This is the annex to our paper *"Appearance Radii in Medial Axis Test Mask for Small Planar Chamfer Norms"* by Jérôme Hulin and Édouard Thiel, to appear in DGCI #15.

This annex is available online at

`http://pageperso.lif.univ-mrs.fr/~hulin/dgci09/`

The annex contains the proofs of Lemmas 1 to 6 and Theorems 2 to 4. Other technical lemmas are needed, they are numbered with small latin numbers (i, ii, etc.).

In this document we introduce some additional notations.
Let us denote by $\mathrm{rad}(B)$ the representable radius of a given ball B. A *G-cone* of $\mathbb{Z}^n$ is the image of $\mathrm{G}(\mathbb{Z}^n)$ by a given symmetry σ in Σ^n. For any set of vectors $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k} \in \mathbb{Z}^n$, $G_{adj}(\overrightarrow{v_1}, \ldots, \overrightarrow{v_k})$ stands for the relation: $\overrightarrow{v_1}, \ldots, \overrightarrow{v_k}$ all lie in a common G-cone of $\mathbb{Z}^n$. Given a vector $\overrightarrow{v} \in \mathbb{Z}^n$, we call $\widetilde{v}$ the *representative* of $\overrightarrow{v}$ in $\mathrm{G}(\mathbb{Z}^n)$, defined to be the (necessarily unique) vector in $\mathrm{G}(\mathbb{Z}^n)$ verifying $\widetilde{v} = \sigma(\overrightarrow{v})$ for some $\sigma \in \Sigma^n(O)$.

Also, we wish to generalize the Frobenius number for non-coprime numbers a, b : let $p = \gcd(a, b)$, we see that all (a, b)-representable integers are multiples of p. We define $g'(a, b)$ to be the largest multiple of p which is not (a, b)-representable. This number exists since a/p and b/p are coprime, and we have

$$g'(a, b) = p * g\Big(\frac{a}{p}, \frac{b}{p}\Big) = \frac{ab}{p} - a - b. \tag{1}$$

For example, if we take $a = 9$ and $b = 15$, we have $\gcd(a, b) = 3$ and $g'(a, b) = 3 * g(3, 5) = 21$. All integers greater than $g(3, 5) = 7$ are $(3, 5)$-representable, and all multiples of 3 greater than $g'(9, 15) = 21$ are $(9, 15)$-representable:

$$\begin{aligned}\big\{[x]_{9,15}\big\}_{x\in\mathbb{N}} &= 9\mathbb{N} + 15\mathbb{N} = \{0,\ 9,\ 15,\ 18,\ 24,\ 27,\ 30,\ 33,\ \ldots\}\ ;\\ &= 3*(3\mathbb{N}+5\mathbb{N}) = 3*\ \{0,\ 3,\ 5,\ 6,\ 8,\ 9,\ 10,\ 11,\ \ldots\}\ .\end{aligned}$$

Lemma i (Covering radius) *Let $\|.\|$ be a norm, $\overrightarrow{v}$ be a vector in $\mathbb{Z}^n$, and B a ball of centre $p \in \mathbb{Z}^n$. We have $\mathcal{R}_{p-\overrightarrow{v}}(B) \leqslant \mathrm{rad}(B) + \|\overrightarrow{v}\|$.*

Proof. Let $q = p - \overrightarrow{v}$, and $r = \mathrm{rad}(B)$ denote the representable radius of B. Let z be a point of B which maximizes the distance to q (see Fig. 2). The representable radius of the ball $H_q(B)$ is $\mathcal{R}_q(B) = d(q, z)$. According to the triangle inequality, we can write $\mathcal{R}_q(B) = d(q, z) \leqslant d(q, p) + d(p, z)$. However, z belongs to B so $d(p, z) \leqslant r$; furthermore $d(q, p) = \|\overrightarrow{v}\|$, so $\mathcal{R}_q(B) \leqslant \|\overrightarrow{v}\| + r$. □

Lemma ii *Let $\overrightarrow{u}, \overrightarrow{v} \in \mathbb{Z}^2$. If $\neg G_{adj}(\overrightarrow{u}, \overrightarrow{v})$ then for any 2-dimensionnal G-symmetrical norm $\|.\|$, we have $\|\overrightarrow{u} + \overrightarrow{v}\| \leqslant \|\widetilde{u} + \widetilde{v}\|$.*

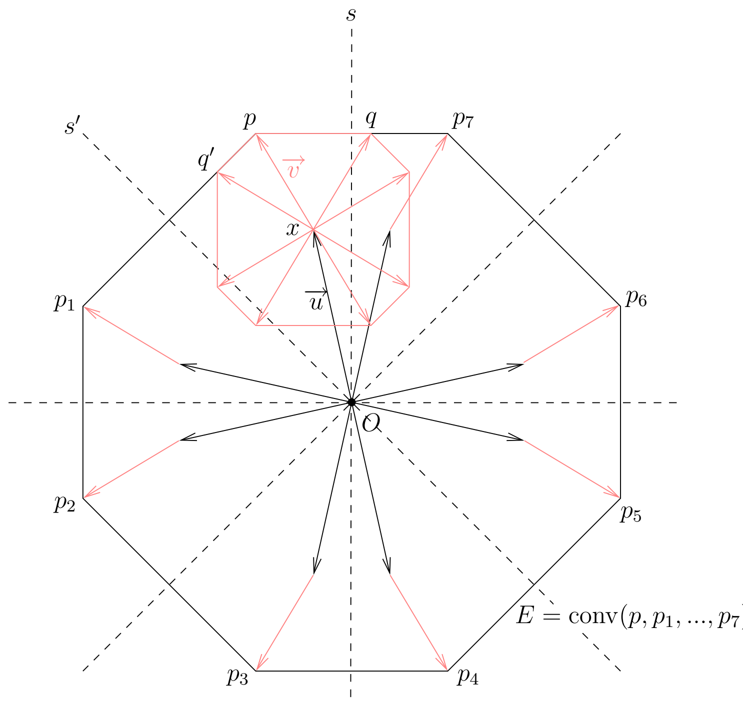

Fig. 11. The 8 G-cones of $\mathbb{Z}^2$ (delimited by dotted lines); two G-adjacent vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. The inverses of $\overrightarrow{u}$ by all the $\sigma \in \Sigma^2$ are depicted by black vectors, the inverses of $\overrightarrow{v}$ by the $\sigma \in \Sigma^2$ are the gray vectors. The points $\{x + \sigma(\overrightarrow{v})\}$ are contained in the convex hull of the $O + \sigma(\overrightarrow{u} + \overrightarrow{v})$.

Proof. Suppose $\overrightarrow{u}$ and $\overrightarrow{v}$ belong to the same G-cone C. We want to show that $\forall \sigma \in \Sigma^2, \|\overrightarrow{u} + \sigma(\overrightarrow{v})\| \leqslant \|\overrightarrow{u} + \overrightarrow{v}\|$. Let $x = O + \overrightarrow{u}, p = x + \overrightarrow{v}$, and $p_1, \dots, p_7$ be the images of p by all the symmetries $\sigma \in \Sigma^2 \setminus \{\mathrm{Id}\}$ (see Fig. 11). In other words, $\{p, p_1, \dots, p_7\} = \{O + \sigma(\overrightarrow{u} + \overrightarrow{v}), \sigma \in \Sigma^2\}$. Finally, let E be the convex hull of the points $p, p_1, \dots, p_7$. Now we use the fact that $\forall \sigma \in \Sigma^2$, the point $z = x + \sigma(\overrightarrow{v})$ is included in E. At worst, two such points belong to the boundary of E : the points q and q' defined by $\overrightarrow{xq} = s(\overrightarrow{xp})$ and $\overrightarrow{xq'} = s'(\overrightarrow{xp})$, with s and s' being the symmetries about each axis surrounding C. According to the symmetries, we have that q belongs to the line segment $[pp_7]$, therefore $\overrightarrow{Oq} = \lambda \overrightarrow{Op} + (1 - \lambda)\overrightarrow{Op_7}$ for some $0 \leqslant \lambda \leqslant 1$. Then, by convexity of the norm, we deduce that $\|\overrightarrow{Oq}\| \leqslant \lambda \|\overrightarrow{Op}\| + (1 - \lambda)\|\overrightarrow{Op_7}\|$. Furthermore, by definition of p_7, we have $Op = Op_7$, hence $\|\overrightarrow{Oq}\| \leqslant \|\overrightarrow{Op}\|$. Similar considerations apply to the point $q' \in [pp_1]$. □

