

## COMPUTING THE JTH SOLUTION OF A FIRST-ORDER QUERY

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**Abstract.** We design algorithms of "optimal" *data complexity* for several natural problems about first-order queries on structures of bounded degree. For that purpose, we first introduce a framework to deal with logical or combinatorial problems  $R \subset I \times O$  whose instances  $x \in I$  may admit of several solutions  $R(x) = \{y \in O : (x, y) \in R\}$ . One associates to such a problem several specific tasks : compute a random (for the uniform probability distribution) solution  $y \in R(x)$ ; enumerate without repetition each solution  $y_j$  in some specific linear order  $y_0 < y_1 < \dots < y_{n-1}$  where  $R(x) = \{y_0, \dots, y_{n-1}\}$ ; compute the solution  $y_j$  of rank  $j$  in the linear order  $<$ .

Algorithms of "minimal" data complexity are presented for the following problems: given any first-order formula  $\varphi(\bar{v})$  and any structure  $S$  of bounded degree:

- (1) compute a random element of  $\varphi(S) = \{\bar{a} : (S, \bar{a}) \models \varphi(\bar{v})\}$ ;
- (2) compute the  $j^{\text{th}}$  element of  $\varphi(S)$  for some linear order of  $\varphi(S)$ ;
- (3) enumerate the elements of  $\varphi(S)$  in lexicographical order.

More precisely, we prove that, for any fixed formula  $\varphi$ , the above problems 1 and 2 (resp. problem 3) can be computed within constant time (resp. with constant delay) after a linear ( $O(|S|)$ ) precomputation. Our essential tool for deriving those complexity results is a *normalization* procedure of first-order formulas on *bijective structures*.

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**Résumé.** Cet article présente des algorithmes de "data complexity" minimale pour plusieurs problèmes naturels de requêtes du premier ordre sur des structures de degré borné. En premier lieu, on introduit un cadre formel pour les problèmes logico-combinatoires  $R \subset I \times O$  dont les instances  $x \in I$  peuvent admettre plusieurs solutions  $R(x) = \{y \in O : (x, y) \in R\}$ . Trois problématiques algorithmiques spécifiques sont étudiées : générer une solution aléatoire  $y \in R(x)$  pour la distribution de probabilité uniforme; énumérer sans répétition les solutions  $y_j$  de  $R(x) = \{y_0, \dots, y_{n-1}\}$  dans un ordre linéaire donné  $y_0 < y_1 < \dots < y_{n-1}$ ; calculer la solution  $y_j$  de rang  $j$  pour l'ordre  $<$ .

On présente des algorithmes de "data complexity" minimale pour les problèmes suivants portant sur une formule du premier ordre  $\varphi(\bar{v})$  et une structure  $S$  de degré borné :

- (1) calculer un élément aléatoire (pour la distribution de probabilité uniforme) de l'ensemble  $\varphi(S) = \{\bar{a} : (S, \bar{a}) \models \varphi(\bar{v})\}$ ;
- (2) calculer le  $j^{eme}$  élément de  $\varphi(S)$ , pour un certain ordre de  $\varphi(S)$ ;
- (3) énumérer les éléments de  $\varphi(S)$  dans l'ordre lexicographique.

Plus précisément, on prouve que, pour toute formule fixée  $\varphi$ , chaque problème (1) et (2) (resp. (3)) peut être résolu en temps constant (resp. à délai constant) après un précalcul en temps linéaire ( $O(|S|)$ ). L'outil essentiel pour prouver ces résultats est une nouvelle procédure de *normalisation* des formules du premier ordre sur les *structures bijectives*.

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*In honour of Serge Grigorieff at the occasion of his 60th birthday.*

*In memoriam of Clemens Lautemann.*

## 1. INTRODUCTION

After the study of the complexity of the logical theories (see [Gri91] for a nice survey), the complexity of query evaluation is a natural and well-studied problem of computer science. Several algorithmic questions may be considered. Classically, given a structure  $S$  in some class  $\mathcal{C}$  and a formula  $\varphi$  in some logic  $\mathcal{L}$  one wants to know how hard it is to evaluate whether  $\varphi$  holds in  $S$ . The complexity can be measured either in terms of the structure size (*data complexity*) or in terms of the structure size *and* the formula size (*combined complexity*). When the formula has free variables then the size of the output also serves in the complexity evaluation. It is well known that the combined complexity of first-order query evaluation is intractable in general [Var95]. Hence, finding restrictions on the structure or on the class of formulas that admit efficient query evaluation is an important problem and has deserved a lot of attention (see, for example, [Yan81, See96, PY99, FFG02, FG01, GLS01, Lin06]).

In this article, we do not consider the global cost of computing the whole result of a query. Instead, query evaluation is studied here from a *dynamical* point of

view. Three kinds of algorithmic tasks are considered: (i) the *Random solution* problem which consists in computing a random tuple of the result (for the uniform probability distribution); (ii) the *Enumeration* problem that aims at generating successively each tuple of the result in some specific linear order  $<$ ; (iii) the  *$j^{\text{th}}$  solution* problem which computes the tuple of rank  $j$  for the linear order  $<$ . In other words, the objective is to compute a single or a new solution (tuple) of the query result. A central notion in algorithms that solve these problems is that of *precomputation*. Before starting each specific task, a (short) precomputation is allowed, based on the structure and the formula only. Afterwards, the algorithm must be able to compute efficiently a particular solution (for example, the  $j^{\text{th}}$  one, whatever  $j$  is, for some order  $<$ ) or to enumerate all the solutions with a small delay between two consecutive ones. Complexity bounds for tasks (ii) or (iii) provide, of course, complexity bounds for the classical query problem : e.g., the total number of steps to compute the entire result of the query is  $\text{time}(\text{precomputation}) + (\text{delay} * \text{number of solutions})$ . Of course, analyzing tasks (i–iii) gives *additional* information on *uniformity* or *regularity* of the query evaluation process.

The complexity of the enumeration problem of (several classes of) logical queries has begun to be studied in some recent articles: see [DO06, Bag06, Cou07] for first-order or monadic second-order logic over structures of bounded tree-width and [BDG07] for acyclic conjunctive queries with disequalities. However, the present article is the first one, to our knowledge, that studies the complexity of the random solution and the  $j^{\text{th}}$  solution problems of queries.

In this work that improves and completes the results of [DG07], we consider first-order queries on *bounded degree structures* (as in [See96, DG07]) and we give complexity results for tasks (i – iii) above. Given a first-order formula  $\varphi(x_1, \dots, x_k)$  and a structure  $S$  of bounded degree,  $\varphi(S)$  denotes the set of tuples  $\{\bar{a} : (S, \bar{a}) \models \varphi(\bar{x})\}$  that constitute the result of query  $\varphi$  over  $S$ . We prove that, for a fixed formula  $\varphi$ , computing a random tuple of  $\varphi(S)$  or computing the  $j^{\text{th}}$  element of  $\varphi(S)$  for some linear order of  $\varphi(S)$  can be done in constant time after a linear ( $O(|S|)$ ) precomputation that depends on formula  $\varphi$  and structure  $S$  only. In other words, the value of  $j$  is given only *after* the precomputation step is completed. For the enumeration task, we prove that one can enumerate all the elements of  $\varphi(S)$  in *lexicographical* order with a constant delay between two consecutive solutions, again after a linear precomputation. On the one hand, it seems necessary to read the whole input structure: this corresponds to the linear time precomputation. On the other hand, the time required to compute one tuple is at least its (constant) size. That means that our results are of *optimal data complexity*.

