On staircase starshapedness in rectilinear spaces

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Abstract. Let $P \subset \mathbb{R}^n$ be a union of a finite number of boxes whose intersection graph is a tree. If every two boundary points of P are visible via staircase paths from a common point of P, then P is starshaped via staircase paths. The same result holds true when P is a cubical polyhedron of $\mathbb{R}^n$, which is the geometric realization of some median graph.

This generalizes the recent result of M. Breen, J. Geometry, 51 (1994), established for simple rectilinear polygons.

Mathematics Subject Classification (1991): Primary 52A30, 51A35.

Keywords: Staircase visibility; Median polyhedron; Rectilinear distance.

1. Introduction

Visibility and *starshapedness* have been introduced by Brunn [5], and the concepts have found numerous applications in geometry of convex sets (e.g. [2],[14]) and computational geometry (e.g. [10],[12],[13]). In 1946 Krasnoselśkii [9] gave the following characterization of compact starshaped sets in $\mathbb{R}^n$.

Krasnosel'skii theorem: *A compact set S in $\mathbb{R}^n$ is starshaped if and only if every $n+1$ points $x_1, ..., x_n$ of S are visible from a common p of S, i.e. all segments $[p, x_i]$ are contained in S.*

Apart from the usual line segment visibility and starshapedness several other notions of starshaped set and visibility have been investigated in the past years. Motivated by some computational geometry problems, the starshapedness via staircase paths in the plane has been considered (e. g. [3],[4],[10], [13]). Let P be a simple rectilinear polygon (i.e. a simple polygon having all edges axis–parallel) A rectilinear path is a polygonal chain consisting of axis–parallel segments. A *staircase* is a monotone rectilinear path, i.e. a rectilinear path that intersects every horizontal and vertical line in a segment, a point, or not at all. It is easy to observe, that the length of any staircase is just the distance between the end–vertices, i.e. staircases are shortest rectilinear paths. For points x and y in P, we say that x *sees*

[1]Research supported by the Alexander von Humboldt Stiftung,
on leave from the Universitatea de stat din Moldova, Chişinău

y *via staircase paths* (x is *visible* from y *via staircase paths*) if there is a staircase path in P which connects x and y. P is called *starshaped via staircase paths* [3,4] if for some point $p \in P$, p sees each point of P via staircase paths. Recently, M. Breen [3,4] proved the next Krasnosel'skii-type theorem for this kind of starshapedness.

Breen theorem: *If every two boundary points of a simple rectilinear polygon P are visible via staircase paths from a common point of P, then P is starshaped via staircase paths.*

The goal of this note is to show that a similar result holds for all rectilinear spaces (l_1–spaces).

Let $\mathbb{R}^n$ be endowed with the rectilinear distance (alias l_1–distance)

$$r(x,y) = \|x-y\|_1 = \sum_{i=1}^{n} |x_i - y_i|.$$

A *staircase* between points $x, y \in \mathbb{R}^n$ is a rectilinear path whose Euclidean length is $r(x,y)$. Let P be a subset of $\mathbb{R}^n$ which is a union of a finitely many boxes (by a *box* we will mean the Cartesian product of segments). For points x and y in P, we say that x *sees* y *via staircase paths* (x is *visible* from y *via staircase paths*) if there is a staircase path in P which connects x and y. P is called *starshaped via staircase paths* (or simply, *staircase–starshaped*) if some point $p \in P$ sees each point of P via staircase paths. A *star* $St(q,P)$ emanating from a point $q \in P$ is the set of all points $p \in P$ which are visible via staircase paths from q. In the sequel we will use terms *visibility* and *starshapedness* for visibility and starshapedness via staircase paths.

2. Some metric notions

Let (X,d) be a metric space. A point $x \in X$ is *between* the points $a, b \in X$ provided

$$d(a,x) + d(x,b) = d(a,b).$$

The set of all points in between two given points $a, b \in X$ is called an *interval*, and is denoted by $I(a,b)$ (in case of rectilinear spaces the interval will be denoted by $I_r(a,b)$). A set $C \subseteq X$ is *convex* provided it contains intervals between all pairs of points from C. The smallest convex set containing a given set $S \subset X$ is the *convex hull* $conv(S)$ of S. A *halfspace* is a convex set with a convex complement.