Lemma iii (Covering the generator) *Let* $\overrightarrow{v}$ *be a vector in* $\mathrm{G}(\mathbb{Z}^2)$*,* $\|.\|$ *be a 2D G-symmetrical norm, and* B *a ball of centre* O*. There is at least one point* p *in* $\mathrm{G}(B)$ *verifying* $\mathcal{R}_{O-\overrightarrow{v}}(B) = \|\overrightarrow{v} + \overrightarrow{Op}\|$*.*

Proof. Let q be a point of B that maximizes the distance to $O - \overrightarrow{v}$, and define $\overrightarrow{u} = \overrightarrow{Oq}$; we have $\mathcal{R}_{O-\overrightarrow{v}}(B) = \|\overrightarrow{v} + \overrightarrow{u}\|$. The ball B is G-symmetrical so the point $p = O + \widetilde{u}$ belongs to B. Furthermore, $\overrightarrow{v} \in \mathrm{G}(\mathbb{Z}^2)$, hence lemma ii states that $\|\overrightarrow{v}+\widetilde{u}\| \geqslant \|\overrightarrow{v}+\overrightarrow{u}\|$. As a consequence, p is a point of $\mathrm{G}(B)$ that maximizes the distance to $O - \overrightarrow{v}$. □

Lemma iv (Corollary of Lemma iii) *Let* $\|.\|$ *be a 2D G-symmetrical norm,* B *be a ball of centre* O *and* B' *a ball of centre* $O - \overrightarrow{v}$ *for some* $\overrightarrow{v} \in \mathbb{Z}^2$*. If* $\mathrm{G}(B) \subseteq B'$ *then* $B \subseteq B'$*.*

Proof. Set $O' = O - \overrightarrow{v}$. According to Lemma iii, there is a point $p \in \mathrm{G}(B)$ for which $d(O', p) = \mathcal{R}_{O'}(B)$. Moreover, $\mathrm{G}(B) \subseteq B'$ implies $p \in B'$, hence $\mathrm{rad}(B') \geqslant O'p = \mathcal{R}_{O'}(B)$. Consequently, $B \subseteq B'$. □

Lemma 1 (Representable radius) *Let* $\mathcal{C}(\overrightarrow{v_1}, \overrightarrow{v_2})$ *be an influence cone of a given 2D chamfer norm, and assume the vectors* $\overrightarrow{v_1}$ *and* $\overrightarrow{v_2}$ *have repective weights* w_1 *and* w_2*. If* r *is* (w_1, w_2)*-representable then for any vector* $\overrightarrow{v}$ *in* $\mathcal{C}(\overrightarrow{v_1}, \overrightarrow{v_2})$*,* $\mathcal{R}_{O-\overrightarrow{v}}\big(\mathcal{B}(O, r)\big) = r + \|\overrightarrow{v}\|$*.*

Proof. Write $B = \mathcal{B}(O, r)$, $B' = H_{O-\overrightarrow{v}}(B)$ and $r' = \mathrm{rad}(B')$. The integer r is (w_1, w_2)-representable, so there is a point p in the cone $\mathcal{C}(O, \overrightarrow{v_1}, \overrightarrow{v_2})$ s.t. $Op = r$ (see Fig. 12). Furthermore, O belongs to the cone $\mathcal{C}(O', \overrightarrow{v_1}, \overrightarrow{v_2})$. Accordingly, there is a minimal chamfer path between O' and p passing through O. Hence $O'p = O'O + Op = \|\overrightarrow{v}\| + r$. since $p \in B'$, we have $r' \geqslant O'p$, and thus $r' \geqslant \|\overrightarrow{v}\| + r$. However, we know from Lemma i that $r' \leqslant \|\overrightarrow{v}\| + r$. Hence $r' = r + \|\overrightarrow{v}\|$. □

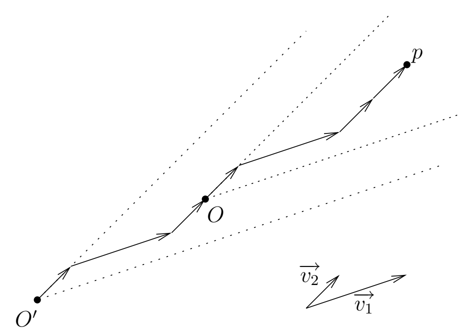

Fig. 12. A point p in the influence cone $\mathcal{C}(O, \overrightarrow{v_1}, \overrightarrow{v_2})$.

Lemma 2 (Covering a cone) *Let $\mathcal{C}(\overrightarrow{v_1}, \overrightarrow{v_2})$ be an influence cone of a given 2D chamfer norm, and B be a ball of centre O. For any vector $\overrightarrow{v}$ in the cone $\mathcal{C}(\overrightarrow{v_1}, \overrightarrow{v_2})$, we have $B \cap \mathcal{C}(O, \overrightarrow{v_1}, \overrightarrow{v_2}) = H_{O-\overrightarrow{v}}(B) \cap \mathcal{C}(O, \overrightarrow{v_1}, \overrightarrow{v_2})$.*
In other words, the balls B and $H_{O-\overrightarrow{v}}(B)$ coincide in the cone $\mathcal{C}(O, \overrightarrow{v_1}, \overrightarrow{v_2})$.

Proof. Write $r = \mathrm{rad}(B)$, $O' = O - \overrightarrow{v}$ and $B' = H_{O'}(B)$. By definition we have $B \subseteq B'$. Now, consider a point $p \in B' \cap \mathcal{C}(O, \overrightarrow{v_1}, \overrightarrow{v_2})$; we want to show that $p \in B$. The point O belongs to the cone $\mathcal{C}(O', \overrightarrow{v_1}, \overrightarrow{v_2})$, and p belongs to the cone $\mathcal{C}(O, \overrightarrow{v_1}, \overrightarrow{v_2})$, hence there is a minimal path between O' and p passing through O. So we can write

$$O'p = O'O + Op = \|\overrightarrow{v}\| + Op. \tag{2}$$

Moreover, p belongs to $H_{O'}(B)$, so $O'p \leqslant \mathrm{rad}(B')$. It follows from (2):

$$\|\overrightarrow{v}\| + Op \leqslant \mathrm{rad}(B'). \tag{3}$$

Furthermore, Lemma i yields

$$\mathrm{rad}(B') \leqslant r + \|\overrightarrow{v}\|. \tag{4}$$

Combining (3) with (4), we deduce that $Op + \|\overrightarrow{v}\| \leqslant r + \|\overrightarrow{v}\|$, and, in consequence, $Op \leqslant r$. □

Lemma 3 *Let $\mathcal{M}$ be a minimal norm mask $\langle a, b\rangle$, then we have: $\forall \overrightarrow{u}, \overrightarrow{v} \in \mathrm{G}(\mathbb{Z}^2_*)$, $\overrightarrow{u} \succ \overrightarrow{u} + \overrightarrow{v}$.*