In this article, we consider first-order logic over the so-called *bijective structures* that consist of bijective unary functions and monadic predicates. First-order logic over such structures is the analog of first-order logic over *structures of bounded degree* when dealing with *functions* instead of *relations*. Then, our algorithmic results will derive from a *quantifier elimination* and *normalization* procedure of first-order formulas over bijective structures. More precisely, the notion of *inductive description* is introduced as a natural way to describe relationships between variables and a new normal form for first-order logic on bijective structures is

proved: every first-order formula is equivalent to an *exclusive* disjunction of inductive descriptions.

The article is organized as follows. In Section 2, the main definitions of our problems and complexity classes are given. Section 3 presents the logical notions and our main normalization result. They are used in Section 4 to prove our complexity bounds for first-order queries over bijective structures. Finally, Section 5 derives similar results for queries over structures of bounded degree.

## 2. PRELIMINARIES

### 2.1. PROBLEMS

We will deal with problems whose instances may admit of several solutions. Given an instance, we can be interested in different actions: counting its solutions, enumerating them all, choosing one of them randomly, and so on. In order to handle this diversity of problems in a uniform way, we introduce the notion of *problem relation* which is merely a binary relations  $R \subseteq I \times O$  over two sets  $I$  and  $O$  respectively called *space of instances* (or *input space*) and *space of solutions* (or *output space*). For each input  $x \in I$ , the set  $R(x) = \{y : (x, y) \in R\}$  is called the *set of solutions* of  $R$  on  $x$ . We will demand this set to be finite for each  $x$ . Now, the standard *decision* and *evaluation* problems associated with  $R$  can be defined as follows:

DECIDE( $R$ )		EVAL( $R$ )	
<i>Input:</i>	an instance $x$ of $R$	<i>Input:</i>	an instance $x$ of $R$
<i>Output:</i>	accept iff $R(x) \neq \emptyset$	<i>Output:</i>	$R(x)$

We will deal with the *counting* and *random* problem associated with  $R$ :

COUNT( $R$ )		RANDOM( $R$ )	
<i>Input:</i>	an instance $x$ of $R$	<i>Input:</i>	an instance $x$ of $R$
<i>Output:</i>	$\text{card}(R(x))$	<i>Output:</i>	a <i>random</i> element of $R(x)$

But we will mainly focus on two other problems whose definition needs some more notation: Let  $<$  be a strict linear order defined on the space of solutions of  $R$ . For any input  $x$ , we denote by  $\text{enum}(R, x, <)$  the strictly increasing enumeration of  $R(x)$  according to  $<$  and by  $\text{jth}(R, x, <, j)$  the element of rank  $j$  in this enumeration. That is: if  $R(x) = \{y_0, \dots, y_{n-1}\}$  with  $y_0 < \dots < y_{n-1}$ , then  $\text{enum}(R, x, <) = y_0 y_1 \dots y_{n-1}$  and  $\text{jth}(R, x, <, j) = y_j$ . The *enumeration* and  $j^{\text{th}}$  *solution* problems related to  $R$  and  $<$  are respectively defined by:

ENUM( $R, <$ )		$j^{\text{th}}$ ( $R, <$ )	
<i>Input:</i>	an instance $x$ of $R$	<i>Input:</i>	an instance $x$ of $R$ and $j \in \mathbb{N}$
<i>Output:</i>	$\text{enum}(R, x, <)$	<i>Output:</i>	$\text{jth}(R, x, <, j)$

Notice that beside the ordering in which the solutions of  $R$  on  $x$  appear in the final output, there is an important difference between the problems EVAL( $R$ ) and ENUM( $R, <$ ) which is not quite explicit in our definitions: in the enumeration

problem, it is essential that the solutions are returned one after another, while it is not required in the evaluation version of the problem. This will be made precise in the forthcoming definition of enumeration algorithms.

## 2.2. COMPLEXITY CLASSES

The computation model used in this article is a *random access machine* model (RAM) with *uniform cost measure*, with addition and subtraction as its basic arithmetic operations (see [GS02]). It has read-only *input registers* (containing the input  $x$  of the problem), read-write *work registers* and write-only *output registers*. The *size* of an input (resp. output)  $x$ , denoted by  $|x|$ , is the number of registers it occupies (in a "reasonable" representation). We suppose that in each computation of the RAM, register values or memory addresses are bounded by  $c \cdot |x|$ , for some constant  $c$ . (Note that this bound applies to each register including input and output registers.) Although this latter assumption may appear to restrict the computational power of RAMs, it is in fact realistic and close to the intuition of what linear time (or related small complexity measure) is. Moreover, one defines a *random RAM* to be a RAM with an additional instruction  $RANDOM(reg)$  whose semantics is: store in register  $reg$  a *random* integer in the interval  $[0, c \cdot |x| - 1]$ .

We say that a (RAM) algorithm  $\mathcal{A}$  computes the enumeration problem  $ENUM(R, <)$  if, for any input  $x$ ,  $\mathcal{A}$  computes one after one the elements of the sequence  $\text{enum}(R, x, <)$  and stops immediately after writing the last one. For such a procedure, we denote by  $\text{time}_j(x)$  the moment when  $\mathcal{A}$  has completed the writing of the  $j^{\text{th}}$ -solution (by convention,  $\text{time}_0(x) = 0$ ). We define  $\text{delay}_j(x) = \text{time}_j(x) - \text{time}_{j-1}(x)$ .

An enumeration algorithm  $\mathcal{A}$  that computes a problem  $ENUM(R, <)$  is *constant delay* if there is a constant  $c$  such that for, any input  $x$ , we have  $\text{delay}_j(x) \leq c$  for any  $j < \text{card}(R(x))$  and if furthermore,  $\mathcal{A}$  uses space less than  $c$  (i.e. at most  $c$  memory registers) during its computation. Then, we say that problem  $ENUM(R, <)$  is *computable with constant delay* and we write  $ENUM(R, <) \in \text{CONSTANT-DELAY}$ .

The problem  $J^{\text{th}}(R, <)$  is *computable in constant time* if there is a constant time and constant space RAM which, for each input  $x$  and each integer  $j$ , computes  $\text{card}(R(x))$  and  $\text{jth}(R, x, <, j)$  if  $j \leq \text{card}(R(x))$ . Similarly,  $\text{RANDOM}(R)$  is *computable in constant time* if there is a random RAM which, for each input  $x$ , computes a random element of  $R(x)$  in constant time and space.

These classes are not computationally robust. However, their respective closures by linear reductions are.

**Definition 2.1.** We write  $ENUM(R, <) \in \text{CONSTANT-DELAY}_{lin}$  if there is a function  $r : x \mapsto r(x)$  computable in linear time and a constant delay algorithm  $\mathcal{A}$  which, when applied to  $r(x)$  for any input  $x$  computes  $\text{enum}(R, x, <)$ .

Similarly, we write  $J^{\text{th}}(R, <) \in \text{CONSTANT-TIME}_{lin}$  or  $\text{RANDOM}(R) \in \text{CONSTANT-TIME}_{lin}$  if there is a function  $r : x \mapsto r(x)$  computable in linear time and a RAM (resp. random RAM)  $\mathcal{A}$  which, when applied to  $(r(x), j)$  (resp.  $r(x)$ ) for any input  $x$  and any integer  $j$

(resp. for any input  $x$ ) runs in constant time and space and computes  $\text{card}(R(x))$  and  $\text{jth}(R, x, <, j)$  (resp. computes a random element of  $R(x)$ ).