A particular instance of metric convexity is the (geodesic) convexity in graphs. In a connected graph a vertex is in between two given vertices if it is on a shortest path joining these vertices.

Some further terminology: a set S of a metric space (X,d) is called *gated* [6,7], provided every $x \in X$ admits a *gate* in S, that is a point $x' \in S$ such that $x' \in I(x,y)$ for all $y \in S$. For l_1–spaces spaces the intervals, as well as compact convex sets, are boxes. These boxes represent gated sets. More generally, compact convex sets of median spaces are gated.

Recall that a metric space (X,d) is called *median* if for any triple x, y, z of points the intersection $I(x,y) \cap I(y,z) \cap I(z,x)$ is a singleton, that is, there is a unique "median" point

$m(x,y,z)$ between each pair of x,y,z [1,8,11,16]. A *median graph* is a graph whose standard graph–metric generates a median space. The median spaces represent a common generalization of different mathematical structures such as median semilattices (including distributive lattices) , median algebras [1,8], median graphs (including trees and hypercubes) [1,16], l_1–spaces, median complexes and median cubical polyhedra [15,16,17], and some classes of convexity structures [16]. Between all these types of mathematical objects, close relationships have been established. For classical results on median spaces the reader is referred to [1], [8], [11], [16].

3. Median polyhedra

A *cubical complex* is a finite set $\mathcal{C}$ of graphic cubes of any dimensions which is closed under taking subcubes and nonempty intersections. *Vertices* of a cubical complex $\mathcal{C}$ are 0–dimensional cubes of $\mathcal{C}$. In the *underlying graph* G of the complex two vertices of $\mathcal{C}$ are adjacent if and only if they constitute a 1–dimensional cube. For each induced graphic cube $Q \subseteq G$ we consider a solid cube $|Q|$ of the same dimension, such that for each pair of graphic cubes Q', Q'' of G,

$$|Q' \cap Q''| = |Q'| \cap |Q''|.$$

The union $|G|$ of these solid cubes is a compact cubical polyhedron, which is called the *(geometric) realization* of G; for further information consult van de Vel's book [16]. We say that G has a realization in the rectilinear space $\mathbb{R}^n$ if $|G|$ can be represented in $\mathbb{R}^n$ in such a way, that every solid cube of $|G|$ is a box of the same dimension of $\mathbb{R}^n$. In the sequel, we will consider only such realizations of graphs of cubical complexes.

A *median polyhedron* in $\mathbb{R}^n$ is the geometric realization of some finite median graph; see Figures 1 and 2 for illustrations. Properties and characterizations of median polyhedra have been considered in [15],[16],[17]. Some of them will be presented in the next section.

Figure 1.

Figure 2.

4. Results

In this section we shall prove the following theorem.

Theorem. *Let P be a median polyhedron of the rectilinear space $\mathbb{R}^n$. If every two points of P are visible via staircase paths from a common point of P, then P is staircase–starshaped.*

Proof. We begin by sketching the proof. First, recall that P can be endowed with the unique median convexity such that all boxes are convex; see [16]. Let d be the *intrinsic* *rectilinear metric* of P, i.e. $d(x,y)$ is the length of the shortest rectilinear path in P between points $x,y \in P$. We shall prove that the resulting metric space (P,d) is median . Observe that d preserves initial rectilinear distances between pairwise visible points of P. Namely, for $x,y \in P$ the equality $d(x,y)=r(x,y)$ holds if and only if x sees y (via staircase paths). In the next step we shall show that the star $St(x,P)$ of any point $x \in P$ is a compact convex set in the median space (P,d). Since the sets $\{St(x,P) : x \in P\}$ meet two by two, applying the Helly theorem for median spaces [16] we obtain that $\cap\{St(x,P) : x \in P\}$ is nonempty. Every point from this intersection sees each point of P, and therefore we are done.