Proof. Let B be a ball of centre O and representable radius r. Since there is only one influence cone $\mathcal{C}(\overrightarrow{a}, \overrightarrow{b})$ in the generator, Lemma 1 gives $\mathcal{R}_{O-\overrightarrow{u}}(B) = r + \|\overrightarrow{u}\|$ and $\mathcal{R}_{O-\overrightarrow{u}-\overrightarrow{v}}(B) = r + \|\overrightarrow{u} + \overrightarrow{v}\|$. Moreover $\overrightarrow{u}$ and $\overrightarrow{v}$ belong to the same influence cone, therefore $\|\overrightarrow{u} + \overrightarrow{v}\| = \|\overrightarrow{u}\| + \|\overrightarrow{v}\|$. Thus, the difference between $\mathcal{R}_{O-\overrightarrow{u}-\overrightarrow{v}}(B)$ and $\mathcal{R}_{O-\overrightarrow{u}}(B)$ is $\|\overrightarrow{v}\|$; so Lemma i leads to $H_{O-\overrightarrow{u}}(b) \subset H_{O-\overrightarrow{u}-\overrightarrow{v}}(B)$. □

Lemma v (Construction of the sequences $\mathcal{R}_{k\overrightarrow{a}}$ and $\mathcal{R}_{k\overrightarrow{b}}$) *Let B be a ball of centre O. For abbreviation, we write $\mathcal{R}_{\overrightarrow{v}}$ instead of $\mathcal{R}_{O-\overrightarrow{v}}(B)$. For any $k \in \mathbb{N}$, we can express $\mathcal{R}_{(k+1)\overrightarrow{a}}$ and $\mathcal{R}_{(k+1)\overrightarrow{b}}$ as*

$$\mathcal{R}_{(k+1)\overrightarrow{a}} = \max\left\{ [\mathcal{R}_{k\overrightarrow{a}}]_{a,c} + a,\ [\mathcal{R}_{k\overrightarrow{a}}]_{b,c} + c - b \right\};$$
$$\mathcal{R}_{(k+1)\overrightarrow{b}} = \max\left\{ [\mathcal{R}_{k\overrightarrow{b}}]_{a,c} + c - a,\ [\mathcal{R}_{k\overrightarrow{b}}]_{b,c} + b \right\}.$$

Proof. Given a vector $\overrightarrow{v} \in \mathrm{G}(\mathbb{Z}^n)$, $\mathcal{R}_{\overrightarrow{v}} = \max_{p \in B}\Big\{d(O - \overrightarrow{v}, p)\Big\}$. Since any norm is translation invariant, we have

$$\mathcal{R}_{\overrightarrow{v}} = \max_{p \in B}\Big\{d(O, p + \overrightarrow{v})\Big\} = \max_{\overrightarrow{u} \in \mathbb{Z}^2, \|\overrightarrow{u}\| \leqslant r}\Big\{\|\overrightarrow{u} + \overrightarrow{v}\|\Big\}. \tag{5}$$

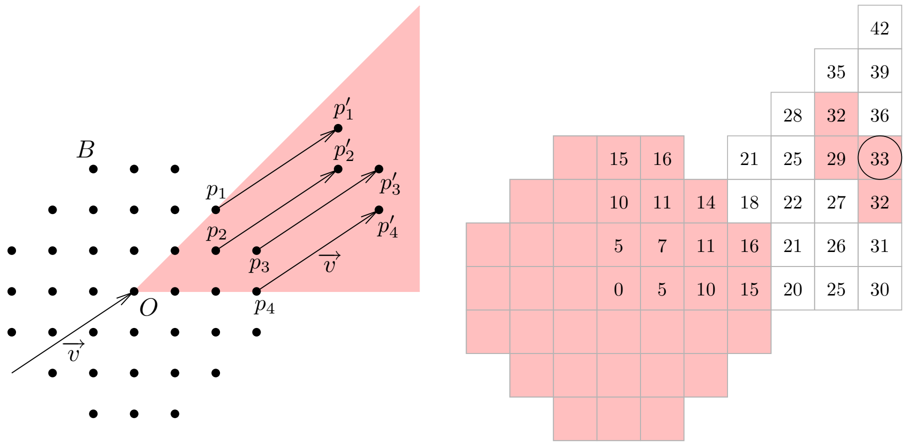

Fig. 13. Covering a ball $B = \mathcal{B}(O, 16)$ — drawn with bullets — in direction $\overrightarrow{v}(3, 2)$, for the norm mask $\langle 5, 7, 11\rangle$. The radius of $H_{O-\overrightarrow{v}}(B)$ is given by the maximum value of the $d(O, p'_i)$: here $\mathcal{R}_{O-\overrightarrow{v}}(B) = Op'_3 = 33$.

Since we consider G-symmetrical masks, it is sufficient by Lemma iv to consider $p \in \mathrm{G}(B)$ and $\overrightarrow{u} \in \mathrm{G}(\mathbb{Z}^2)$ in equation (5), see Fig. 13 for an example, so

$$\mathcal{R}_{\overrightarrow{v}} = \max_{\overrightarrow{u} \in \mathrm{G}(\mathbb{Z}^2), \|\overrightarrow{u}\| \leqslant r} \left\{ \|\overrightarrow{u} + \overrightarrow{v}\| \right\} . \tag{6}$$

Now Consider the case where $\overrightarrow{v} = \overrightarrow{a}$. We decompose (6) depending on whether $\overrightarrow{u}$ belongs or not to the influence cone $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$. We obtain

$$\mathcal{R}_{\overrightarrow{a}} = \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u} + \overrightarrow{a}\|\}, \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b}),\, \overrightarrow{u} \notin \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} \right\} . \tag{7}$$

If $\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$, then $\overrightarrow{u} + \overrightarrow{a} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$, and since the elementary displacement δ_x is $\delta_x = a$ in this cone, we get $\|\overrightarrow{u} + \overrightarrow{a}\| = \|\overrightarrow{u}\| + a$. If $\overrightarrow{u} \notin \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$, then $\overrightarrow{u}$ and $\overrightarrow{u} + \overrightarrow{a}$ both belong to $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$ ($\overrightarrow{u} + \overrightarrow{a}$ may also belong to $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$); since $\delta_x = c - b$ in this cone, we have $\|\overrightarrow{u} + \overrightarrow{a}\| = \|\overrightarrow{u}\| + c - b$.

Hence we can deduce from (7) that

$$\mathcal{R}_{\overrightarrow{a}} = \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u}\|\} + a, \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b}),\, \overrightarrow{u} \notin \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u}\|\} + c - b \right\} .$$

But $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c}) \cap \mathcal{C}(\overrightarrow{c}, \overrightarrow{b}) = \overrightarrow{c}\mathbb{N}$, so

$$\mathcal{R}_{\overrightarrow{a}} = \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u}\|\} + a, \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b}),\, \overrightarrow{u} \notin \overrightarrow{c}\mathbb{N}} \{\|\overrightarrow{u}\|\} + c - b \right\} . \tag{8}$$

We necessarily have $\|\overrightarrow{u}\| + a > \|\overrightarrow{u}\| + c - b$ because $a > c - b$ for any minimal norm $\langle a, b, c\rangle$. Since the term $\|\overrightarrow{u}\| + a$ appears for $\overrightarrow{u} \in \overrightarrow{c}\mathbb{N}$ in the second max of equation (8), we can also take $\overrightarrow{u} \in \overrightarrow{c}\mathbb{N}$ in the third max:

$$\mathcal{R}_{\overrightarrow{a}} = \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u}\|\} + a,\ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b})} \{\|\overrightarrow{u}\|\} + c - b \right\}. \tag{9}$$

The norm of a vector $\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$ can be expressed as a positive linear combination of a and c, therefore

$$\max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c}),\, \|\overrightarrow{u}\| \leqslant r} \{\|\overrightarrow{u}\|\} = \max_{t \in a\mathbb{N}+c\mathbb{N},\, t \leqslant r} \{t\} = [r]_{a,c}\,.$$

