounds.

Finally, to decide which connected components of the graph $H_k$ are contained in the neighbourhood $N(v)$ of some vertex $v \in N_{k-1}(u)$, we do the following. In total $O(deg(v))$ time, for each connected component $F_i$ $(i = 1, \ldots, r)$ with $N(v) \cap F_i \neq \emptyset$, we compute the value $n_i$ which is the number of vertices of $F_i$ from $N(v)$. If $n_i = |F_i|$ then we put $v_{F_i} := v$. Thus, this line of the procedure can be implemented in $O(\sum_{v \in N_{k-1}(u)} deg(v))$ time, too.

Summarizing, the whole procedure requires only

$$O(\sum_{k=1}^{q} ( \sum_{v \in N_k(u)} deg(v) + |N_k(u)|)) = O(|V| + |E|)$$

time.

A strongly chordal graph together with some additive tree 3–spanner produced by our procedure is given in Figure 3. Note that this strongly chordal graph has no additive tree 2–spanner [16]. The following theorem shows that not only this graph but every strongly chordal graph has an additive tree 3–spanner. So this result is best possible.

THEOREM 2. *Let $G$ be a chordal graph which does not contain induced suns $S_3$ and $S_4$. The tree $T = (V, E')$ is a multiplicative 4–spanner as well as an additive 3–spanner of $G$.*