### 2.3. ALGORITHMS WITH PRECOMPUTATION

It is convenient to reformulate the above defined classes by using *RAM algorithms with precomputation*:  $\text{jth}(R, <)$  belongs to  $\text{CONSTANT-TIME}_{lin}$  if there is a RAM algorithm  $\mathcal{A}$  which, for any input  $x$  and any integer  $j$ , decomposes into the following two successive phases:

- (a)  $\mathcal{A}.\text{precomp}(x)$  which performs linear time precomputation that computes  $r(x)$  and  $\text{card}(R(x))$  from  $x$  (and independently of  $j$ ).
- (b)  $\mathcal{A}.\text{output}(r(x), j)$  which computes  $\text{jth}(R, x, <, j)$  (if  $j < \text{card}(R(x))$ ) in constant time and constant space.

By abuse of notation, phase (b) will be called  $\mathcal{A}.\text{output}(x, j)$  in the next sections.

#### Remark 2.2.

- Note the crucial assumption that the precomputation phase  $\mathcal{A}.\text{precomp}$  does not use the part  $j$  of the input.
- The phase  $\mathcal{A}.\text{output}$  uses constant space, that is, a fixed number of registers. Those "work" registers are disjoint from the "input" registers that contain  $(r(x), j)$  and are read-only.

A  $\text{CONSTANT-TIME}_{lin}$  algorithm  $\mathcal{A}$  for  $\text{RANDOM}(R)$  is similarly defined by replacing the phase  $\mathcal{A}.\text{output}(r(x), j)$  by the new phase  $\mathcal{A}.\text{random}(r(x))$  which computes a random element of  $R(x)$  in constant time and constant space by using the instruction  $\text{RANDOM}(reg)$ .

Finally, a  $\text{CONSTANT-DELAY}_{lin}$  algorithm  $\mathcal{A}$  for  $\text{ENUM}(R, <)$  is defined in the same way by replacing the phase  $\mathcal{A}.\text{output}$  by a procedure  $\mathcal{A}.\text{enum}(r(x))$  which computes  $\text{enum}(R, x, <)$  in constant delay and constant space.<sup>1</sup>

**Remark 2.3.** In some of the forthcoming algorithms (see  $\text{COUNT}(\varphi, S)$  and  $\text{JTH}(\varphi, S)$  in the proof of Proposition 4.1) we fully use the robustness of the linear time complexity and of the  $\text{CONSTANT-TIME}_{lin}$  and  $\text{CONSTANT-DELAY}_{lin}$  classes. These classes are not affected by the enrichment by arithmetical operations (multiplication, exponentiation, ...) of the set of basic operations that equip our RAM model. In particular, an algorithm  $\mathcal{A}$  designed to prove the belonging to one of these classes can freely refer to  $u \text{ div } v$  or  $u \text{ mod } v$  for some fixed  $v$  smaller than the size of the input and for any  $u$  which is polynomial in this size. This is because we can build and store the tables of these operations in the precomputation phase of

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<sup>1</sup>Note that imposing constant space for enumeration algorithms is a very strong assumption. From an algorithmic point of view, this corresponds to "oblivious" enumeration algorithms that output new solutions without memoization of previously enumerated solutions (and without repetition). A more liberal use of space (from linear to exponential restrictions) would define seemingly larger complexity classes which are also worth to be studied.

$\mathcal{A}$ , so that their results on any operands can be evaluated in constant time during the output phase.

The following lemmas are easy consequences of our definitions and their proofs are left to the reader.

**Lemma 2.4.** *If  $j^{\text{th}}(R, <)$  belongs to  $\text{CONSTANT-TIME}_{\text{lin}}$  then  $\text{ENUM}(R, <) \in \text{CONSTANT-DELAY}_{\text{lin}}$  and  $\text{RANDOM}(R) \in \text{CONSTANT-TIME}_{\text{lin}}$ .*

**Lemma 2.5.** *Let  $R, R' \subseteq I \times O$  be two problem relations over the same input and output spaces. Let furthermore  $<$  be a linear order on  $O$ . If both problems  $\text{ENUM}(R, <)$  and  $\text{ENUM}(R', <)$  belong to  $\text{CONSTANT-DELAY}_{\text{lin}}$ , then so does the union problem  $\text{ENUM}(R \cup R', <)$ .*

### 3. LOGICAL FRAMEWORK

Basics of first-order logic for finite structures can be found in [Lib04]. A *signature*  $\sigma$  is a finite set of relation and function symbols of given arities (0-ary function symbols are constant symbols). A (finite)  $\sigma$ -structure consists of a domain  $D$  together with an interpretation of each symbol of  $\sigma$  over  $D$  (we do not distinguish between a symbol and its interpretation). We denote by FO the class of first-order formulas, and by FO(qf) its quantifier-free fragment.

#### 3.1. QUERY PROBLEMS

We are interested in the problems described in the previous section in the context of first-order queries. For each  $\sigma$ -formula  $\varphi(\bar{x})$  with  $k$  free variables  $\bar{x} = (x_1, \dots, x_k)$  and for each  $\sigma$ -structure  $S$  of domain  $D$ , we denote by  $\varphi(S)$  the set  $\{\bar{a} \in D^k : S \models \varphi(\bar{a})\}$ . Moreover, we denote by  $\text{enum}(\varphi, S, <)$  the increasing enumeration of  $\varphi(S)$  according to a given linear order  $<$  defined on  $D^k$ . The  $j^{\text{th}}$  element of such an enumeration is denoted by  $\text{jth}(\varphi, S, <, j)$ . Given a set  $\mathcal{C}$  of structures and a linear order  $<$  over  $k$ -tuples, we define the following problems.

<p>COUNT(<math>\varphi, \mathcal{C}</math>)</p> <p><i>Input:</i> <math>S \in \mathcal{C}</math></p> <p><i>Output:</i> <math>\text{card}(\varphi(S))</math></p>	<p>RANDOM(<math>\varphi, \mathcal{C}</math>)</p> <p><i>Input:</i> <math>S \in \mathcal{C}</math></p> <p><i>Output:</i> a random element of <math>\varphi(S)</math></p>
<p>ENUM(<math>\varphi, \mathcal{C}, &lt;</math>)</p> <p><i>Input:</i> <math>S \in \mathcal{C}</math></p> <p><i>Output:</i> <math>\text{enum}(\varphi, S, &lt;)</math></p>	<p><math>j^{\text{th}}(\varphi, \mathcal{C}, &lt;)</math></p> <p><i>Input:</i> <math>S \in \mathcal{C}</math> and <math>j \in \mathbb{N}</math></p> <p><i>Output:</i> <math>\text{jth}(\varphi, S, &lt;, j)</math></p>

The most natural linear order for the enumeration and  $j^{\text{th}}$  solution problems is the lexicographic order  $<_{\text{lex}}$  on  $D^k$  associated with the natural linear order of the domain  $D = \{0, \dots, n-1\}$  of the input structure  $S$ . It will also happen that the linear order involved in these problems depends on the formula  $\varphi$ . We will make this dependence explicit in our notation. Typically, we will later enumerate

$\varphi(S)$  according to some special lexicographical order denoted by  $<_{lex}^\varphi$  that depends on  $\varphi(S)$  and we will study the corresponding problem  $J^{th}(\varphi, \mathcal{C}, <_{lex}^\varphi)$  (see Definition 4.4).

### 3.2. LOGICAL DEFINITIONS

A central notion studied in this article as in [DG07] is that of *bijective* structure. It is defined as follows.