First we show that (P,d) is a median space. Let G be the underlying graph of P. We proceed by induction on the number of convex splits of G (recall that a *convex split* is a pair of complementary halfspaces of G); for properties of convex splits of median graphs consult [1,11,16]. Our proof follows the proof of Theorem 3.16 of [16]. If G is a cube, then P is a box and we are done, because then $d \equiv r$. Otherwise, there is a pair of non-disjoint halfspaces H_1, H_2 covering G [15,16]. Then each of H_1, H_2 is a median graph with less convex splits than G. By the induction assumption, $(|H_1|, d_1)$ and $(|H_2|, d_2)$ are median spaces (denote by I_1 and I_2 their interval operators). From [16, p. 215] we know that

$$|H_1| \cap |H_2| = |H_1 \cap H_2|, |H_1| \cup |H_2| = |G| = P.$$

The set $|H_1 \cap H_2|$ is a common gated subpolyhedron of $|H_1|$ and $|H_2|$. Notice also that d_1 and d_2 coincide on $|H_1 \cap H_2|$. Pick two points $x \in |H_1|$ and $y \in |H_2|$ and let x' and y' be their gates in the set $|H_1 \cap H_2|$. We claim that

$$d(x,y) = d_1(x,x') + d_1(x',y') + d_2(y',y).$$

Indeed, consider an arbitrary rectilinear path $L(x,y)$ between x and y. Necessarily, $L(x,y)$ intersects $|H_1 \cap H_2|$ in some point z. Since $|H_1 \cap H_2|$ is gated, the length of the subpath of $L(x,y)$ between x and z is at least $d_1(x,x')+d_1(x',z)$, while the length of the second subpath is at least $d_2(z,y')+d_2(y',y)$. Therefore $L(x,y)$ is a shortest rectilinear path if and only if $d_1(x',z)+d_2(z,y')=d_1(x',y')$, i.e. $z \in I_1(x',y')=I_2(x',y')$. In this case the length of $L(x,y)$ is exactly $d_1(x,x')+d_1(x',y')+d_2(y',y)$. Hence all shortest paths in P between x and y are contained in $I_1(x,y') \cup I_2(x',y)$, whence $I(x,y) = I_1(x,y') \cup I_2(x',y)$. The Amalgamation Theorem for median spaces [15,16,17] yields that I is the unique median operator on P extending I_1 and I_2, and thus (P,d) is a median space.

We say that a convex split (H_1,H_2) of the underlying graph G *separates* the points $p,q \in P$ if $p \in |H_1|$ and $q \notin |H_1|$.

Next we will prove that for every point $x \in P$ the star $St(x,P)$ is convex in the median space (P,d) (compactness is evident). Notice that $p \in St(x,P)$ if and only if $d(x,p)=r(x,p)$. Then $I(x,p) \subseteq I_r(x,p) \cap P$ and x sees each point of the interval $I(x,p)$. Let y and z be two arbitrary points of $St(x,P)$ and let m denotes the median of the triple x,y,z in (P,d). The point x sees m, because $m \in I(x,y)$. In order to establish that the whole interval $I(y,z)$ belongs to $St(x,P)$ we proceed by induction on the number α of convex splits of G which separate the points y and z. If $\alpha=0$, necessarily y and z belong to a common solid cube of $|G|$, whence $I(y,z)$ coincides with the box $I_r(y,z)$. Then m is the median for x,y,z in the l_1–space $\mathbb{R}^n$, too. Since every two points of a box see each other, for arbitrary $w \in I(y,z)$ we can write

$$d(x,w) = d(x,m) + d(m,z) = r(x,m) + r(m,z) = r(x,z),$$

i.e. x sees w.