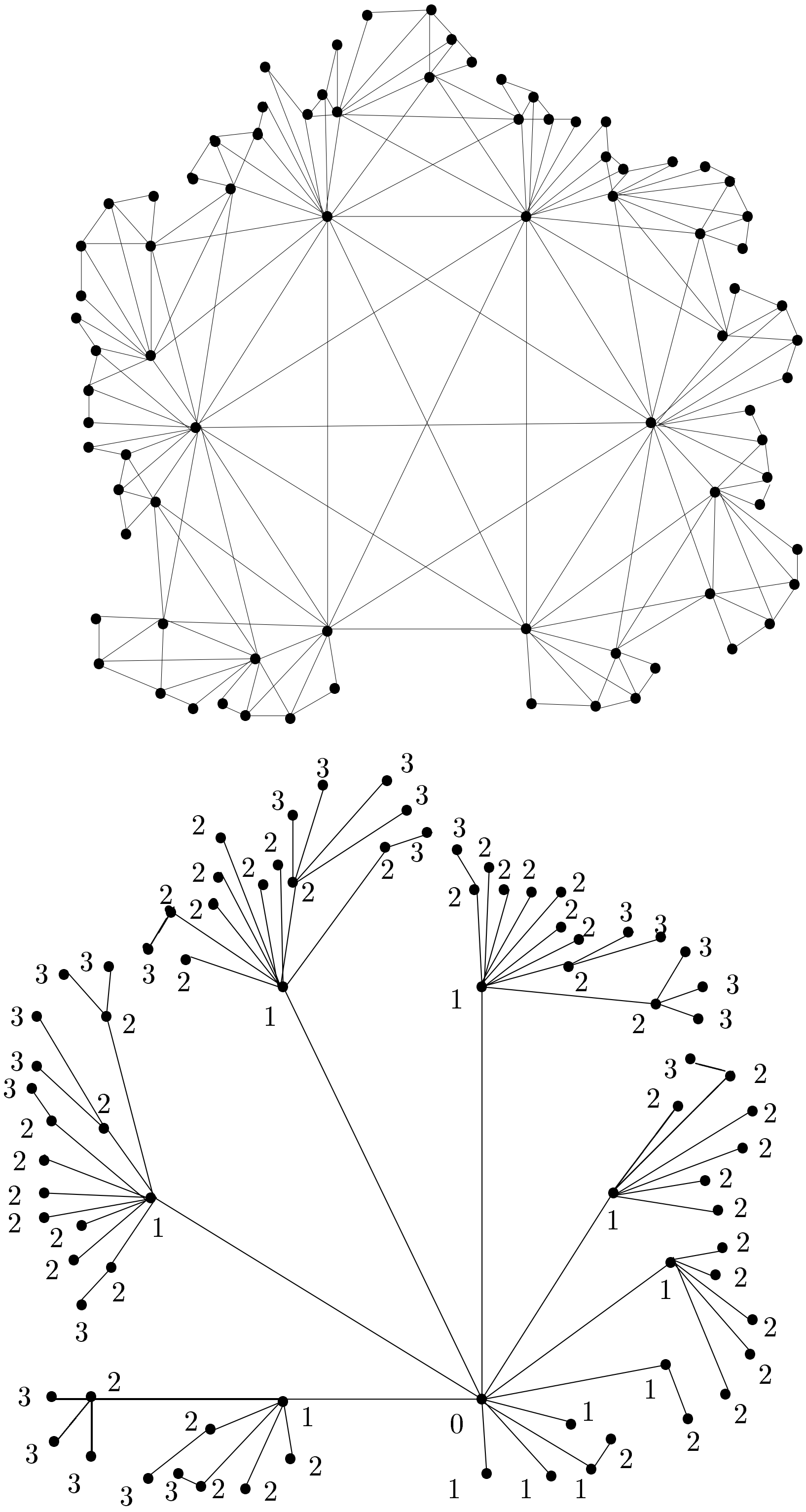

Figure 3. A strongly chordal graph and an additive tree 3-spanner of it.

Proof. First we will show that $d_T(x,y) \leq 4$ holds for any edge $xy$ of $G$. Suppose that $d_G(u,x) = d_G(u,y) = k$, say $x,y \in S_j^k$. If $x$ and $y$ belong to a common connected component $F$ of the graph $H_k$, then in $T$ they have a common father $v_F$, and thus $d_T(x,y) = 2$. Otherwise, let $x'$ and $y'$ be the fathers in $T$ of $x$ and $y$, respectively. Since $x', y' \in C_j$, in $H_{k-1}$ the vertices $x'$ and $y'$ lie in a common connected component. As we already showed $d_T(x',y') = 2$. Therefore, in this case $d_T(x,y) = 4$.

Now suppose that $x \in N_k(u)$ and $y \in N_{k-1}(u)$. Let $x'$ be the father of $x$ in $T$. If $x' = y$ we are done. Otherwise, by Lemma 3 $x'$ and $y$ are adjacent and thus belong to a common connected component of the graph $H_{k-1}$. According to the algorithm, for an edge of this type, $d_T(x',y) = 2$ holds. Hence $d_T(x,y) = 3$.

Now consider two arbitrary vertices $v$ and $w$ of $G$ and a shortest $(v,w)$–path. Applying to every edge $xy$ of this path the inequality $d_T(x,y) \leq 4$, we will get $d_T(v,w) \leq 4 \cdot d_G(v,w)$, i.e., $T$ is a multiplicative 4–spanner for $G$.

That $T$ is an additive 3–spanner of $G$ follows already from the previous part of our proof and from [24, Lemma 1]. We present here another proof. For this suppose that $T$ is rooted at the vertex $u$. From the algorithm it follows that the distances in $G$ and $T$ between a vertex and any of its ancestors are the same. Pick two vertices $v, w \in V$ and proceed by induction on $d_G(v,w)$. If $v$ and $w$ are adjacent, then we are done, because then $d_T(v,w) \leq 4$. Now suppose that $d_G(v,w) = s \geq 2$ and let $z$ be a neighbour of $v$ on a shortest path between $v$ and $w$. From the induction assumption we have $d_T(z,w) \leq s-1+3 = s+2$ and $d_T(v,z) \leq 4$. Let $a = nca(v,z)$ be the nearest common ancestor of $v$ and $z$ in the tree $T$. Since $d_T(a,v) = d_G(a,v)$, $d_T(a,z) = d_G(a,z)$ and $vz \in E$ we obtain that $|d_T(v,a) - d_T(z,a)| \leq 1$. We can additionally assume that $d_T(z,w) < d_T(v,w) - 1$, otherwise we immediately conclude that $d_T(v,w) \leq d_G(v,w)+3$. ¿From this and the previous inequality we deduce that the vertex $nca(w,z)$ lies on the path of $T$ between the vertices $a$ and $z$. Therefore, $a$ is an ancestor of $w$, and thus $d_T(a,w) = d_G(a,w)$. Notice that the distance sums $d_T(v,w) + d_T(a,z)$ and $d_T(v,z) + d_T(a,w)$ are equal. Hence