**Definition 3.1.** The *arity of a signature* is the maximal arity of its symbols. A *unary signature* has arity 1. Thus, a unary signature is made of constant, monadic predicate and unary function symbols. By a *unary structure* we mean a structure over a unary signature. A *bijective structure* is a unary structure in which all the unary function symbols are interpreted as permutations of the domain. We will assume that any bijective structure contains *at least one* unary function. The class of all bijective structures is denoted by BIJ.

As in [DG07], we will focus on first-order logic over bijective structures. For convenience, we will handle functions and their reciprocals directly in the syntax. This suggests the following definition.

**Definition 3.2.** A *bijective term*  $\tau(x)$  is of the form  $f_1^{\epsilon_1} \dots f_l^{\epsilon_l}(x)$  where  $l \geq 0$ ,  $\epsilon_i = \pm 1$ ,  $x$  is a variable and each  $f_i^{\epsilon_i}$  is either the function symbol  $f_i$  or its inverse  $f_i^{-1}$ .

The notions of *bijective atomic formulas*, *bijective literals* and *bijective first-order formulas* are built as usual from bijective terms.

We will often denote by  $\tau$  or  $\tau(x)$  a bijective term, according to our intention to ignore or to specify its constitutive variable. We will also make an extensive use of the intermediate notation  $\tau(\bar{x})$ , which indicates that the constitutive variable of  $\tau$  *belongs* to the tuple  $\bar{x}$ . Alternatively, we will point up this fact through the expression " $\tau$  is a bijective term *on*  $\bar{x}$ ".

### 3.3. FIRST-ORDER FORMULAS VS. INDUCTIVE DESCRIPTIONS

**Definition 3.3.** Let  $\psi_1(\bar{x}), \dots, \psi_s(\bar{x})$  be first-order formulas with the same variables. The disjunction  $\bigvee_{i=1}^s \psi_i(\bar{x})$  is said *exclusive* if its disjuncts are mutually exclusive, that is, if  $\psi_i(\bar{x})$  implies  $\neg\psi_j(\bar{x})$ , for all  $1 \leq i < j \leq s$ .

We will denote by  $\bigvee_{i=1}^s \psi_i$  a disjunction to emphasize that it is exclusive. We won't argue for the use of such a writing when its justification is obvious from the context. Clearly, the exclusive disjunction inherits the associativity and commutativity properties from the usual disjunction. In particular, a formula of the form  $\bigvee_{i \in I} \bigvee_{j \in J} \psi_i^j$  can be written  $\bigvee_{i \in I, j \in J} \psi_i^j$ .

We will need a normalization of quantifier-free formulas (Proposition 3.5) in order to prove that first-order formulas fulfil a kind of quantifier elimination result over bijective structures (Corollary 3.6). This normalization involves the notion of



*inductive description.* Roughly speaking, an inductive description is a conjunction of bijective literals in which a free variable  $y$  can be singled out in such a way that the formula can be written  $\psi(\bar{x}) \wedge \theta(\bar{x}, y)$  where  $\psi$  is an inductive description and where  $\theta$  describes in a very simple way the connection between  $y$  and the other variables. Let us detail this notion:

**Definition 3.4.** The set of *inductive descriptions*, denoted by  $\text{ID}$ , is the smallest set of quantifier-free conjunctive bijective formulas that contains:

- (i) quantifier-free conjunctive bijective formulas  $\delta(x)$  of arity 1;
- (ii) formulas of arity  $d + 1$  (with  $d > 0$ ) of the form

$$\varphi(\bar{x}, y) \equiv \psi(\bar{x}) \wedge y = \tau(\bar{x}),$$

where  $\psi(\bar{x})$  is an inductive description of arity  $d$  and  $\tau(\bar{x})$  is a bijective term built on one variable of  $\bar{x}$ ;

- (iii) formulas of arity  $d + 1$  (with  $d > 0$ ) of the form

$$\varphi(\bar{x}, y) \equiv \psi(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{i \leq m} y \neq \tau_i(\bar{x}),$$

where  $\delta$  and  $\psi$  are inductive descriptions of respective arity 1 and  $d$ , the  $\tau_i(\bar{x})$  are bijective terms, and  $\psi(\bar{x})$  logically implies both  $\text{PWD}_{i \in [m]}(\tau_i(\bar{x}))$  and  $\bigwedge_{i \in [m]} \delta(\tau_i(\bar{x}))$ .

(We abbreviate by  $\text{PWD}_{i \in [k]}(t_i)$  the formula  $\bigwedge_{i \neq j \in [k]} t_i \neq t_j$  stating that the  $t_i$  are pairwise distinct.)

Given two sets  $\mathcal{L}, \mathcal{L}'$  of formulas and a set  $\mathcal{C}$  of structures, we write " $\mathcal{L} \subseteq \mathcal{L}'$  on  $\mathcal{C}$ " if for all  $\varphi \in \mathcal{L}$  there exists  $\varphi' \in \mathcal{L}'$  such that  $\varphi(S) = \varphi'(S)$  for all  $S \in \mathcal{C}$ . We write " $\mathcal{L} = \mathcal{L}'$  on  $\mathcal{C}$ " when both  $\mathcal{L} \subseteq \mathcal{L}'$  and  $\mathcal{L}' \subseteq \mathcal{L}$  hold on  $\mathcal{C}$ . Recall that  $\text{FO}(qf)$  denotes the set of quantifier free first-order formulas. Besides, we denote by  $\Upsilon \text{ID}$  the set of formulas that are exclusive disjunctions of inductive descriptions.

**Proposition 3.5.** *On bijective structures,  $\text{FO}(qf) = \Upsilon \text{ID}$ .*

*Proof.* It is easily seen, by standard methods in logic, that each quantifier-free formula is equivalent to an exclusive disjunction  $\Upsilon_{i=1}^s \varphi_i$  where each  $\varphi_i$  is a conjunctive formula (that is, a conjunction of literals). Furthermore, since the logic  $\Upsilon \text{ID}$  is closed under exclusive disjunction, we can consider without loss of generality that the quantifier-free formula to be written in  $\Upsilon \text{ID}$  is conjunctive. So let  $\varphi$  be a quantifier-free conjunctive formula. We prove that  $\varphi$  is equivalent to an exclusive disjunction of inductive descriptions by induction on its arity. If  $\text{arity}(\varphi) = 1$ , the result is clear. Otherwise, distinguish one (arbitrary) free variable  $y$  in  $\varphi$  and write  $\varphi \equiv \varphi(\bar{x}, y)$  with  $y \notin \bar{x}$ . Now  $\varphi$  has the shape:

$$\varphi \equiv \bigwedge \pm \alpha_i(\bar{x}, y) = \beta_i(\bar{x}, y) \wedge \bigwedge \pm U_i(\gamma_i(\bar{x}, y)) \quad (1)$$

where all the  $\alpha_i, \beta_i, \gamma_i$ 's are terms built on one variable of  $\bar{x}, y$  and where the  $U_i$ 's are unary relation symbols. By authorizing *bijective* terms, we can rewrite  $\varphi$  in such a way that each equality (resp. inequality) involving  $y$  and  $\bar{x}$  has the shape  $y = \tau(\bar{x})$  (resp.  $y \neq \sigma(\bar{x})$ ), where  $\tau$  and  $\sigma$  are now bijective terms on  $\bar{x}$ . Moreover, by sorting the atomic formulas of  $\varphi$  according to the variables involved in them, we can finally write  $\varphi$  under the form:

$$\varphi \equiv \theta(\bar{x}) \wedge \delta(y) \wedge \bigwedge y = \tau_i(\bar{x}) \wedge \bigwedge y \neq \sigma_i(\bar{x}) \quad (2)$$

where both  $\theta$  and  $\delta$  are quantifier-free conjunctive bijective formulas. (The conjuncts  $U_i(\gamma_i)$  occurring in (1) have been incorporated in  $\theta$  or  $\delta$ , according to the fact that the term  $\gamma_i$  is built on  $\bar{x}$  or on  $y$ .)

