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Tverberg numbers for cellular bipartite graphs

By

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1. Introduction. The well-known theorem of Radon (1921) says that each set S of more than $n+1$ points in an n-dimensional linear space $\mathbb{R}^n$ can be partitioned into two disjoint subsets such that their convex hulls have a point in common. In 1966 Tverberg [16] has given a far-reaching generalization by taking partitions of S into a finite number of subsets.

Tverberg Theorem. *Each set of $m(n+1)-n$ points $\mathbb{R}^n$ can be partitioned into m subsets whose convex hulls have a point in common.*

(The case $n=2$ was settled by Birch [4]. For new proofs of the Tverberg theorem see [13] and [17].) These theorem of Radon and Tverberg are formulated completely in terms of intersections of convex hulls and this suggests to formulate corresponding statements for more general kinds of convexities. A family $\mathscr{C}$ of subsets of a set X is called a *convexity* on X if $\mathscr{C}$ contains $\emptyset$ and X and is closed under arbitrary intersections and directed unions; see Soltan [14] and van de Vel [18]. The members of $\mathscr{C}$ are called *convex sets*. The *convex hull* $co(S)$ of any subset S of X is defined to be the intersection of all convex sets which contain S. For instance, every connected graph is endowed with the *geodesic convexity*, consisting of all those subsets S of the vertex set which include each shortest path of G joining two vertices of S. As to the definition of classical convex invariants (such as the Helly, Carathédory, Radon and Tverberg numbers) of a convexity we adopt the convention of van de Vel [18]. The *Radon number* of a convexity $\mathscr{C}$ is the smallest integer r (if it exists) such that any finite set S with $|S|>r$ admits a partition $\{S_1, S_2\}$ with

$$co(S_1) \cap co(S_2) \neq \emptyset .$$

In these circumstances, $\{S_1, S_2\}$ is called a *Radon partition* and every point of $co(S_1) \cap co(S_2)$ is called a *Radon point*. The *Helly number* is the smallest integer h (if it exists) such that for each finite set S with $|S|>h$

$$\bigcap_{a \in S} co(S \setminus \{a\}) \neq \emptyset .$$

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The *Carathéodory number* is the smallest integer c (if it exists) such that

$$co(S) \subseteqq \bigcup_{a \in S} co(S \setminus \{a\})$$

for each finite set S with $|S| > c$.

Let S be a non-empty subset of X. A partition $\{S_1, \ldots, S_m\}$ of S is called a *Tverberg m-partition* provided

$$\bigcap_{i=1}^{m} co(S_i) \neq \emptyset .$$

The mth *partition number* alias mth *Tverberg number* or mth *Radon number* of X is the smallest number p_m (if it exists) such that each finite set with more than p_m points has a Tverberg partition into $m+1$ parts. Note that the first partition number is just the Radon number. Tverberg's theorem asserts that $p_m = m(n+1) = mr$ in $\mathbb{R}^n$. One of the main questions concerning convexity numbers is to decide whether Tverberg's theorem is of a purely combinatorial nature, or in other words: does the inequality $p_m \leqq mr$ hold for all $m \geqq 1$ and for all convexities? This problem is known in the literature as the Eckhoff conjecture [10]. Jamison [12] has shown that it is valid when the Radon number equals 2. In the same paper, the Eckhoff conjecture was verified for a class of convexity spaces which include ordered sets, trees, Cartesian products of two trees and subspaces of these. The next formula holds for all convexities:

$$p_m \leqq c(mh - 1) + 1 ;$$

see Doignon, Reay and Sierksma [6], Jamison [12], and Sierksma and Boland [14].

In this note we show that the inequality $p_m \leqq mr$ holds for the geodesic convexity on cellular bipartite graphs. These graphs are obtained from even cycles and edges by successive applications of special amalgams. Among them are all trees, Cartesian products of two trees and isometric subgraphs of these, and more generally cube-free median graphs. We prove that the convex invariants of cellular bipartite graphs behave like the classical invariants of plane convexity:

$$c \leqq 2, h \leqq r \leqq 3, p_m \leqq 3m .$$

In contrast to convexity in cellular graphs, geodesic convexities in general graphs seem to behave like arbitrary convexities; see [8]. So one cannot expect a general Helly or Radon theorem involving a convenient dimension parameter. In some cases (for chordal graphs or Helly graphs) the Radon number is given by the clique number: see [3], [5]. The same holds for minimal path convexities [8]. Duchet and Meyniel [9] have shown that $r \leqq 2\eta - 1$ holds for any graph G, where η is the Hadwiger number of G. The stronger upper bound $r \leqq \eta$ was established for $K_{1,3}$-free graphs in [11].