Now, assume that $I(y,z)$ is not a box of $\mathbb{R}^n$. Again, pick an arbitrary point $w \in I(y,z)$. Denote by y' and z' the medians in (P,d) of the triples y,m,w and z,m,w, respectively. Since $y' \in I(m,y) \cap I(w,y)$ and $z' \in I(m,z) \cap I(w,z)$, we infer that $w,z \in I(y',z')$. If, say, $y' \neq y$, then we can replace y by y'. Indeed, since $y' \in I(y,m) \subseteq I(y,x)$, the point y' is visible from x. On the other hand, each convex split of G separating y' and z separates the points y and z, too. Therefore, we may assume without loss of generality that $y' = y$ and $z' = z$, i.e. $y,z \in I(m,w)$. This leads us to the next metric equalities:

$$d(y,m) = d(w,z), d(y,w) = d(m,z).$$

Consider the smallest convex subgraph C of G such that $I(y,z) \subseteq |C|$. Since $I(y,z) \neq I_r(y,z)$, we have that C is not a (graphic) cube. According to [11] C contains a convex path (a,b,c) of length two. Let (H_1', H_2') and (H_1'', H_2'') be the convex splits of G which separate the vertices a,b and b,c, respectively. Assume, without loss of generality, that $a \in H_1', b \in H_1'' \cap H_2'$ and $c \in H_2''$. Then $H_1' \subset H_1''$ and $H_2'' \subset H_2'$. Indeed, if the contrary holds, and, say, $u \in H_1' \cap H_2''$, then the median of the triple a,c and u is a common neighbour of a and c distinct from b. This contradicts the convexity of the set $\{a,b,c\}$. Hence, $|H_1'| \subset |H_1''|$ and $|H_2''| \subset |H_2'|$.

From the choice of the subgraph C we conclude that the points y and z are separated by each of the convex splits (H_1', H_2') and (H_1'', H_2''). So, we can suppose that $y \notin |H_2'|$ and $z \notin |H_1''|$. Applying the metric relations between the points y,z,m and w, we infer that the same holds true for the points m and w. Moreover, either of the points m and w cannot belong to $|H_2' \cap H_1''|$, otherwise $y,z \in I(m,w)$ implies that all four points y,z,m and w were located either in $|H_2'|$ or in $|H_1''|$, which is impossible. So, suppose that $m \notin |H_2'|$ and $w \notin |H_1''|$. Denote by m' and y' the gates of points m and y in the gated set $|H_2'|$. The point m' is visible from x, because $m' \in I(m,z) \subset I(x,z)$ and x sees z. Consider the points y, m' and $y' \in I(y,m')$. Since $I(y,m') \subset I(y,z)$, each convex split of G which separates y and m' will separate the points y and z, too. On the other hand, the split (H_1'', H_2'') does not separate the points y and m'. Indeed, suppose by way of contradiction that $m' \notin |H_1''|$. Since m' is the gate for m in $|H_2'|$, we have $m' \in I(m,b)$. However, $m \in |G| - |H_2'| \subset |H_1''|$ and $b \in H_1''$, yielding a contradiction with the convexity of the set $|H_1''|$. Therefore the pair y,m' is separated by less than α convex splits. By virtue of the induction assumption, x sees $y' \in I(y,m')$.

Now consider the pair of points y' and z. Since $y' \in I(y,w)$ and $w \in I(y,z)$, we conclude that $w \in I(y',z)$. Again, we assert that y' and z are separated by less than α convex splits of G, namely, the split (H_1', H_2') does not separate these points. This is so, because y' is the gate of y in $|H_2'|$, and, therefore, $y', w, z \in |H_2'|$. By the induction hypothesis w is visible from x. This concludes the proof of convexity of the set $St(x,P)$ and of the whole theorem. □

Corollary 1. *Let P be a median polyhedron of the rectilinear space $\mathbb{R}^n$. If every two boundary points of P are visible via staircases from a common point of P, then P is staircase–starshaped.*