Two cases occur, according to the emptiness of the conjunction of equalities. **If it is not empty, it contains a conjunct  $y = \tau_j(\bar{x})$ .** Thus, by substituting the term  $\tau_j(\bar{x})$  for each occurrence of  $y$  in  $\varphi$ , but that of the conjunct  $y = \tau_j(\bar{x})$ , we get the equivalent form:

$$\varphi \equiv \psi(\bar{x}) \wedge y = \tau_j(\bar{x})$$

where  $\psi$  is a quantifier-free conjunctive bijective formula and  $\tau_j(\bar{x})$  is a bijective term. By induction hypothesis,  $\psi$  is equivalent to an exclusive disjunction of inductive descriptions, say  $\psi \equiv \bigvee_i \psi_i(\bar{x})$ . Thus

$$\varphi \equiv \bigvee_i (\psi_i(\bar{x}) \wedge y = \tau_j(\bar{x}))$$

and we are done, since each disjunct above is an inductive description, by item (ii) of Definition 3.4.

**If  $\varphi$  does not contain any conjunct of the form  $y = \tau(\bar{x})$ ,** Equation (2) has the form:

$$\varphi(\bar{x}, y) \equiv \theta(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{i=1}^m y \neq \sigma_i(\bar{x}). \quad (3)$$

Here,  $\theta$  and  $\delta$  are quantifier-free conjunctive bijective formulas and the  $\sigma_i(\bar{x})$  are bijective terms. Let's keep in mind this last formula, that has to be written in  $\Upsilon$  ID. To carry on with the normalization of  $\varphi$ , we are going to concentrate for a while on its subformula  $\delta(y) \wedge \bigwedge_{i=1}^m y \neq \sigma_i(\bar{x})$ . More precisely, we are going to evaluate this subformula with respect to the number of distinct  $\sigma_i$  that satisfy  $\delta$ . Actually, in order to isolate the heart of our reasoning from minor features, we will focus on formulas of the form  $\delta(y) \wedge \bigwedge_{i=1}^m y \neq v_i$ , where the  $v_i$  are new first-order variables. Let us first introduce some notations.

For  $m > 0$ , we denote by  $2^{[m]}$  the set of subsets of  $[m] = [1, m]$ , by  $\min P$  the minimal element of a non empty  $P \in 2^{[m]}$ , and by  $\bigcup \mathcal{P}$  the union set  $\bigcup_{P \in \mathcal{P}} P$  of any  $\mathcal{P} \subseteq 2^{[m]}$ . If the elements of some  $\mathcal{P} \subseteq 2^{[m]}$  are pairwise disjoint, we say that

$\mathcal{P}$  is a set of disjoint  $m$ -subsets and we note  $\mathcal{P} \in \text{SDS}[m]$ . Equivalently,  $\mathcal{P} \in \text{SDS}[m]$  if  $\{(P)_{P \in \mathcal{P}}, [m] \setminus \bigcup \mathcal{P}\}$  is a partition of  $[m]$ .

Let  $\delta$  be a first-order formula of arity 1 and  $\bar{v} = v_1, \dots, v_m$  be a tuple of first-order variables. A  $\delta$ -type for  $\bar{v}$  is a formula  $\delta_{\mathcal{P}}^m(\bar{v})$ , associated with a given  $\mathcal{P} \subseteq 2^{[m]}$ , that comprehensively describes a particular behaviour of the variables with respect to  $\delta$ . Such a formula asserts that, for any  $i, j \in [m]$ :

- (i) If  $i, j$  belong to a same  $P \in \mathcal{P}$ , then  $v_i, v_j$  must be interpreted by a same element;
- (ii) If  $i, j$  belong to different components of  $\mathcal{P}$ , then  $v_i, v_j$  must be interpreted by distinct elements;
- (iii) If  $i \in \bigcup \mathcal{P}$ , then  $v_i$  must be interpreted by some elements that fulfil  $\delta$ ;
- (iv) If  $i \in [m] \setminus \bigcup \mathcal{P}$ , then  $v_i$  must be interpreted by some element that does not fulfil  $\delta$ .

For convenience, we formalize these assertions as follows:

$$\begin{aligned} (i) \quad & \bigwedge_{P \in \mathcal{P}} \bigwedge_{i \in P} v_i = v_{\min P} & (ii) \quad & \bigwedge_{P \neq Q \in \mathcal{P}} v_{\min P} \neq v_{\min Q} \\ (iii) \quad & \bigwedge_{P \in \mathcal{P}} \delta(v_{\min P}) & (iv) \quad & \bigwedge_{i \in [m] \setminus \bigcup \mathcal{P}} \neg \delta(v_i) \end{aligned}$$

and we set:  $\delta_{\mathcal{P}}^m(v_1, \dots, v_m) \equiv (i) \wedge (ii) \wedge (iii) \wedge (iv)$ . Now, the following remarks follow from the definition of  $\delta_{\mathcal{P}}^m$ : First, the formula  $\bigvee_{\mathcal{P} \in \text{SDS}[m]} \delta_{\mathcal{P}}^m(\bar{v})$  is a tautology and its disjuncts are mutually exclusive. Therefore,  $\delta(y) \wedge \bigwedge_{i \in [m]} y \neq v_i$  is equivalent to

$$\left( \bigvee_{\mathcal{P} \in \text{SDS}[m]} \delta_{\mathcal{P}}^m(\bar{v}) \right) \wedge \delta(y) \wedge \bigwedge_{i \in [m]} y \neq v_i,$$

and hence to

$$\bigvee_{\mathcal{P} \in \text{SDS}[m]} \left( \delta_{\mathcal{P}}^m(\bar{v}) \wedge \delta(y) \wedge \bigwedge_{i \in [m]} y \neq v_i \right).$$

Now, each disjunct of the above formula can be written

$$\delta_{\mathcal{P}}^m(\bar{v}) \wedge \delta(y) \wedge \bigwedge_{P \in \mathcal{P}} y \neq v_{\min P}.$$

This is because, under the hypothesis  $\delta_{\mathcal{P}}^m(\bar{v}) \wedge \delta(y)$ , each  $v_i$  such that  $i \notin \bigcup \mathcal{P}$  clearly differs from  $y$ , since it doesn't satisfy  $\delta$  while  $y$  does, and because for each  $P \in \mathcal{P}$ , the sets  $\{v_i, i \in P\}$  and  $\{v_{\min P}\}$  coincide. Therefore we get:

$$\delta(y) \wedge \bigwedge_{i \in [m]} y \neq v_i \sim \bigvee_{\mathcal{P} \in \text{SDS}[m]} \left( \delta_{\mathcal{P}}^m(\bar{v}) \wedge \delta(y) \wedge \bigwedge_{P \in \mathcal{P}} y \neq v_{\min P} \right) \quad (4)$$

Notice that  $\delta_{\mathcal{P}}^m(\bar{v})$  implies  $\text{PWD}_{P \in \mathcal{P}}(v_{\min P}) \wedge \bigwedge_{P \in \mathcal{P}} \delta(v_{\min P})$  (immediately follows from the definition of  $\delta_{\mathcal{P}}^m$ ).