2. Cellular bipartite graphs. In what follows let $G = (V, E)$ denote a finite connected graph endowed with the standard graph metric $d(x, y)$. For arbitrary vertices $x, y \in V$ let

$$I(x, y) = \{v \in V : d(x, v) + d(v, y) = d(x, y)\}$$

denote the (metric) *interval* between x and y. By an *isometric cycle* we will mean a cycle of G which is also a metric subspace. A subset S (or the subgraph induced by S) of G

is *gated* if for every vertex $v \in V$ there exists a (unique) vertex $v' \in S$ (the *gate* for v in S) such that $d(v,x) = d(v,v') + d(v',x)$ for all $x \in S$; see Dress and Scharlau [7] for further information on gated subsets. A graph G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 are (isomorphic to) two intersecting gated subgraphs of G whose union is all of G. A bipartite graph G is called *cellular* [1] if it can be obtained by successive applications of gated amalgamations from even cycles and edges. Cellular graphs were introduced in order to characterize bipartite graphs whose metrics are totally decomposable in the sense of Bandelt and Dress [2]. They show that any metric d on a finite set X admits a specific additive decomposition $d = d_1 + \ldots + d_p + d'$ into split metrics d_i, $i = 1, \ldots, p$, associated with splits (i.e. bipartitions) of the set X, and a split-prime residue d'. If in such a decomposition there is no split-prime reminder, i.e., $d' = 0$, then the metric d is called *totally decomposable*; for precise definitions consult [2]. Bandelt and Dress characterize totally decomposable metrics in terms of a "five-point condition". In [1] a structural characterization of bipartite graphs with totally decomposable metric is presented.

Theorem A [1, Theorem 1]. *For a bipartite graph $G = (V, E)$ with at least two vertices the following conditions are equivalent:*

(i) *G is cellular;*
(ii) *the metric d of G is totally decomposable;*
(iii) *for any subset $S \subset V$,*

$$co(S) = \bigcup_{u, v \in S} I(u, v);$$

(iv) *every isometric cycle of G is gated and G does not contain any three isometric cycles C_1, C_2, C_3 and three distinct edges e_1, e_2, e_3 sharing a common vertex such that e_i belongs to C_j exactly when $i \neq j$.*

The structure of cellular bipartite graphs can be specified further. A *cutset* R of a connected graph G is any subset (or subgraph) for which $G - R$ is disconnected. Evidently, if R is a gated cutset, then G can be represented as a gated amalgam of two gated subgraphs G_1 and G_2 along R.

Theorem B [1, Theorem 3]. *Every cellular bipartite graph either is indecomposable (i.e., comprises a single vertex, or a single edge, or an even cycle) or possesses a gated cutset that is a tree.*

3. Convex invariants of cellular bipartite graphs. In this section we present the main results of this note.

Proposition 1 [2, Proposition 3 and its proof]. *If d is a totally decomposable metric on a finite set X, then (in the usual metric convexity) $c \leqq 2$. In particular, $c \leqq 2$ for all cellular bipartite graphs.*

Proposition 2. *If G is a cellular bipartite graph then $h \leqq r \leqq 3$.*

Proof. The inequality $h \leqq r$ holds for all convexities [18]. Each of the characterizations of cellular bipartite graphs presented in Theorem A provides a method of proving the inequality $r \leqq 3$. For instance, using Theorem A (iii) one can establish that any 4-vertex subset $A = \{a_1, a_2, a_3, a_4\}$ of a cellular bipartite graph G has a Radon partition of the form $\{\{a_i, a_j\}, \{a_k, a_l\}\}$ (see [1]). We will outline the proof of this property based on Theorem B to give an idea of how the gated amalgamation along a tree can be employed. We proceed by induction on the number of vertices of G. The assertion is evident when G is a tree or an even cycle. Moreover, the Radon partitions in trees have the following additional property.

Observation. Every 4-vertex subset $A = \{a_1, a_2, a_3, a_4\}$ of a tree T has at least two Radon partitions of the form $\{\{a_i, a_j\}, \{a_k, a_l\}\}$ whose Radon points coincide.

Indeed, the convex hull of A is a subtree of T which can be represented as in Figure 1 (some of the labeled vertices of this tree can coincide). Then $\{a_1, a_3\}$ and $\{a_1, a_4\}$ together with their complements in A constitute two Radon partitions of A. The Radon points of both partitions form the path between the vertices u and v.