Proof. According to the Theorem it is enough to establish that every two points y and z of P are visible from a common point of P. Pick a shortest rectilinear path between y and z in P. Let $[y,u]$ and $[v,z]$ be the first and the last links of this path. Denote by y' and z' the first intersections of the rays $[u,y)$ and $[v,z)$ with the boundary of P. We claim that replacing the links $[u, y]$ and $[v, z]$ with $[u, y']$ and $[v, z']$ we again obtain a shortest rectilinear path (between y' and z'). First we show that its part from y to z' obeys this property. Let m be the median of the triple y, v and z'. Necessarily this point belongs to the segment $[v, z'] = I(v, z')$. Since $v \in I(y, z)$ and $z \in [v, z']$, this is possible only if the points m and v coincide. This means that the points $y, u, \ldots, v, z, z'$ lie on a common shortest path. Using similar arguments, we can prove that the whole extended path is a shortest path between y' and z'. Thus $y, z \in I(y', z')$. By the initial conditions, the points y' and z' are visible from a common point $x \in P$. As $St(x, P)$ is convex, we deduce that x sees both y and z. □

Recall, that the *intersection graph* of a family of sets is defined by letting the edges be the pairs of intersecting sets.

Corollary 2. *Let P be a union of a finite number of boxes in $\mathbb{R}^n$ whose intersection graph is a tree. If every two boundary points of P are visible via staircases from a common point of P, then P is staircase–starshaped.*

Proof. It is sufficient to prove that P is a median polyhedron. As in the proof of the Theorem, we show that the intrinsic metric d of P transforms P into a median space. To establish this, we proceed by induction on the number k of boxes. Let $P = \cup_{i=1}^{k} B_i$, and suppose that the box B_k is a pendant vertex of the intersection graph. Denote $P' = \cup_{i=1}^{k-1} B_i$. By the induction hypothesis (P', d) is a median space, as well as the box B_k endowed with the rectilinear metric. The intersection $B_k \cap P'$ is a box, because B_k intersects only one box of P'. As $B_k \cap P'$ is gated in both P' and B_k, we can apply the Amalgamation Theorem for median spaces [15,16]. The proof that the resulting metric is just the intrinsic metric of P is the same as in the proof of the Theorem.

In the next step we show that P is a cubical polyhedron. For this purpose we subdivide P by taking the intersections of P with the planes which pass trough the faces of the boxes $B_1, \ldots, B_k$ and are parallel to the coordinate planes. Let G be the underlying graph of the obtained cubical complex. Evidently, $P = |G|$. Observe also that all boxes of the subdivision are convex in (P, d), because each of them is a subbox of some initial box. So, we can apply Theorem 3.13 [16, p. 211]. According to this theorem G is a median graph, and, therefore, P is a median polyhedron. □

Every simple rectilinear polygon P can be represented as a union of rectangles, whose intersection graph is a tree. Indeed, consider the subdivision of P into rectangles by using only horizontal cuts of P. The intersection graph of the obtained family of rectangles if the dual graph of the subdivision, and is a tree.

5. Conclusion

We have established a Krasnosel'skii type theorem for staircase starshapedness in median polyhedra, which constitute a special class of cubical polyhedra. Fortunately, in our case all stars of points are convex in the intrinsic metric, whence we can employ the Helly theorem. This approach cannot be directly generalized to get a Krasnosel'skii type characterization of staircase–starshaped compact sets or rectilinear polyhedra (even in the plane). The structure of staircase–starshaped polyhedra is much more complex than that of starshaped polyhedra. First, even in the two–dimensional case they are not necessarily simply connected. As an example consider the union of all black squares of the chess–board. The obtained figure is staircase–starshaped with respect to any of the 64 corners of the squares. Another example: take the surface of a 3–dimensional cube. It is staircase–starshaped and its kernel is exactly the skeleton of the cube. The following example shows that for starshapedness via staircases the Krasnosel'skii number (if it exists) exceeds the usual one. In $\mathbb{R}^2$ consider a square minus the interior of a small cross located in the center of this square. Then any three points are visible via staircases from a vertex of the square, however, no point sees fully the figure. Notice that the same construction in $\mathbb{R}^3$ (i.e. a cube minus a three–dimensional cross) is staircase–starshaped. We conclude with the following common generalization of both concepts of starshapedness. We already mentioned that any staircase path is a shortest l_1–path between its end–points. Now, call a set $S \subset \mathbb{R}^n$ l_1*–starshaped* if there is a point $p \in S$ which can be connected by a shortest l_1–path lying in S with any other point of S. Since segments are shortest paths for any norm in $\mathbb{R}^n$, starshaped sets are l_1–starshaped. The authors of [2], where this notion is presented in the framework of normed spaces, mentioned: "Finding criteria for this kind of starshapedness is probably a very difficult problem."