In the same manner in $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$, we can write

$$\max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b}),\, \|\overrightarrow{u}\| \leqslant r} \{\|\overrightarrow{u}\|\} = \max_{t \in b\mathbb{N}+c\mathbb{N},\, t \leqslant r} \{t\} = [r]_{b,c}\,.$$

Then we can rewrite (9) using representable integers only:

$$\mathcal{R}_{\overrightarrow{a}} = \max\Big\{ [r]_{a,c} + a,\ [r]_{b,c} + c - b \Big\}. \tag{10}$$

The next step is to compute $\mathcal{R}_{2\overrightarrow{a}}$ by

$$\mathcal{R}_{2\overrightarrow{a}} = \max_{\overrightarrow{u} \in \mathbb{Z}^2 : \|\overrightarrow{u}\| \leqslant r} \left\{ \|\overrightarrow{u} + 2\overrightarrow{a}\| \right\} = \max_{\overrightarrow{u} \in \mathbb{Z}^2 : \|\overrightarrow{u}\| \leqslant r} \left\{ \|\overrightarrow{u} + \overrightarrow{a} + \overrightarrow{a}\| \right\} \tag{11}$$

that is decomposed whether $\overrightarrow{u} + \overrightarrow{a}$ belongs to $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$:

$$\begin{aligned} \mathcal{R}_{2\overrightarrow{a}} &= \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u}+\overrightarrow{a} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u} + \overrightarrow{a}\| + a\},\ \max_{\overrightarrow{u}+\overrightarrow{a} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b}),\, \overrightarrow{u}+\overrightarrow{a} \notin \overrightarrow{c}\mathbb{N}} \{\|\overrightarrow{u} + \overrightarrow{a}\| + c - b\} \right\} \\ &= \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u}+\overrightarrow{a} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} + a,\ \max_{\overrightarrow{u}+\overrightarrow{a} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b}),\, \overrightarrow{u}+\overrightarrow{a} \notin \overrightarrow{c}\mathbb{N}} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} + c - b \right\}. \end{aligned} \tag{12}$$

Again, since $a > c - b$, we can insert the case $\overrightarrow{u} + \overrightarrow{a} \in \overrightarrow{c}\mathbb{N}$ in the third max:

$$\begin{aligned} \mathcal{R}_{2\overrightarrow{a}} &= \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u}+\overrightarrow{a} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} + a,\ \max_{\overrightarrow{u}+\overrightarrow{a} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b})} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} + c - b \right\} \\ &= \max\left\{ \max_{\|\overrightarrow{u}\| \leqslant r,\, \overrightarrow{u}+\overrightarrow{a} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} + a,\ \max_{\|\overrightarrow{u}\| \leqslant r,\, \overrightarrow{u}+\overrightarrow{a} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b})} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} + c - b \right\}. \end{aligned} \tag{13}$$

Then, using the notation of representable integer, we obtain

$$\begin{aligned} \mathcal{R}_{2\overrightarrow{a}} &= \max\left\{ \max_{\|\overrightarrow{u}\| \leqslant r,\, \overrightarrow{u}+\overrightarrow{a} \in a\mathbb{N}+c\mathbb{N}} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} + a,\ \max_{\|\overrightarrow{u}\| \leqslant r,\, \overrightarrow{u}+\overrightarrow{a} \in b\mathbb{N}+c\mathbb{N}} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} + c - b \right\} \\ &= \max\left\{ \left[\max_{\|\overrightarrow{u}\| \leqslant r} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} \right]_{a,c} + a,\ \left[\max_{\|\overrightarrow{u}\| \leqslant r} \{\|\overrightarrow{u} + \overrightarrow{a}\|\} \right]_{b,c} + c - b \right\}. \end{aligned} \tag{14}$$

By definition of $\mathcal{R}_{\overrightarrow{v}}$ in (5), we deduce

$$\mathcal{R}_{2\overrightarrow{a}} = \max\big\{ [\mathcal{R}_{\overrightarrow{a}}]_{a,c} + a,\ [\mathcal{R}_{\overrightarrow{a}}]_{b,c} + c - b \big\}. \tag{15}$$

In that manner, we can construct the sequence $\mathcal{R}_{k\overrightarrow{a}}$ by induction:

$$\mathcal{R}_{(k+1)\overrightarrow{a}} = \max\big\{ [\mathcal{R}_{k\overrightarrow{a}}]_{a,c} + a,\ [\mathcal{R}_{k\overrightarrow{a}}]_{b,c} + c - b \big\}. \tag{16}$$

In order to compute the sequence $\mathcal{R}_{k\overrightarrow{b}}$, we revisit equation (6), replacing $\overrightarrow{v}$ by $\overrightarrow{b}$. Then we decompose in two cases, whether $\overrightarrow{u}$ belongs to the cone $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$ or not:

$$\mathcal{R}_{\overrightarrow{b}} = \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b})} \big\{ \|\overrightarrow{u} + \overrightarrow{b}\| \big\},\ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c}),\ \overrightarrow{u} \notin \mathcal{C}(\overrightarrow{c}, \overrightarrow{b})} \big\{ \|\overrightarrow{u} + \overrightarrow{b}\| \big\} \right\}. \tag{17}$$

In the influence cones $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$ and $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$, the displacements $\delta_x + \delta_y$ are $c - a$ and b, respectively. Hence, in the same way as in (9), we can write

$$\mathcal{R}_{\overrightarrow{b}} = \max_{\|\overrightarrow{u}\| \leqslant r} \left\{ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b})} \big\{ \|\overrightarrow{u}\| \big\} + b,\ \max_{\overrightarrow{u} \in \mathcal{C}(\overrightarrow{a}, \overrightarrow{c})} \big\{ \|\overrightarrow{u}\| \big\} + c - a \right\}. \tag{18}$$

Next, we replace each $\max\{\|\overrightarrow{u}\|\}$ by its arithmetical expression, obtaining

$$\mathcal{R}_{\overrightarrow{b}} = \max\big\{ [r]_{a,c} + c - a,\ [r]_{b,c} + b \big\}. \tag{19}$$

Using a similar reasoning as the one for $\mathcal{R}_{k\overrightarrow{a}}$, by distinguishing the two cases $\overrightarrow{u} + \overrightarrow{b} \in \mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$ and $\overrightarrow{u} + \overrightarrow{b} \notin \mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$, we can finally deduce by induction the terms of the sequence $\mathcal{R}_{k\overrightarrow{b}}$:

$$\mathcal{R}_{(k+1)\overrightarrow{b}} = \max\big\{ [\mathcal{R}_{k\overrightarrow{b}}]_{a,c} + c - a,\ [\mathcal{R}_{k\overrightarrow{b}}]_{b,c} + b \big\}. \tag{20}$$

□

The construction of the sequence $\mathcal{R}_{k\overrightarrow{b}}$ is illustrated in Fig. 14 for a ball B of radius 46, with the norm $\|.\|_{\langle 9,12,19\rangle}$. The integer 46 is $(9,19)$-representable, but is not $(12,19)$-representable: By (19) and (20) it is sufficient to consider, for each influence cone, the propagation of the maximal (representable) distance value within B. In $\mathcal{C}(O, \overrightarrow{a}, \overrightarrow{c})$, this maximal value is $[46]_{9,19} = 46$, while in $\mathcal{C}(O, \overrightarrow{c}, \overrightarrow{b})$ this maximal value is $[46]_{12,19} = 43$.