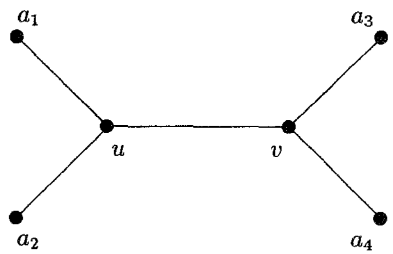

Figure 1.

Now suppose that G is a gated amalgam of two subgraphs G_1 and G_2 along a gated tree T. We can assume that A is not completely contained in G_1 or G_2, otherwise we can apply the induction assumption. So let $a_1 \notin G_2$ and $a_4 \notin G_1$. Denote by a'_1, a'_2, a'_3, a'_4 the gates of the vertices a_1, a_2, a_3, a_4 in the tree T. First, suppose that a_2 and a_3 belong to G_2. By the induction hypothesis there exists a Radon partition $\{a'_1, a_3\}$, $\{a_2, a_4\}$ of the set $\{a'_1, a_2, a_3, a_4\}$. As $a'_1 \in I(a_1, a_3)$, we conclude that $I(a_1, a_3) \cap I(a_2, a_4) \neq \emptyset$. So assume that $a_2 \in G_1$ and $a_3 \in G_2$. By the Observation, the set $\{a'_1, a'_2, a'_3, a'_4\}$ has at least two Radon partitions. One of them necessarily consists of two pairs of vertices from different subgraphs G_1 and G_2, say, it has the form $\{a'_1, a'_3\}$, $\{a'_2, a'_4\}$. We assert that $I(a'_1, a'_3) \subseteqq I(a_1, a_3)$. Since a'_1 and a'_3 are the gates of a_1 and a_3 in T and T separates the vertices a_1 and a_3, we deduce that $a'_1 \in I(a_1, a_3)$ and $a'_3 \in I(a_3, a'_1)$. Hence the vertices a'_1 and a'_3 lie on a common shortest path between a_1 and a_3 and thus $I(a'_1, a'_3) \subseteqq I(a_1, a_3)$. Similarly, $I(a'_2, a'_4) \subseteqq I(a_2, a_4)$. Therefore, $\{a_1, a_3\}, \{a_2, a_4\}$ is a Radon partition of the set A. □

Note that the equality $r = 3$ holds for all cellular bipartite graphs except paths and the 4-cycle.

From Propositions 1 and 2 and the inequality $p_m \leqq c(mh - 1) + 1$ we conclude that $p_m \leqq 6m - 1$ holds for all cellular bipartite graphs. On the other hand, the Eckhoff conjecture asserts that $p_m \leqq 3m$. First we verify this inequality for cycles.

Lemma 3. *If G is an even cycle, then $p_m \leqq 3m$; this inequality is sharp.*

Proof. We proceed by induction on m. For $m = 1$ we apply Proposition 2. So assume that $m > 1$. Let A be a $(3m+1)$-vertex subset of G. Then either all vertices of A lie on a common shortest path between two vertices u, v of A or G is a convex hull of three vertices $u, v, w \in A$. Removing from A the vertices u and v in the first case and the vertices u, v and w in the second case we obtain a new set A' with at least $3(m-1)+1$ vertices. By the induction assumption A' has a Tverberg m-partition $\{A_1, \ldots, A_m\}$. Let $x \in \bigcap_{i=1}^{m} co(A_i)$. In the first case $x \in I(u,v)$ and we get a Tverberg $(m+1)$-partition $\{A_1, \ldots, A_m, \{u,v\}\}$. Otherwise, if $co(u,v,w)$ is the whole graph G we obtain a Tverberg $(m+1)$-partition $\{A_1, \ldots, A_m, \{u,v,w\}\}$.

In order to show that the inequality is sharp we consider a cycle of length $6m+6$ whose vertices are distributed into 6 disjoint paths $P_1, \ldots, P_6$. Here the paths P_1, P_3 and P_5 contain m vertices each, while the remaining paths contain $m+2$ vertices each. Let A be the set of vertices of P_1, P_3 and P_5. Suppose that A has a Tverberg $(m+1)$-partition $\{A_1, \ldots, A_{m+1}\}$ with $x \in \bigcap_{i=1}^{m+1} co(A_i)$. Assume without loss of generality that $x \in P_1 \cup P_2$. Since $(P_1 \cup P_2) \cap co(B) = \emptyset$ for any subset $B \subseteq P_3 \cup P_5$, we conclude that each A_i has at least one vertex from P_1, contrary to the assumption that P_1 contains only m vertices. □

Theorem. *If G is a cellular bipartite graph, then $p_m \leqq 3m$ for all $m \geqq 1$.*