Let us now come back to the initial formula  $\varphi(\bar{x}, y)$  described in (3), that we aim to normalize. By (4), it can be written:

$$\theta(\bar{x}) \wedge \bigvee_{\mathcal{P} \in \text{SDS}[m]} \left( \delta_{\mathcal{P}}^m(\sigma_1(\bar{x}), \dots, \sigma_m(\bar{x})) \wedge \delta(y) \wedge \bigwedge_{P \in \mathcal{P}} y \neq \sigma_{\min P}(\bar{x}) \right),$$

or equivalently:

$$\bigvee_{\mathcal{P} \in \text{SDS}[m]} \left( \theta(\bar{x}) \wedge \delta_{\mathcal{P}}^m(\sigma_1(\bar{x}), \dots, \sigma_m(\bar{x})) \wedge \delta(y) \wedge \bigwedge_{P \in \mathcal{P}} y \neq \sigma_{\min P}(\bar{x}) \right).$$

Let us denote by  $\psi_{\mathcal{P}}(\bar{x})$  the formula  $\theta(\bar{x}) \wedge \delta_{\mathcal{P}}^m(\sigma_1(\bar{x}), \dots, \sigma_m(\bar{x}))$ . By induction hypothesis,  $\psi_{\mathcal{P}}$  can be written as an exclusive disjunction of inductive descriptions, say  $\bigvee_{i \leq k} \psi_{\mathcal{P}}^i$ . Of course, each  $\psi_{\mathcal{P}}^i$  implies  $\psi_{\mathcal{P}}$ , and hence  $\delta_{\mathcal{P}}^m(\sigma_1(\bar{x}), \dots, \sigma_m(\bar{x}))$ . Therefore, according to the remark that follows Equation (4), each  $\psi_{\mathcal{P}}^i$  implies  $\text{PWD}_{P \in \mathcal{P}}(\sigma_{\min P}(\bar{x})) \wedge \bigwedge_{P \in \mathcal{P}} \delta(\sigma_{\min P}(\bar{x}))$ . Finally, the initial formula (3) can be written

$$\bigvee_{\mathcal{P} \in \text{SDS}[m]} \bigvee_{i \leq k} \left( \psi_{\mathcal{P}}^i(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{P \in \mathcal{P}} y \neq \sigma_{\min P}(\bar{x}) \right)$$

where each  $\psi_{\mathcal{P}}^i$  is an inductive description that implies both  $\text{PWD}_{P \in \mathcal{P}}(\sigma_{\min P}(\bar{x}))$  and  $\bigwedge_{P \in \mathcal{P}} \delta(\sigma_{\min P}(\bar{x}))$ . The conclusion follows from item (iii) of Definition 3.4.  $\square$

**Corollary 3.6.** *For each  $(\varphi, S) \in \text{FO} \times \text{BIJ}$ , there exists  $\varphi' \in \bigvee \text{ID}$  such that  $\varphi'(S) = \varphi(S)$ . Furthermore, there is some function  $f$  such that  $|\varphi'| \leq f(|\varphi|)$  (upper bound independent of  $S$ ) and  $\varphi'$  is computable from  $(\varphi, S)$  in time  $f(|\varphi|)|S|$ .*

*Proof.* We can assume, without loss of generality, that the formula  $\varphi$  of the statement has the form  $\exists z \gamma(\bar{x}, z)$ , where  $\gamma$  is a quantifier-free formula. By Proposition 3.5, such a  $\gamma$  can be written as a disjunction of inductive descriptions. Now, recall the three forms that a formula of ID can fit, according to Definition 3.4:

- (i)  $\delta(y)$  with  $\text{arity}(\delta) = 1$ ;
- (ii)  $\psi(\bar{x}) \wedge y = \tau(\bar{x})$ , where  $\psi(\bar{x}) \in \text{ID}$ ;
- (iii)  $\psi(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{i \leq m} y \neq \tau_i(\bar{x})$ , where  $\delta, \psi \in \text{ID}$  and  $\psi(\bar{x})$  logically implies  $\text{PWD}_{i \in [m]}(\tau_i(\bar{x})) \wedge \bigwedge_{i \in [m]} \delta(\tau_i(\bar{x}))$ .

Moreover, although it is not emphasized in the statement of Proposition 3.5, this rewriting of  $\gamma$  in  $\bigvee \text{ID}$  can be done in such a way that in each disjunct that fits the form (ii) or (iii) above, the variable  $y$  singularized in both these forms can be

arbitrarily chosen<sup>2</sup>. In particular, when we write  $\exists z\gamma(\bar{x}, z)$  as  $\exists z\bigvee_i \gamma_i$ , with the  $\gamma_i$ 's in ID, we can ensure that in each  $\gamma_i$  of the form (ii) or (iii), the singularized variable  $y$  coincides with  $z$ . Consequently, the formula  $\exists z\gamma(\bar{x}, z)$  can be written  $\bigvee_i \exists y\gamma_i(\bar{x}, y)$ , where each  $\gamma_i$  fits one of the forms (i), (ii) or (iii) of Definition 3.4, with  $y$  as the singularized variable.

Given a bijective structure  $S$ , it is now easy to translate each formula  $\exists y\gamma_i(\bar{x}, y)$  into a quantifier-free formula  $\tilde{\gamma}_i(\bar{x})$ , equivalent on  $S$ . This is done according to the form of  $\gamma_i$ :

- (i)  $\exists y\delta(y)$  becomes false if  $\delta(S) = \emptyset$ , true otherwise;
- (ii)  $\exists y(\psi(\bar{x}) \wedge y = \tau(\bar{x}))$  becomes  $\psi(\bar{x})$ ;
- (iii)  $\exists y(\psi(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{i \leq m} y \neq \tau_i(\bar{x}))$  becomes false if  $\delta(S)$  contains less than  $m$  elements,  $\psi(\bar{x})$  otherwise.

The complexity of this procedure amounts to that of computing  $\text{card}(\delta(S))$  for some first-order formula  $\delta(y)$  of arity 1. Hence, it is  $O(|S|)$ . So,  $\varphi$  can be transformed into a quantifier-free formula  $\varphi'$  so that  $\varphi(S) = \varphi'(S)$  and, clearly,  $|\varphi'| \leq |\varphi|$ . We conclude by applying again Proposition 3.5.  $\square$

#### 4. COMPLEXITY RESULTS ON BIJECTIVE STRUCTURES

The enumeration, counting and  $j^{\text{th}}$  solution problems are very easy for queries defined by inductive descriptions as the following proposition asserts.

**Proposition 4.1.** *For each function  $\varphi \in \text{ID}$ ,*

1.  $\text{COUNT}(\varphi, \text{BIJ})$  is computable in linear time ;
2.  $\text{J}^{\text{th}}(\varphi, \text{BIJ}, <_{\text{lex}}) \in \text{CONSTANT-TIME}_{\text{lin}}$  ;
3.  $\text{ENUM}(\varphi, \text{BIJ}, <_{\text{lex}}) \in \text{CONSTANT-DELAY}_{\text{lin}}$ .

*Proof.* Thanks to Lemma 2.4, we just have to prove 1 and 2. The algorithms described in this proof are recursive. Their behaviour on an input  $(\varphi, S)$  (resp.  $(\varphi, S, j)$ ) is directed by the shape of the inductive description  $\varphi$ . That is, the algorithms operate according to modalities that depend on the form (i), (ii) or (iii) of Definition 3.4 that  $\varphi$  matches.

1.  $\text{COUNT}(\varphi, \text{BIJ})$ . The following recursive procedure computes  $\text{card}(\varphi(S))$  in time  $O(|S|)$  for any  $\varphi \in \text{ID}$  and  $S \in \text{BIJ}$ . As mentioned above, it uses the inductive definition of ID.