$$d_T(v,w) = d_T(a,w) - d_T(a,z) + d_T(v,z) = d_G(a,w) - d_G(a,z) + d_T(v,z)$$

$$\leq d_G(w,z) + 4 \leq d_G(v,w) + 3,$$

concluding the proof. □

## 4. Spanners of dually chordal graphs

Below we show that the procedure from the previous section can be applied to produce multiplicative 4–spanners and additive 3–spanners in dually chordal graphs, another generalization of strongly chordal graphs.

To define dually chordal graphs, we need some notions from the theory of hypergraphs [1]. Let $\mathcal{E}$ be a hypergraph with underlying set $V$, i.e. $\mathcal{E}$ is a collection of subsets of $V$. The *dual hypergraph* $\mathcal{E}^*$ has $\mathcal{E}$ as its vertex set and for every $v \in V$ a hyperedge $\{e \in \mathcal{E} : v \in e\}$. The *line graph* $L(\mathcal{E}) = (\mathcal{E}, E)$ of $\mathcal{E}$ is the intersection graph of $\mathcal{E}$, i.e. $ee' \in E$ if and only if $e \cap e' \neq \emptyset$. A *Helly hypergraph* is one whose edges satisfy the Helly property, that is, any subfamily $\mathcal{E}' \subseteq \mathcal{E}$ of pairwise intersecting edges has a nonempty intersection. A hypergraph $\mathcal{E}$ is a *hypertree* if there is a tree $T$ with vertex set $V$ such that every edge $e \in \mathcal{E}$ induces a subtree in $T$. Equivalently, $\mathcal{E}$ is a hypertree if and only if the line graph $L(\mathcal{E})$ is chordal and

$\mathcal{E}$ is a Helly hypergraph. A hypergraph $\mathcal{E}$ is a *dual hypertree* (*$\alpha$–acyclic hypergraph*) if there is a tree $T$ with vertex set $\mathcal{E}$ such that, for every vertex $v \in V$, $T_v = \{e \in \mathcal{E} : v \in e\}$ induces a subtree of $T$. Observe that $\mathcal{E}$ is a hypertree if and only if $\mathcal{E}^*$ is a dual hypertree.

For a graph $G$ by $\mathcal{C}(G) = \{C : C$ is a maximal clique in $G\}$ we denote the *clique hypergraph*. Let also $\mathcal{D}(G) = \{D_k(v) : v \in V,\ k$ a nonnegative integer$\}$ be the *disk hypergraph* of $G$. A graph $G$ is called *dually chordal* if the clique hypergraph $\mathcal{C}(G)$ is a hypertree [2, 9, 26]. In [2, 9] it is shown that dually chordal graphs are the graphs $G$ whose disk hypergraphs $\mathcal{D}(G)$ are hypertrees (see [2, 9] for other characterizations, in particular in terms of certain elimination schemes, and [3] for their algorithmic use). From the definition of hypertrees we deduce that for dually chordal graphs the line graphs of the clique and disk hypergraphs are chordal. Conversely, if $G$ is a chordal graph then $\mathcal{C}(G)$ is a dual hypertree, and, therefore, the line graph $L(\mathcal{C}(G))$ is a dually chordal graph, justifying the term "dually chordal graphs". Finally note that graphs being both chordal and dually chordal were dubbed *doubly chordal* and investigated in [2, 9, 20].