Lemma 4 (Domination along $\overrightarrow{a}\mathbb{N}$ and $\overrightarrow{b}\mathbb{N}$) *For any minimal norm mask $\langle a, b, c\rangle$ and any $k \in \mathbb{N}_*$, we have $k\overrightarrow{a} \succ (k+1)\overrightarrow{a}$ and $k\overrightarrow{b} \succ (k+1)\overrightarrow{b}$.*

Proof. Let B be a ball of centre O and radius r, where r is (a,c)- or (b,c)-representable. Let $k \in \mathbb{N}_*$, we set $O' = O - k\overrightarrow{a}$, $O'' = O - (k+1)\overrightarrow{a} = O' - \overrightarrow{a}$, $B' = H_{O'}(B)$ and $B'' = H_{O''}(B)$ (see Fig. 15). Moreover, we set $r' = \mathcal{R}_{O'}(B)$ and $r'' = \mathcal{R}_{O''}(B)$. Our aim is to show that $B' \subseteq B''$: consider a point q in the

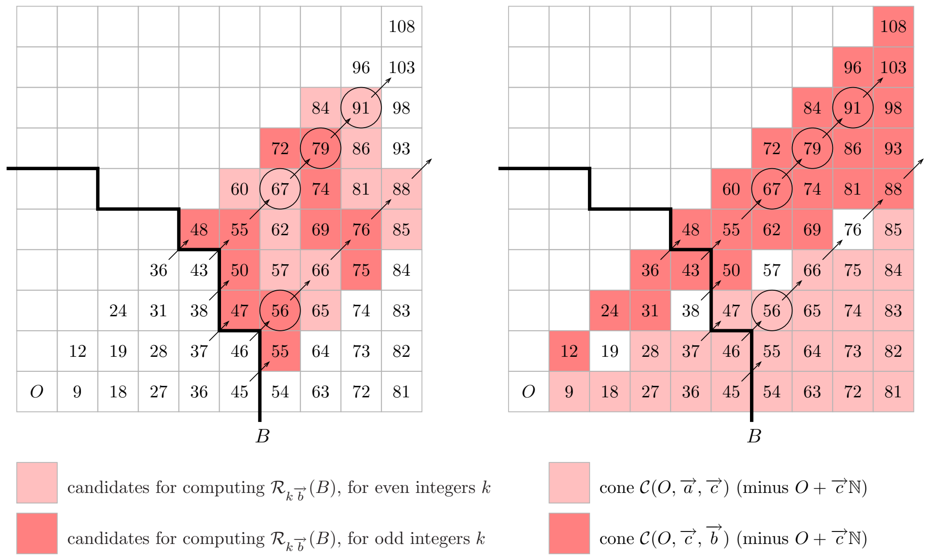

Fig. 14. Construction of the radii $\mathcal{R}_{k\overrightarrow{b}}$ (circled values) for a ball B of radius 46 (delimited by the thick line), with the norm $\|.\|_{\langle 9,12,19\rangle}$. By induction: $\mathcal{R}_{\overrightarrow{b}} = 56$, $\mathcal{R}_{2\overrightarrow{b}} = 67$, $\mathcal{R}_{3\overrightarrow{b}} = 79$, etc. Left: induction steps. Right: The two influence cones within the generator.

ball B', we are going to show that $q \in B''$, i.e., $O''q \leqslant r''$.

Let B'_g denote the generator of B' about centre O', i.e., $B'_g = B' \cap \mathcal{C}(O', \overrightarrow{a}, \overrightarrow{b})$. Given that $\overrightarrow{a} \in \mathrm{G}(\mathbb{Z}^2)$, Lemma iv tells us that if all points from B'_g belong to B'' then $B' \subseteq B''$. It is therefore sufficient to consider $q \in B'_g$. We know from Lemma v that

$$r'' = \max\left\{ [r']_{a,c} + a,\ [r']_{b,c} + c - b \right\} . \tag{21}$$

Let us evaluate $O''q$ knowing that $\overrightarrow{O''q} = \overrightarrow{O'q} + \overrightarrow{a}$. Two cases are to be considered:

- ▷ If $q \in \mathcal{C}(O', \overrightarrow{a}, \overrightarrow{c})$, then $O''q = O'q + a$ (elementary displacement $\delta_x = a$ in the influence cone $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$). But we also have $O'q \in a\mathbb{N} + c\mathbb{N}$, so

$$O''q \leqslant [r']_{a,c} + a. \tag{22}$$

  Combining (21) and (22) gives $O''q \leqslant r''$.
- ▷ If $q \notin \mathcal{C}(O', \overrightarrow{a}, \overrightarrow{c})$, then $\overrightarrow{O'q}$ and $\overrightarrow{O'q} + \overrightarrow{a}$ both belong to the cone $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$, and so $O''q = O'q + c - b$ (elementary displacement $\delta_x = c - b$ in the influence cone $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$). Since we also have $O'q \in b\mathbb{N} + c\mathbb{N}$, it comes

$$O''q \leqslant [r']_{b,c} + c - b. \tag{23}$$

  In the same way, equations (21) and (23) give $O''q \leqslant r''$.

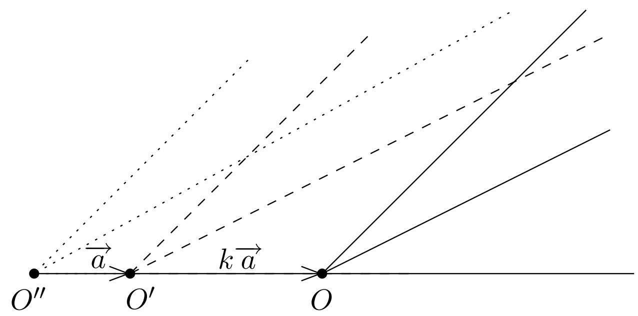

Fig. 15. Influence cones about centres O, $O' = O - k\overrightarrow{a}$ and $O'' = O - (k+1)\overrightarrow{a}$.

We can rewrite the proof of the lemma by exchanging $(\overrightarrow{a}, a)$ with $(\overrightarrow{b}, b)$. The elementary displacement to be considered are $\delta_x + \delta_y = b$ in the cone $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$ for the first case, and $\delta_x + \delta_y = c - a$ in the cone $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$ for the second case. Thus we obtain the domination relation along the $\overrightarrow{b}\mathbb{N}$ axis. □

Lemma 5 (Domination by addition of $\overrightarrow{c}$) *For any minimal norm mask $\langle a, b, c\rangle$ and any $\overrightarrow{v} \in \mathrm{G}(\mathbb{Z}^2_*)$, we have $\overrightarrow{v} \succ \overrightarrow{v} + \overrightarrow{c}$.*

Proof. Consider a ball B of centre O and radius $r = \mathrm{rad}(B)$. Set $O' = O - \overrightarrow{v}$, $O'' = O - \overrightarrow{v} - \overrightarrow{c}$, $r' = \mathcal{R}_{O'}(B)$, $r'' = \mathcal{R}_{O''}(B)$, $B' = H_{O'}(B)$ and $B'' = H_{O''}(B)$. We are reduced to proving $B' \subseteq B''$. Let p be a point in $\mathrm{G}(B)$ which maximizes the distance to O' (see Fig. 16). The vector $\overrightarrow{v}$ belongs to $\mathrm{G}(\mathbb{Z}^2_*)$, so by Lemma iii we have $O'p = r'$.
We observe that all minimal paths from O'' to p contain at least one occurrence of $\overrightarrow{c}$. Actually, these paths are expressed as $\alpha\overrightarrow{a} + \beta\overrightarrow{c}$ or $\alpha\overrightarrow{b} + \beta\overrightarrow{c}$, but they can not be composed of $\overrightarrow{a}$ only or $\overrightarrow{b}$ only. Hence, there is a minimal path from O'' to p passing through O', and so

$$O''p = O''O' + O'p = c + r'. \tag{24}$$

By definition, the point p belongs to B, therefore p also belongs to B'', and so

$$O''p \leqslant r''. \tag{25}$$

Combining (24) and (25) yields $c + r' \leqslant r''$. Furthermore we know from Lemma i that $\mathcal{R}_{O''}(B') \leqslant r' + c$, this implies that $\mathcal{R}_{O''}(B') \leqslant r''$, and so $H_{O''}(B') \subseteq B''$. Since by definition $B' \subseteq H_{O''}(B')$, we finally have $B' \subseteq B''$. □