The last instruction of this procedure is justified by the fact that  $\psi$  fulfils the implication  $\psi(\bar{x}) \rightarrow \text{PWD}(\tau_i, i \leq m) \wedge \bigwedge_{i \leq m} \delta(\tau_i)$ .

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<sup>2</sup>This possibility is explicitly mentioned in the first paragraph of the proof of Proposition 3.5: when  $\text{arity}(\varphi) > 1$ , the normalization procedure "distinguishes one (arbitrary) free variable  $y$  and writes  $\varphi \equiv \varphi(\bar{x}, y)$  with  $y \notin \bar{x}$ "

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procedure COUNT( $\varphi, S$ )
  if arity( $\varphi$ ) = 1 then
    compute  $\varphi(S)$  ; return card( $\varphi(S)$ )
  if  $\varphi \equiv \psi(\bar{x}) \wedge y = \tau(\bar{x})$  then
    return COUNT( $\psi, S$ )
  if  $\varphi \equiv \psi(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{i < m} y \neq \tau_i$  then
    return COUNT( $\psi, S$ ) * (COUNT( $\delta, S$ ) -  $m$ )

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2.  $J^{th}(\varphi, \text{BIJ}, <_{\text{lex}})$ . We design an algorithm that, given  $\varphi \in \text{ID}$  and  $S \in \text{BIJ}$ , performs a preprocessing step in time  $O(|S|)$  in such a way that  $\text{jth}(\varphi, S, <_{\text{lex}}, j)$  can be afterwards computed in constant time for any  $j \in \mathbb{N}$ . Both the preprocessing and the output phase are recursive. The following easy considerations will make the proof of the algorithm immediate.

(i) If  $\text{arity}(\varphi) = 1$ , then computing and sorting  $\varphi(S)$  can be done in time  $O(|S|)$ . Afterwards,  $\text{jth}(\varphi, S, <_{\text{lex}}, j)$  can be returned in constant time for any  $j$ .

(ii) If  $\varphi(\bar{x}, y) \equiv \psi(\bar{x}) \wedge y = \tau(\bar{x})$ , then  $\text{jth}(\varphi, S, <_{\text{lex}}, j) = (\bar{a}, \tau(\bar{a}))$  where  $\bar{a} = \text{jth}(\psi, S, <_{\text{lex}}, j)$ .

(iii) If  $\varphi(\bar{x}, y) \equiv \psi(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{i \leq m} y \neq \tau_i(\bar{x})$ , then the result rests on the two following remarks:

- If  $(A, <_A)$  and  $(B, <_B)$  are two linear orders and  $<_{\text{lex}}$  is the associated lexicographic ordering for  $A \times B$ , then for any  $j < \text{card}(A) \cdot \text{card}(B)$ :

$$\text{jth}(A \times B, <_{\text{lex}}, j) = (\text{jth}(A, <_A, q), \text{jth}(B, <_B, r)),$$

where  $q = j \text{ div } \text{card}(B)$  and  $r = j \text{ mod } \text{card}(B)$ .

- Let  $(A, <)$  be a linear order and  $F$  a subset of  $A$ . Denote by  $a_0, \dots, a_{p-1}$  (resp.  $f_0, \dots, f_{m-1}$ ) the strictly increasing enumeration of  $A$  (resp.  $F$ ) according to  $<$ . Then for any  $r < p - m$ :

$$\text{jth}(A \setminus F, <, r) = \text{jth}(A, <, s),$$

where  $s$  is the result of the following procedure:

```

 $s \leftarrow r$ ;  $i \leftarrow 0$ ; while ( $f_i \leq a_s$  and  $i < m$ ) { $i++$ ;  $s++$ }; return  $s$ .

```

The precomputation and output steps of an algorithm JTH for the problem  $J^{th}(\varphi, \text{BIJ}, <_{\text{lex}})$  are described below. We use some instruction of the form  $T \leftarrow \text{sort } X$ . It means "sort the set  $X$  and store it in increasing order in a table  $T[0, \dots, \text{card}(X) - 1]$ ". The soundness of JTH proceeds from the above remarks. Besides, it is clearly a  $\text{CONSTANT-TIME}_{lin}$  algorithm. □

**Corollary 4.2.** For each  $\varphi \in \text{FO}$ ,  $\text{ENUM}(\varphi, \text{BIJ}, <_{\text{lex}}) \in \text{CONSTANT-DELAY}_{lin}$ .

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procedure JTH.precomp( $\varphi, S$ )
  compute  $card(\varphi(S))$ 
  if  $\varphi$  is a formula  $\delta$  of arity 1 then
    compute  $\delta(S)$  ;  $\Delta \leftarrow \text{sort } \delta(S)$ 
  if  $\varphi \equiv \psi(\bar{x}) \wedge y = \tau(\bar{x})$  then
    JTH.precomp( $\psi, S$ )
  if  $\varphi \equiv \psi(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{i < m} y \neq \tau_i$  then
    JTH.precomp( $\delta, S$ ) ; JTH.precomp( $\psi, S$ )

procedure JTH.output( $\varphi, S, j$ )
  if  $j \geq card(\varphi(S))$  then return error
  else
    if  $\varphi$  is a formula  $\delta$  of arity 1 then return  $\Delta[j]$ 
    if  $\varphi \equiv \psi(\bar{x}) \wedge y = \tau(\bar{x})$  then
       $\bar{a} \leftarrow \text{JTH.output}(\psi, S, j)$  ; return  $(\bar{a}, \tau(\bar{a}))$ 
    if  $\varphi \equiv \psi(\bar{x}) \wedge \delta(y) \wedge \bigwedge_{i < m} y \neq \tau_i$  then
       $\beta \leftarrow card(\delta(S)) - m$ 
       $\bar{a} \leftarrow \text{JTH.output}(\psi, S, j \text{ div } \beta)$ 
       $\Theta \leftarrow \text{sort } \{\tau_1(\bar{a}), \dots, \tau_m(\bar{a})\}$ 
       $i \leftarrow 0$  ;  $s \leftarrow j \bmod \beta$  ;
      while  $\Theta[i] \leq \Delta[s]$  and  $i < m$  do  $\{i++, s++\}$ 
      return  $(\bar{a}, \Delta[s])$ 

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*Proof.* Derives from Corollary 3.6, Lemma 2.5 and Proposition 4.1.  $\square$

**Remark 4.3.** This corollary improves Theorem 7 of [DG07] which states a similar result for FO, but for some unspecified linear order  $<$ .

We are not able to prove that, for each  $\varphi \in \text{FO}$ , problem  $\text{J}^{th}(\varphi, \text{BIJ}, <_{\text{lex}})$  belongs to the class  $\text{CONSTANT-TIME}_{lin}$  and we conjecture it does not. However, a similar result holds for a less canonical linear order that depends on the input formula  $\varphi$ .

**Definition 4.4.** Let  $\varphi \equiv \bigvee_{i=1}^s \varphi_i$  be an exclusive disjunction of inductive descriptions  $\varphi_i$  of signature  $\sigma$  and let  $S$  be a  $\sigma$ -structure. Let  $<_{\text{lex}}^{\varphi}$  denote the "lexicographical" order on  $\varphi(S)$  defined as follows: for any  $\bar{a}, \bar{b} \in \varphi(S)$ :  $\bar{a} <_{\text{lex}}^{\varphi} \bar{b}$ , if

- either  $\bar{a} \in \varphi_i(S)$  and  $\bar{b} \in \varphi_j(S)$  for some  $i < j \leq s$ ,
- or  $\bar{a}, \bar{b} \in \varphi_i(S)$  for some  $i \leq s$  and  $\bar{a} <_{\text{lex}} \bar{b}$ .