Proof. We prove the inequality $p_m \leqq 3m$ by induction on the number of vertices of a cellular graph G. Consider an arbitrary $(3m+1)$-vertex set A of G. In view of Lemma 3 we can suppose that G is decomposable. According to Theorem B it can be represented as a gated amalgam of two graphs G_1 and G_2 along a gated tree T. Let V_1 and V_2 be the vertex-sets of the graphs G_1 and G_2, respectively. Put $k = \min\{|A \cap (V_1 - V_2)|, |A \cap (V_2 - V_1)|\}$. We proceed by induction on k. If $k = 0$, i.e. A is contained in one of the sets V_1 or V_2, say $A \subset V_2$, then we can apply the induction hypothesis to the subgraph G_2. Next suppose that $k = |A \cap (V_1 - V_2)| > 0$. Pick any vertex $u \in A \cap (V_1 - V_2)$ and let u' be the gate of u in T. By the induction assumption the set $A' = (A \backslash \{u\}) \cup \{u'\}$ has a Tverberg $(m+1)$-partition $\{A_1, \ldots, A_m, A_{m+1}\}$. In view of condition (iii) of Theorem A, we can suppose that this partition consists of 2-vertex subsets only. Let $x \in \bigcap_{j=1}^{m+1} co(A_j)$. We may assume that u' belongs to some set A_t, otherwise $\{A_1, \ldots, A_m, A_{m+1}\}$ represents a Tverberg $(m+1)$-partition of the initial set A. Let $A_t = \{u', v\}$. If $v \in V_2$ then $u' \in I(u,v)$ and replacing in A_t the vertex u' by u we get a Tverberg $(m+1)$-partition of A. So assume that $v \in V_1 - V_2$. Then $x \in I(u', v) \subset V_1$.

We claim that there exists at least one vertex $w \in A \cap V_2$ such that $w \notin \bigcup_{j=1}^{m+1} A_j$. Assume the contrary. Since $|A \cap (V_2 - V_1)| \geqq |A \cap (V_1 - V_2)|$ necessarily at least one pair $A_q = \{y, z\}$ consists of vertices of V_2, where $y \in V_2 - V_1$. Hence $x \in I(y,z) \subset V_2$, i.e. $x \in T = V_1 \cap V_2$. Let v', y' and z' be the gates in T of the vertices v, y and z, respectively. Since $x \in I(u', v)$ and $v' \in I(x, v)$, there must be a shortest path between u' and v which passess through the vertices x and v'. On the other hand, since $y' \in I(y,x)$ and $z' \in I(z,x)$, there must be a shortest path between y and z which contains the vertices y', x and z. Therefore $x \in I(u',v') \cap I(y',z')$. According to the Observation (cf. the proof of Proposition 2) there is another Radon partition of the set $\{u',v',y',z'\}$ which has x as a Radon point. Let, say, $x \in I(u',y') \cap I(v',z')$. Since $u,v \in V_1$ and $y,z \in V_2$, we conclude that $x \in I(u,y) \cap I(v,z)$. Replacing the sets A_t and A_q by the pairs $\{u,y\}$ and $\{v,z\}$ we get a Tverberg $(m+1)$-partition of the set A. Therefore there is a vertex $w \in A \cap V_2$ with $w \notin \bigcap_{j=1}^{m} A_j$. Then $u' \in I(u,w)$ and $x \in I(u',v)$, i.e. $x \in co(u,v,w)$. In particular, replacing set A_t by the set $A_t^+ = \{u,v,w\}$ we get a Tverberg $(m+1)$-partition of the set A. Since $co(A_t^+) = I(u,v) \cup I(v,w) \cup I(w,u)$ we can replace in this partition the set A_t^+ by that pair of vertices u, v and w whose interval contains the vertex x. The proof of the theorem is now complete. $\square$

We conclude with an illustration of the result of the Theorem for $m = 2$. In fact, the set $\{a_1,a_2,a_3,a_4,a_5,a_6\}$ has a unique Tverberg 3-partition $\{a_1,a_4\}$, $\{a_2,a_5\}$, $\{a_3,a_6\}$.

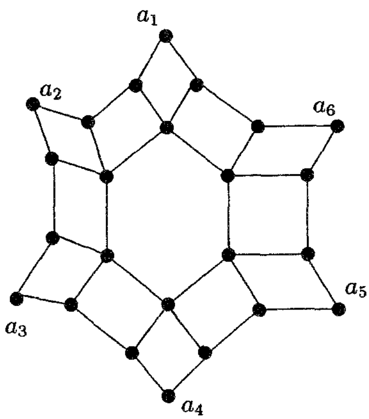

Figure 2.

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