Theorem 2 *For any minimal norm mask $\mathcal{M} = \langle a, b, c\rangle$, we have*

$$\{\overrightarrow{a}, \overrightarrow{b}\} \subseteq \mathcal{T}_{\mathcal{M}} \subseteq \{\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\}.$$

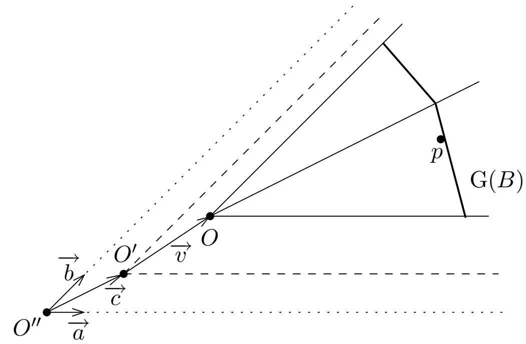

Fig. 16. One occurrence of $\overrightarrow{c}$ in any minimal path linking O'' to p.

Proof. Let $\overrightarrow{v} \in \mathrm{G}(\mathbb{Z}^2_*)$, different from $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$. If $\overrightarrow{v}$ is a multiple of $\overrightarrow{c}$, then we know from Lemma 5 that $\overrightarrow{v} \prec \overrightarrow{c}$. Otherwise, there are two possibilities: either $\overrightarrow{v}$ belongs to $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$, in that case $\overrightarrow{v} = k\overrightarrow{a} + l\overrightarrow{c}$ for some $k, l \in \mathbb{N}_*$. If $l \neq 0$ then Lemma 5 gives $k\overrightarrow{a} \succ \overrightarrow{v}$; and by Lemma 4 we deduce $\overrightarrow{a} \succ \overrightarrow{v}$. Or, $\overrightarrow{v}$ belongs to $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$, in that case $\overrightarrow{v} = k\overrightarrow{b} + l\overrightarrow{c}$ for some $k, l \in \mathbb{N}_*$. If $l \neq 0$ then Lemma 5 yields $k\overrightarrow{b} \succ \overrightarrow{v}$; and by Lemma 4 we obtain $\overrightarrow{b} \succ \overrightarrow{v}$. □

Lemma 6 (Arithmetical expression of $R_{app}(\overrightarrow{c})$) *For any minimal norm mask $\langle a, b, c\rangle$, we have*

$$R_{app}(\overrightarrow{c}) \;=\; \min\left\{\, r \in \mathbb{N} \;:\; \begin{cases} r + c - b < \Big[[r]_{a,c} + a\Big]_{b,c} \\ r + c - a < \Big[[r]_{b,c} + b\Big]_{a,c} \end{cases} \right\} + c\,.$$

Proof. To shorten the notation, let $H_a(B) = H_{O-\overrightarrow{a}}(B)$, $H_b(B) = H_{O-\overrightarrow{b}}(B)$ and $H_c(B) = H_{O-\overrightarrow{c}}(B)$. We also set $O_a = O - \overrightarrow{a}$, $O_b = O - \overrightarrow{b}$ and $O_c = O - \overrightarrow{c}$. Similarly, we write $\mathcal{R}_a(B) = \mathcal{R}_{O-\overrightarrow{a}}(B)$, etc.

Given that only $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ may appear in $\mathcal{T}$, we deduce that $\overrightarrow{c} \in \mathcal{T}$ iff there is a ball B of centre O which satisfies the following conditions:

$$\begin{cases} H_a(B) \not\subseteq H_c(B) & \text{(i)} \\ H_b(B) \not\subseteq H_c(B). & \text{(ii)} \end{cases} \tag{26}$$

Let r be the smallest radius of B satisfying this system (assuming there is such a r), then $R_{app}(\overrightarrow{c}) = \mathcal{R}_c(B) = r + c$. With the notation $B_r = \mathcal{B}(O, r)$, we can write

$$R_{app}(\overrightarrow{c}) = \min\Big\{\, r \in \mathbb{N} \;:\; H_a(B_r) \not\subseteq H_c(B_r) \text{ and } H_b(B_r) \not\subseteq H_c(B_r) \,\Big\} + c\,. \tag{27}$$

Considering (27), we first examine the set $\{r \in \mathbb{N} \,:\, H_a(B_r) \not\subseteq H_c(B_r)\}$: it consists in finding which $r \in \mathbb{N}$ implies the existence of a point q in $H_a(B_r) \backslash H_c(B_r)$.

It suffices to look for such a point q in the cone $\mathcal{C}(O_a, \overrightarrow{a}, \overrightarrow{b})$: actually $O_a - O_c = \overrightarrow{b} \in \mathrm{G}(\mathbb{Z}^2)$ so by Lemma iii, $H_a(B_r) \subseteq H_c(B_r)$ iff $H_a(B_r) \cap \mathcal{C}(O_a, \overrightarrow{a}, \overrightarrow{b}) \subseteq H_c(B_r)$. Moreover, such a point q can not belong to the cone $\mathcal{C}(O, \overrightarrow{a}, \overrightarrow{c})$, since Lemma 2 states that $H_a(B_r) \cap \mathcal{C}(O, \overrightarrow{a}, \overrightarrow{c}) = H_c(B_r) \cap \mathcal{C}(O, \overrightarrow{a}, \overrightarrow{c}) = B_r \cap \mathcal{C}(O, \overrightarrow{a}, \overrightarrow{c})$.
On account of these two observations, it is sufficient to look for q in the cone $\mathcal{C}(O_a, \overrightarrow{c}, \overrightarrow{b})$, see Fig. 17. Therefore,

$$H_a(B_r) \nsubseteq H_c(B_r) \;\Leftrightarrow\; \exists q \in \mathcal{C}(O_a, \overrightarrow{c}, \overrightarrow{b}) \;:\; \begin{cases} q \in H_a(B_r) \\ q \notin H_c(B_r)\;; \end{cases}$$

that is to say:

$$H_a(B_r) \nsubseteq H_c(B_r) \;\Leftrightarrow\; \exists q \in \mathcal{C}(O_a, \overrightarrow{c}, \overrightarrow{b}) \;:\; \begin{cases} O_a q \leqslant \mathcal{R}_a(B_r) & (\alpha) \\ O_c q > \mathcal{R}_c(B_r)\;. & (\beta) \end{cases} \tag{28}$$

Suppose there is such a point q, and set $p = q + \overrightarrow{a}$. The point p belongs to the cone $\mathcal{C}(O, \overrightarrow{c}, \overrightarrow{b})$. The vectors $\overrightarrow{O_a O}$ and $\overrightarrow{qp}$ are equal; since all chamfer distances are translation invariant, $(28.\alpha)$ is equivalent to

$$Op \leqslant \mathcal{R}_a(B_r)\;. \tag{29}$$

We now turn to the inequality $(28.\beta)$: we know that r is either (a, c)-representable, or (b, c)-representable. Furthermore $\overrightarrow{c}$ belongs to both cones $\mathcal{C}(\overrightarrow{a}, \overrightarrow{c})$ and $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$.