**Corollary 4.5.** Let  $\varphi \in \text{YID}$ . Then  $\text{J}^{th}(\varphi, \text{BIJ}, <_{\text{lex}}^{\varphi}) \in \text{CONSTANT-DELAY}_{lin}$ .

*Proof.* For any exclusive disjunction of inductive descriptions  $\varphi \equiv \bigvee_{i=1}^s \varphi_i$  and any  $S \in \text{BIJ}$ , it holds:  $\text{jth}(\varphi, S, <_{\text{lex}}^{\varphi}, j) = \text{jth}(\varphi_i, S, <_{\text{lex}}, k)$ , where  $i$  is the smallest index such that  $j < \sum_{u=1}^i card(\varphi_u(S))$  and where  $k = j - \sum_{u=1}^{i-1} card(\varphi_u(S))$ . Together

with the algorithm JTH for  $J^{th}(\text{ID}, \text{BIJ}, <_{\text{lex}})$  described above (see page 15), this remark justifies the following algorithm  $\mathcal{A}$  for  $J^{th}(\varphi, \text{BIJ}, <_{\text{lex}}^{\varphi})$ :

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procedure  $\mathcal{A}.\text{precomp}(\varphi, S)$ 
  write  $\varphi$  under the form  $\bigvee_{i=1}^s \varphi_i$ , with each  $\varphi_i \in \text{ID}$ 
  for  $i = 1$  to  $s$  do
     $\text{JTH.precomp}(\varphi_i, S)$ 
    compute  $v_i = \sum_{j=1}^i \text{card}(\varphi_j(S))$ 
procedure  $\mathcal{A}.\text{output}(\varphi, S, j)$ 
  if  $j \geq v_s$  then return error
  else
    Find  $i \leq s$  such that  $v_{i-1} \leq j < v_i$ 
    return  $\text{JTH.output}(\varphi_i, S, j - v_{i-1})$ 

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It is easily seen that  $\mathcal{A}$  runs in  $\text{CONSTANT-TIME}_{\text{lin}}$ . □

**Corollary 4.6.** *For each  $\varphi \in \text{FO}$ ,  $\text{RANDOM}(\varphi, \text{BIJ}) \in \text{CONSTANT-TIME}_{\text{lin}}$ .*

*Proof.* Immediate, from Corollary 3.6, Corollary 4.5 and Lemma 2.4. □

## 5. COMPLEXITY RESULTS ON STRUCTURES OF BOUNDED DEGREE

Let  $S$  be a structure of domain  $D$  over a relational signature  $\sigma$ . We say that the *degree* of  $S$  is *bounded by  $d$*  if, for each symbol  $R \in \sigma$ , each  $a \in D$  belongs to at most  $d$  tuples of  $R$ . We denote by  $\text{DEG}_d$  the class of relational structures of degree bounded by  $d$ . The complexity results we obtain for our query problems on bijective structures immediately imply the same results on structures of bounded degree because of the following lemma of [DG07]:

**Lemma 5.1** (Durand, Grandjean). *Let  $d > 0$  be a fixed integer. For each  $(\varphi, S) \in \text{FO} \times \text{DEG}_d$ , there exists  $(\varphi', S') \in \text{FO} \times \text{BIJ}$  such that  $\varphi(S) = \varphi'(S')$ . Moreover, the size  $|\varphi'|$  only depends on  $|\varphi|$  and  $d$ , and  $(\varphi', S')$  is computable from  $(\varphi, d, S)$  in time  $O(|S|)$ , that means, in time  $\leq f(|\varphi|, d) \cdot |S|$ , for some function  $f$ .*

*Proof.* See [DG07], Proposition 10 and Lemma 11. □

**Corollary 5.2.** *For each  $d > 0$  and each  $\varphi \in \text{FO}$ , we have:*

- (1)  $\text{ENUM}(\varphi, \text{DEG}_d, <_{\text{lex}}) \in \text{CONSTANT-DELAY}_{\text{lin}}$ ;
- (2)  $J^{th}(\varphi, \text{DEG}_d, <_{\text{lex}}^{\varphi, d}) \in \text{CONSTANT-TIME}_{\text{lin}}$ , for some linear order  $<_{\text{lex}}^{\varphi, d}$ ;
- (3)  $\text{RANDOM}(\varphi, \text{DEG}_d) \in \text{CONSTANT-TIME}_{\text{lin}}$ .

Let  $H = (V_H, E_H)$  be a fixed graph over the set  $V_H = \{1, \dots, k\}$ . For any graph  $G = (V_G, E_G)$ , the  *$H$ -subgraphs* of  $G$  are the tuple  $(x_1, \dots, x_k) \in V_G^k$  such that  $\forall i, j \in V_H: ij \in E_H \rightarrow x_i x_j \in E_G$ . Similarly, the  *$H$ -induced subgraphs* of  $G$  are the tuples  $(x_1, \dots, x_k) \in V_G^k$  such that  $ij \in E_H \leftrightarrow x_i x_j \in E_G$ . The results above can be applied to prove that the  $H$ -(induced) subgraphs of any graph  $G$



of degree  $\leq d$  can be enumerated with constant delay. More precisely, denote by  $\text{GRAPH}_d$  the class of graphs of degree bounded by  $d$ , and by  $\text{SUBGRAPH}_d^H$  the problem relation (see p. 4) whose input space is  $\text{GRAPH}_d$  and whose set of solutions on a given input  $G$  is the set of  $H$ -subgraphs of  $G$ . Define similarly the counterpart problem  $\text{IND-SUBGRAPH}_d^H$  related to induced subgraphs. Given  $H, G \in \text{GRAPH}_d$ , the set of tuples  $\text{SUBGRAPH}_d^H(G)$  and  $\text{IND-SUBGRAPH}_d^H(G)$  are easily expressed by the respective first-order formulas:

$$\varphi_H(x_1, \dots, x_k) \equiv \text{PWD}(x_1, \dots, x_k) \wedge \bigwedge_{ij \in E_H} E(x_i, x_j) \quad \text{and}$$

$$\psi_H(x_1, \dots, x_k) \equiv \text{PWD}(x_1, \dots, x_k) \wedge \bigwedge_{ij \in E_H} E(x_i, x_j) \wedge \bigwedge_{ij \notin E_H} \neg E(x_i, x_j),$$

where  $k = \text{card}(V_H)$ . Therefore, the results of Corollary 5.2 apply to the enumeration,  $j^{\text{th}}$  solution and random solution problems associated with the problem relations  $\text{SUBGRAPH}_d^H$  and  $\text{IND-SUBGRAPH}_d^H$ .

**Corollary 5.3.** *For each  $d > 0$  and each graph  $H$  of degree  $\leq d$ , we have:*

- (1)  $\text{ENUM}(\text{SUBGRAPH}_d^H, <_{lex}) \in \text{CONSTANT-DELAY}_{lin}$ ,
- (2)  $j^{\text{th}}(\text{SUBGRAPH}_d^H, <^{H,d}) \in \text{CONSTANT-TIME}_{lin}$ , for some linear order  $<^{H,d}$ ,
- (3)  $\text{RANDOM}(\text{SUBGRAPH}_d^H) \in \text{CONSTANT-TIME}_{lin}$ ,

and the same results hold for  $\text{IND-SUBGRAPH}_d^H$ .

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