Henceforward, we will suppose that $G$ is a dually chordal graph. To prove that the previous procedure indeed constructs an additive 3–spanner of $G$, we need some properties of 2–sets and graphs $H_k$ of $G$. We start with a result similar to Lemma 5.

Lemma 7. *For any two–set $M \subseteq N_k(u)$ of $G$ there exists a vertex $v \in N_{k-1}(u)$ which is adjacent to all vertices of $M$.*

Proof. Indeed, consider the disks $\{D_1(w) : w \in M\}$ and $D_{k-1}(u)$. Since these disks pairwise intersect, by the Helly property they share a common vertex $v$. Necessarily, $v \in N_{k-1}(u)$ because $M \subseteq N_k(u)$. □

Lemma 8. *Let $S$ be a connected component of the subgraph of $G$ induced by $N_k(u)$. Then $M := N(S) \cap N_{k-1}(u)$ is a 2–set, and, moreover, any two nonadjacent vertices of $M$ have a common neighbour in $M$. In particular, the graph $\Gamma$, defined in the section 2, is a tree.*

Proof. Pick two arbitrary nonadjacent vertices $x, y \in M$. Then we can find two vertices $v, w \in S$ such that $x$ is adjacent to $v$ and $y$ is adjacent to $w$. Consider a path $P$ of $S$ connecting the vertices $v$ and $w$. The disk $D_{k-2}(u)$ together with the disks of the family $\mathcal{M} := \{D_1(z) : z \in P \cup \{x, y\}\}$ forms a cycle in the line graph $L(\mathcal{D}(G))$. From chordality of $L(\mathcal{D}(G))$ and since $D_{k-2}(u) \cap D_1(z) = \emptyset$ for all $z \in P$, we deduce that $D_1(x) \cap D_1(y) \neq \emptyset$, i.e. $d_G(x, y) = 2$. (Here we used the following property of chordal graphs: for each vertex $t$ on a cycle $C$ of length at least four, either the neighbours of $t$ in $C$ are adjacent or $C$ has a chord incident to $t$.) Applying once more this property to the cycle of $L(\mathcal{D}(G))$ formed by the disks of $\mathcal{M}$, we will find in this cycle a disk $D_1(z')$ which together with $D_1(x)$ and $D_1(y)$ induces a triangle in $L(\mathcal{D}(G))$. By the Helly property the pairwise intersecting disks $D_1(z')$, $D_1(x)$ and $D_1(y)$ have a common vertex $s$. If $s \in N_{k-1}(u)$ we are done. So assume that $s \in S$. The disks $D_{k-1}(u)$, $D_1(s)$, $D_1(x)$ and $D_1(y)$ pairwise intersect. Again, by the Helly property and since $d_G(s, u) = k$, we can find in $M$ a common neighbour of the vertices $x, y$ and $s$. □

As in the procedure, let $S_1^{k+1}, \ldots, S_{p_{k+1}}^{k+1}$ denote the connected components of the subgraph of $G$ induced by $N_{k+1}(u)$, $k \geq 1$. Set $M_j = N(S_j^{k+1}) \cap N_k(u)$, $j = 1, \ldots, p_{k+1}$ (the 2–sets $M_j$ play the role of the cliques $C_j$ in the case of chordal graphs without $S_3$ and $S_4$). Denote by $H_k$ the graph with $\cup_{j=1}^{p_{k+1}} M_j$ as the vertex set and two vertices $x, y$ are adjacent in $H_k$ if and only if they belong to a common set $M_j$. We continue with the following result analogous to Lemma 6.