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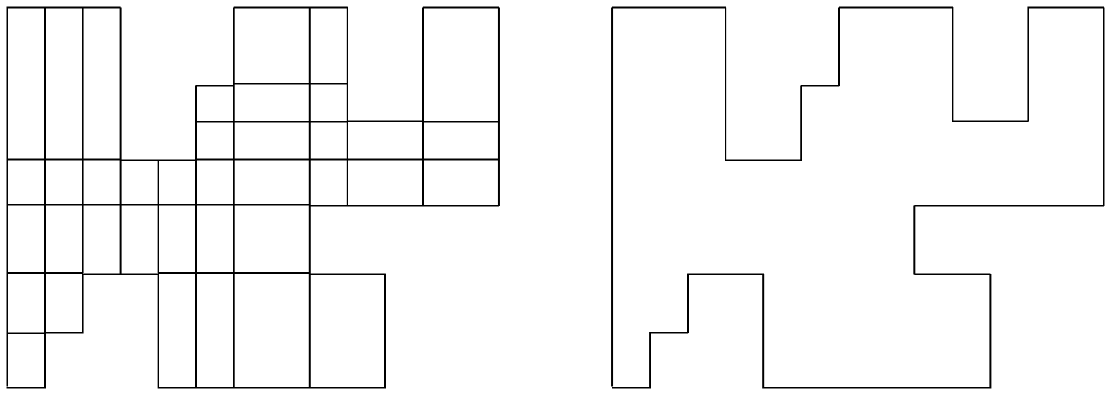

Fig. 1. A median grid and its realization in R^2.

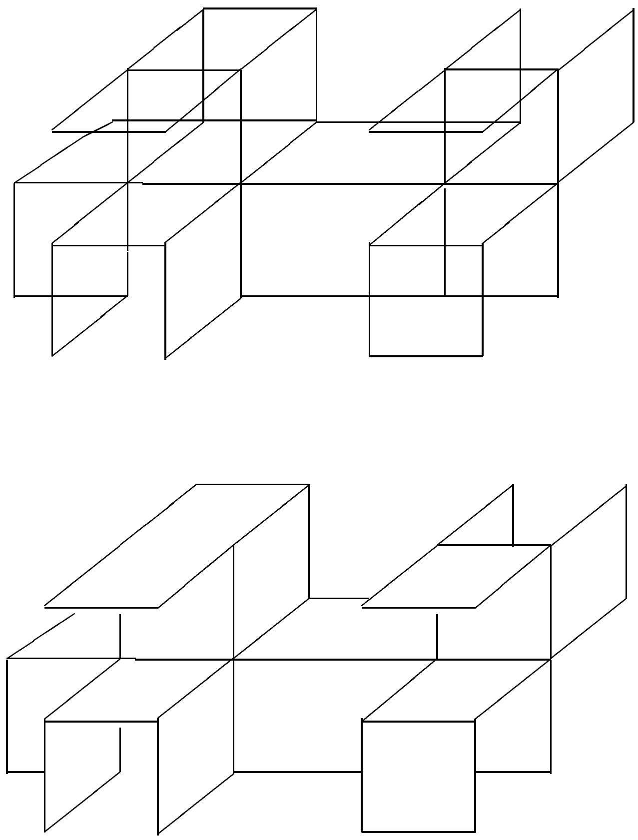

Fig. 2. A median graph and its realization in R^3.