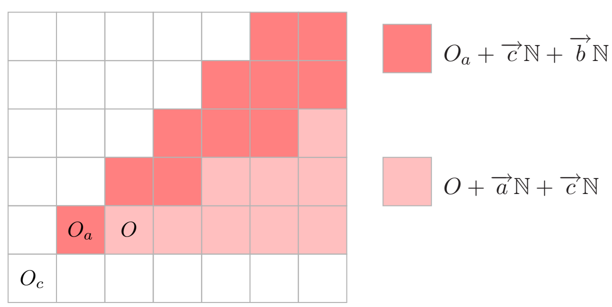

Fig. 17. The cone $\mathcal{C}(O_a, \overrightarrow{c}, \overrightarrow{b})$ (top), and the cone $\mathcal{C}(O, \overrightarrow{a}, \overrightarrow{c})$ (bottom).

By Lemma 1, we deduce

$$\mathcal{R}_c(B_r) = r + c\;. \tag{30}$$

Besides, we have

$$\overrightarrow{O_c q} = \overrightarrow{O_c O_a} + \overrightarrow{O_a q} = \overrightarrow{b} + \overrightarrow{Op}\;. \tag{31}$$

However, $\overrightarrow{Op} = \overrightarrow{O_a q}$ belongs to the cone $\mathcal{C}(\overrightarrow{c}, \overrightarrow{b})$, as does $\overrightarrow{b}$; it follows from (31) that

$$O_c q = b + Op \ . \tag{32}$$

Applying (29), (30) and (32), we can reformulate (28):

$$H_a(B_r) \nsubseteq H_c(B_r) \ \Leftrightarrow\ \exists p \in \mathcal{C}(O, \overrightarrow{c}, \overrightarrow{b}) \ : \begin{cases} Op \leqslant \mathcal{R}_a(B_r) \\ b + Op > r + c \ . \end{cases}$$

$$\Leftrightarrow\ \exists p \in \mathcal{C}(O, \overrightarrow{c}, \overrightarrow{b}) \ : \ r + c - b < Op \leqslant \mathcal{R}_a(B_r) \ . \tag{33}$$

Besides, we know that $\mathcal{R}_a(B_r) = \max\big\{[r]_{a,c} + a, [r]_{b,c} + c - b\big\}$ (see Lemma v). The inequality in (33) can not be satisfied if $\mathcal{R}_a(B_r) = [r]_{b,c} + c - b$, for $[r]_{b,c} \leqslant r$. Accordingly, we can rewrite (33):

$$H_a(B_r) \nsubseteq H_c(B_r) \ \Leftrightarrow\ \exists p \in \mathcal{C}(O, \overrightarrow{c}, \overrightarrow{b}) \ : \ r + c - b < Op \leqslant [r]_{a,c} + a \ . \tag{34}$$

For any p in the influence cone $\mathcal{C}(O, \overrightarrow{c}, \overrightarrow{b})$, the distance Op can be expressed as a non-negative linear combination of b and c. Hence, the inequality in (34) has a solution iff the greatest (b, c)-representable integer no greater than $[r]_{a,c} + a$ is larger than $r + c - b$:

$$H_a(B_r) \nsubseteq H_c(B_r) \ \Leftrightarrow\ r + c - b < \Big[[r]_{a,c} + a\Big]_{b,c} \ . \tag{35}$$

This provides an arithmetical description of the set $\{r \in \mathbb{N} \ : \ H_a(B_r) \nsubseteq H_c(B_r)\}$.

Similar arguments apply to describe the set $\{r \in \mathbb{N} \ : \ H_b(B_r) \nsubseteq H_c(B_r)\}$, while interchanging $(\overrightarrow{a}, a)$ and $(\overrightarrow{b}, b)$: we look for a point q in $H_b(B_r) \setminus H_c(B_r)$, and show that it suffices to look for q in the cone $\mathcal{C}(O_b, \overrightarrow{a}, \overrightarrow{c})$. Assuming q to exist, we set $p = q + \overrightarrow{b}$ and observe that $\mathcal{R}_b(B)$ must be equal to $[r]_{b,c} + b$, thus we obtain

$$H_b(B_r) \nsubseteq H_c(B_r) \ \Leftrightarrow\ r + c - a < \big[[r]_{b,c} + b\big]_{a,c}. \tag{36}$$

We finally complete the proof by substituting (35) and (36) into (27).

□

Lemma vi *Let a, c be two positive integers and let $p = \gcd(a, c)$. For any $x \geqslant g'(a, c) + p$ and any $k \in \mathbb{N}$ we have $[x + kp]_{a,c} = [x]_{a,c} + kp$.*

Proof. Let $x \geqslant g'(a, c) + p$, we denote by α and β the quotient and the remainder of the Euclidean division of x by p. By definition of g', we have $[x]_{a,c} = \alpha p$. Besides, we can write $[x + kp]_{a,c} = [(\alpha + k)p + \beta]_{a,c}$. The remainder of the Euclidean division of $x + p$ by p is β, so we have $[(\alpha + k)p + \beta]_{a,c} = (\alpha + k)p$, and thus $[x + kp]_{a,c} = \alpha p + kp = [x]_{a,c} + kp$. □

Theorem 3 (Appearance of $\overrightarrow{c}$) *For any minimal norm mask $\mathcal{M} = \langle a, b, c\rangle$,*

$$\overrightarrow{c} \in \mathcal{T}_{\mathcal{M}} \ \Leftrightarrow\ \gcd(a, c) + \gcd(b, c) \ \leqslant\ 2\,(a + b - c) \ .$$

Proof. $\overrightarrow{c}$ belongs to $\mathcal{T}$ iff the system in Lemma 6 has a solution. If this system is satisfied for a given r, then it is also satisfied for $r+c$, since by definition of $[.]_{a,c}$ we have $[r]_{a,c}+c \leqslant [r+c]_{a,c}$ and $[r]_{b,c}+c \leqslant [r+c]_{b,c}$. This shows that if the system admits a solution, then it admits an arbitrary large solution. Set $p=\gcd(a,c)$, $q=\gcd(b,c)$ and $M=\max\{g'(a,c)+p,\ g'(b,c)+q\}$, we can write

$$\overrightarrow{c} \in \mathcal{T} \ \Leftrightarrow\ \exists r > M : \begin{cases} \exists \alpha \in b\mathbb{N}+c\mathbb{N} : r+c-b < \alpha \leqslant [r]_{a,c}+a \\ \exists \beta \in a\mathbb{N}+c\mathbb{N} : r+c-a < \beta \leqslant [r]_{b,c}+b\,. \end{cases} \tag{37}$$

According to the definition of g', any integer multiple of q and greater than $g'(b,c)+q$ is (b,c)-representable, so $r > M$ implies that the smallest integer (b,c)-representable larger than r is $[r]_{b,c}+q$. Likewise, $r > g'(a,c)+p$ implies that the smallest integer (a,c)-representable larger than r is $[r]_{a,c}+p$. We can thereby reformulate (37):

$$\overrightarrow{c} \in \mathcal{T} \ \Leftrightarrow\ \exists r > M : \begin{cases} [r+c-b]_{b,c}+q \leqslant [r]_{a,c}+a \\ [r+c-a]_{a,c}+p \leqslant [r]_{b,c}+b\,. \end{cases} \tag{38}$$

Given that $r > M$ and that q (resp. p) divides $c-b$ (resp. $c-a$), Lemma vi gives $[r+c-b]_{b,c}=[r]_{b,c}+c-b$ (resp. $[r+c-a]_{a,c}=[r]_{a,c}+c-a$). Consequently,

$$\overrightarrow{c} \in \mathcal{T} \ \Leftrightarrow\ \exists r > M : \begin{cases} [r]_{b,c}+q \leqslant [r]_{a,c}+a+b-c \\ [r]_{a,c}+p \leqslant [r]_{b,c}+a+b-c\,. \end{cases} \tag{39}$$