LEMMA 9. *Every connected component $F$ of the graph $H_k$ ($k \geq 1$) is a two–set of $G$.*

PROOF. Let $x, y$ be two nonadjacent vertices of a connected component $F$ of the graph $H_k$. Then we can find a collection of 2–sets $M_{i_1}, M_{i_2}, \ldots, M_{i_h}$ such that $x \in M_{i_1}$, $y \in M_{i_h}$ and $M_{i_j} \cap M_{i_{j+1}} \neq \emptyset$ for all $j = 1, \ldots, h-1$. Pick $z_j \in M_{i_j} \cap M_{i_{j+1}}$, $j = 1, \ldots, h-1$ and let $z_0 := x$ and $z_h := y$. Since $z_{j-1}, z_j \in M_{i_j}$, $j = 1, \ldots, h$, we can find two vertices $v'_j, v''_j \in S^{k+1}_{i_j}$ adjacent to $z_{j-1}$ and $z_j$, respectively. Let $P_j$ be a path of $S^{k+1}_{i_j}$ connecting the vertices $v'_j$ and $v''_j$. The disk $D_{k-1}(u)$ together with $D_1(x), D_1(y)$ and the disks of the family $\{D_1(z) : z \in \cup_{j=1}^{h} P_j\}$ forms a cycle in the line graph $L(\mathcal{D}(G))$. From chordality of this graph and since $D_{k-1}(u) \cap D_1(z) = \emptyset$ holds for all $z \in \cup_{j=1}^{h} P_j$, we deduce that $D_1(x) \cap D_1(y) \neq \emptyset$, i.e. $d_G(x, y) = 2$. $\square$

To find the connected components of the graph $H_k$ we construct again a special bipartite graph $B_k = (W, K; U)$. In this graph $W = \{s_1, \ldots, s_{p_{k+1}}\}$ (a vertex $s_j$ represents a set $S^{k+1}_j$), and $K$ is the vertex set of $H_k$. A vertex $s_j \in W$ and a vertex $v \in K$ are adjacent in $B_k$ if and only if $v \in N(S^{k+1}_j)$. The graph $B_k$ can be constructed in $O(\sum_{v \in N_{k+1}(u) \cup N_k(u)} deg(v))$ time. Note that a vertex $v \in N_k(u)$ belongs to $H_k$ if and only if it has a neighbour in $N_{k+1}(u)$. The connected components of $H_k$ are exactly the intersections of the connected components of $B_k$ with the set $K$, and thus can be found within the same time bound. All other steps of the procedure can be implemented in the same way as for chordal graphs. Summarizing, the whole procedure for a dually chordal graph $G$ requires

$$O(\sum_{k=1}^{q} (\sum_{v \in N_k(u)} deg(v) + |N_k(u)|)) = O(|V| + |E|)$$

time.

THEOREM 3. *Let $G$ be a dually chordal graph. The tree $T = (V, E')$ constructed by the second procedure is a multiplicative 4–spanner as well as an additive 3–spanner of $G$.*

PROOF. The proof is the same as the proof of Theorem 2. First, from Lemma 8 we conclude that $T$ is a spanning tree of $G$. We will prove only that $d_T(x, y) \leq 4$ for any edge $xy$ of $G$. First suppose that $x, y \in S^k_j$. If $x$ and $y$ belong to a common connected component of $H_k$, by Lemmas 7 and 9 we deduce that $x$ and $y$ have a common father in $T$, and thus $d_T(x, y) = 2$. Otherwise, let $x'$ and $y'$ be their fathers in $T$. By Lemma 8 we obtain that $x'$ and $y'$ belong to a common connected component of $H_{k-1}$. But we know that in this case $d_T(x', y') = 2$, hence $d_T(x, y) = 4$.

Now suppose that $x \in N_k(u)$ and $y \in N_{k-1}(u)$. Let $x' \neq y$ be the father of $x$ in $T$ (if $x' = y$ we are done). By Lemma 8 $x'$ and $y$ either are adjacent or have a common neighbour in $N_{k-1}(u)$. In any case they both belong to a common connected component of $H_{k-1}$. Then $d_T(x', y) = 2$ and hence $d_T(x, y) = 3$. $\square$

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