With the notation $\Delta = a+b-c$, we obtain

$$\overrightarrow{c} \in \mathcal{T} \ \Leftrightarrow\ \exists r > M : q-\Delta \leqslant [r]_{a,c}-[r]_{b,c} \leqslant \Delta - p\,. \tag{40}$$

To finish the proof, we beforehand need a result on representable integers:

Lemma vii *Let a, b and c be three positive integers, $p=\gcd(a,c)$ and $q=\gcd(b,c)$. If p and q are coprime, then for any integer $x \in [1-p, q-1]$, there is an arbitrary large integer r verifying $[r]_{a,c}-[r]_{b,c}=x$.*

Proof. At first, we establish the proof for $x \in [0, q-1]$. The integers p and q are coprime, so we know by Bezout's theorem that there are two positive integers α and β such that $\alpha p - \beta q = x$. Moreover, α and β can be chosen arbitrarily large, take any such couple (α,β) verifying $\beta q > \max\{g'(a,c), g'(b,c)\}$, and set $r = \alpha p = \beta q + x$. We have $p|r$ and $r > g'(a,c)$, it follows that $[r]_{a,c}=r$. On the other hand, $r=\beta q + x$ belongs to $[\beta q, (\beta+1)q[$, hence $[r]_{b,c}=\beta q$. Consequently, $[r]_{a,c}-[r]_{b,c}=\alpha p-\beta q = x$. The proof concerning $x \in [1-p, 0]$ is obtained in the same manner, exchanging (a,p) with (b,q). □

Let us consider (40) again; remember we want to establish $\overrightarrow{c} \in \mathcal{T} \Leftrightarrow p+q \leqslant 2\Delta$. It is obvious from (40) that $\overrightarrow{c} \in \mathcal{T}$ implies $q-\Delta \leqslant \Delta - p$, that is to say, $p+q \leqslant 2\Delta$. Conversely, if $p+q \leqslant 2\Delta$, then there is at least one integer in the interval $[q-\Delta, \Delta-p]$. Besides, given that the mask $\langle a,b,c\rangle$ is minimal, we have

$\Delta = a + b - c \geqslant 1$, and so $1 - p \leqslant \Delta - p$ and $q - \Delta \leqslant q - 1$. We can find an integer x which belongs to both intervals $[1-p, q-1]$ and $[q-\Delta, \Delta-p]$, for instance $x = \max\{1-p, q-\Delta\}$. The fact that $x \in [1-p, q-1]$ allows us to claim (thanks to Lemma vii) that there is an integer r arbitrarily large s.t. $[r]_{a,c} - [r]_{b,c} = x$. The fact that $x \in [q-\Delta, \Delta-p]$ proves the converse. □

Theorem 4 *Let $\mathcal{M} = \langle a,b,c \rangle$ be a minimal norm mask. If $\overrightarrow{c} \in \mathcal{T}_{\mathcal{M}}$ then $R_{app}(\overrightarrow{c}) < bc$.*

Proof. From Lemma 6, we search an upper bound for the smallest r satisfying

$$r - b + c < \Big[[r]_{a,c} + a\Big]_{b,c} \quad (i) \quad \text{and} \qquad r - a + c < \Big[[r]_{b,c} + b\Big]_{a,c} \quad (ii)\ . \tag{41}$$

We have already seen at the beginning of the proof of Thm. 3 that if this system has a solution, then it admits an arbitrary large solution too. Set $p = \gcd(a,c)$, $q = \gcd(b,c)$ and $M = \max\big\{g'(a,c) + p,\ g'(b,c) + q\big\}$.

First, we show that if (41) is satisfied for a given $r \geqslant M$, then it is also satisfied for all $r - kpq$ s.t. $k \in \mathbb{N}$ and $r - kpq \geqslant M$. Suppose $r - kpq \geqslant M$; $p|kpq$ so Lemma vi gives $\Big[[r-kpq]_{a,c}+a\Big]_{b,c} = \Big[[r]_{a,c}+a-kpq\Big]_{b,c}$. Furthermore, kpq is a multiple of q and $[r]_{a,c}+a \geqslant r$ so again, Lemma vi yields $\Big[[r]_{a,c}+a-kpq\Big]_{b,c} = \Big[[r]_{a,c}+a\Big]_{b,c} - kpq$. Inequality (i) is therefore equivalent to $(r-kpq) - b + c < \Big[[r-kpq]_{a,c}+a\Big]_{b,c}$. The same reasoning applies to (ii), exchanging a with b; this shows that $r - kpq$ satisfies (41). The smallest $r - kpq \geqslant M$ belongs to the interval $[M, M+pq-1]$. Adding the term $+c$ of Lemma 6, we obtain the bound

$$R_{app}(\overrightarrow{c}) \leqslant M + c + pq - 1. \tag{42}$$

Now, let us expand $g'(a,c)$ and $g'(b,c)$ in (42):

$$R_{app}(\overrightarrow{c}) < \max\Big\{\frac{ac}{p} - a - c + p,\ \frac{bc}{q} - b - c + q\Big\} + c + pq,$$

$$\text{and so}\ \ R_{app}(\overrightarrow{c}) < \max\Big\{\frac{ac}{p} - a + p(q+1),\ \frac{bc}{q} - b + q(p+1)\Big\}. \tag{43}$$

Consider the continuous functions $f(p) = \frac{ac}{p} - a + p(q+1)$ for $p \in [1,a]$, and $h(q) = \frac{bc}{q} - b + q(p+1)$ for $q \in [1,b]$. We first study the variations of f in $[1,a]$. We have $\frac{\partial f}{\partial p} = -\frac{ac}{p^2} + q + 1 = \frac{p^2(q+1)-ac}{p^2}$, which vanishes at $p_0 = \sqrt{\frac{ac}{q+1}}$. We always have $p_0 > 1$, and distinguish two cases: if $p_0 \geqslant a$, then f is decreasing from 1 to a, and $\forall p \in [1,a]$, $f(p) \leqslant f(1)$; if $p_0 < a$, then f is decreasing over $[1,p_0]$ and is increasing over $[p_0,a]$. Hence $\forall p \in [1,a]$, $f(p) \leqslant \max\{f(1), f(a)\}$. We can easily find upper bounds for $f(1)$ and $f(a)$: $f(1) = ac - a + q + 1 \leqslant ac - a + b + 1$, but one of the norm conditions is $3b \leqslant 2c$, so $ac - a + b + 1 \leqslant c(a + 2/3) - a + 1 < bc - a + 1 < bc$. $f(a) = aq + c$, but another norm condition

is $2a \leqslant c$, so $aq + c \leqslant c(q/2+1) \leqslant c(b/2+1)$. Since any minimal norm mask $\langle a, b, c\rangle$ satisfies $b \geqslant 4$, we have $b/2+1 < b$, hence $f(a) < bc$.
The same upper bound for $h(q)$ is obtained by exchanging (a, p) with (b, q). $\frac{\partial h}{\partial q}$ vanishes for $q_0 = \sqrt{\frac{bc}{p+1}} > 1$. The two cases are: if $q_0 \geqslant b$, then h is decreasing from 1 to b and $\forall q \in [1, b]$, $h(q) \leqslant h(1)$; if $q_0 < b$, then h is decreasing over $[1, q_0]$ and increasing over $[q_0, b]$. Hence $\forall q \in [1, b]$, $h(q) \leqslant \max\{h(1), h(b)\}$. An upper bound for $h(1)$ is $h(1) = bc - b + p + 1 \leqslant bc - b + a + 1 \leqslant bc$. Concerning $h(b)$, we can write $h(b) = bp + c$, the norm condition $3b \leqslant 2c$ then leads to $h(b) \leqslant c(2p/3+1) \leqslant c(2b/3+1)$. Since any minimal norm mask $\langle a, b, c\rangle$ satisfies $b \geqslant 4$, we have $2b/3+1 < b$, and so $h(b) < bc$. □