

Combinatorics of Boolean automata circuits dynamics^{*}

Jacques Demongeot^{1,4}, Mathilde Noual^{2,4**}, and Sylvain Sené^{3,4}

¹ Université Joseph Fourier – Grenoble 1, AGIM, CNRS FRE 3405, 38706 La Tronche, France

² Université de Lyon, ÉNS-Lyon, LIP, CNRS UMR 5668, 69007 Lyon, France

³ Université d'Évry – Val d'Essonne, IBISC, ÉA 4526, 91000 Évry, France

⁴ IXXI, Institut rhône-alpin des systèmes complexes, 69007 Lyon, France

Abstract. In line with fields of theoretical computer science and biology that study Boolean automata networks to model regulation networks, we present some results concerning the dynamics of networks whose underlying structures are oriented cycles, that is, *Boolean automata circuits*. In the context of biological regulation, former studies have highlighted the importance of circuits on the asymptotic dynamical behaviour of the biological networks that contain them. Our work focuses on the number of attractors of Boolean automata circuits whose elements are updated in parallel. In particular, we give the exact value of the total number of attractors of a circuit of arbitrary size n as well as, for every positive integer p , the number of its attractors of period p depending on whether the circuit has an even or an odd number of inhibitions. As a consequence, we obtain that both numbers depend only on the parity of the number of inhibitions and not on their distribution along the circuit. We also relate the counting of attractors of Boolean automata circuits to other known combinatorial problems and give intuition about how circuits interact by studying their dynamics when they intersect one another in one point.

Keywords: Discrete dynamical system, Boolean automata network, positive and negative circuit, asymptotic behaviour, attractor.

1 Introduction

This article is set in the general framework of complex dynamical systems and, more precisely, in that of regulation networks modeled by means of discrete mathematical tools. Since [1] proposed *threshold Boolean automata networks* to represent formally neural networks, and, later, [2] and [3] introduced the first models of genetic regulation networks, many other studies based on the same

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** Corresponding author: mathilde.noual@ens-lyon.fr, Laboratoire d'informatique du parallélisme de l'ÉNS Lyon, 46 allée d'Italie, 69364 Lyon Cedex 07, France, tel : +33-(0)4-26-23-38-04, fax: +33-(0)4-26-23-38-20.

or different formalisms were carried out to determine theoretical properties of such networks [4,5,6,7,8,9,10,11,12,13,14,15,16]. One of the main motivations of many of them was to better understand those emergent dynamical behaviours that networks display and that cannot be explained or predicted by a simple analysis of the local interactions existing between the components of the networks. In particular, later works by [17] and [18] yielded conjectures and gave rise to problematics that are still relevant in the field of regulation networks beyond the particular definition of the models one may choose to use. For instance, Thomas highlighted the importance of specific patterns, namely circuits, on the dynamics of discrete regulation networks and Kauffman gave an approximation of the number of different possible asymptotic behaviours of Boolean networks.

Thus, from the point of view of theoretical biology but also from that of discrete mathematics and theoretical computer science, it is relevant to address the question of the number of attractors in the dynamics of a network. Close to the 16th Hilbert problem concerning the number of limit cycles of dynamical systems [19], this question has already been considered in some works [20,21,22,15,16]. Driven by a similar will to understand the dynamical properties of (regulation) networks modeled by Boolean automata networks, we have decided to first focus our attention on a simple instance of Boolean automata networks, that is, Boolean automata circuits (which also happen to be a simple instance of threshold Boolean automata networks [1]). The reason for this choice is that circuits are known to play an important part in the dynamics of a network that contains them. One way to see this is to note that a network whose underlying interaction graph has no circuits can only eventually end up in a configuration that will never change over time. A network with retroactive loops, on the contrary, will exhibit more diverse dynamical behaviour patterns. [17] noted the importance of underlying circuits in networks and formulated conjectures concerning the role of positive (*i.e.*, with an even number of inhibitions) and negative (*i.e.*, with an odd number of inhibitions) circuits in the dynamics of regulation networks. Besides the fact that they are known to be decisive patterns for the dynamics of arbitrary biological networks, circuits are also relevant because they may be regarded specifically as internal layers of feedforward networks⁵. Identifying the dynamics of circuits is thus a first step in the process of understanding and formalising the dynamics of such networks which are known to model the architecture of many biological systems [23,24,25,26].

In this paper, we give a new combinatorial characterisation of the asymptotic dynamical behaviour of Boolean automata circuits evolving synchronously (*i.e.*, at each time step, every node executes its transition function). Because our study of Boolean automata circuits takes its meaning when these particular networks are related to more general networks, Section 2 introduces the general context of Boolean automata networks first and then defines Boolean automata circuits. Section 3 then deals with the dynamics of positive and negative circuits updated synchronously. For both types of circuits, we obtain the exact values of the

⁵ Feedforward networks are networks whose structure can be represented by a layered graph with no feedback loops between layers.

total number of attractors of these circuits and of their number of attractors of period p for every positive integer p . These values happen to be terms of integer sequences defined by different combinatoric problems that are isomorph to the problem of counting attractors of a circuit. Section 4 mentions this. The last section, Section 5, makes a first step towards understanding how circuits may interact together by studying the dynamics of circuits that intersect in one point. The conclusion discusses perspectives of this work.

2 Definitions and notations

Boolean automata networks

A Boolean automata network updated synchronously is a couple $N = (G, F)$. $G = (V, A)$ is a digraph called the *interaction graph* of the network. Its set of nodes $V = \{0, \dots, n-1\}$ is assimilated to the set of automata of the network. N is said to be of size n if $|V| = n$. In this case, Boolean vectors $x \in \{0, 1\}^n$ are seen as *configurations* of N and their coefficients $x_i \in \{0, 1\}$ ($i \in V$) correspond to the states of the automata of N . F is the *global transition function* of the network defined by a set of *local transition functions* $\{f_i : \{0, 1\}^n \rightarrow \{0, 1\} \mid i \in V\}$ such that:

$$\forall x \in \{0, 1\}^n, \forall i \in V: F(x)_i = f_i(x). \quad (2.1)$$

When there is no ambiguity as to what network is being considered, given a configuration $x \in \{0, 1\}^n$, we write

$$x = x(0) = F^0(x)$$

and

$$\forall t \in \mathbb{N}, t \geq 1: x(t) = F(F^{t-1}(x)),$$

so that $x(t) = F^t(x)$ can be seen as the configuration of the network N at time step t when it started in the initial configuration $x(0) = x$. We suppose that arcs of A convey *real* interactions in the sense that

$$\forall (j, i) \in A, \exists x \in \{0, 1\}^n: f_i(x) \neq f_i(\bar{x}^j), \quad (2.2)$$

where $\forall k \neq j \in V: \bar{x}_k^j = x_k$ and $\bar{x}_j^j = \neg x_j$. In the sequel, abusing our own notations, we will consider that local transition functions f_i are actually defined on the set $\{0, 1\}^{deg^-(i)}$, where $deg^-(i) = |\{j \in V \mid (j, i) \in A\}|$. We also suppose that all local transition functions are *locally monotone*, that is, $\forall i, j \in V$, either

$$\begin{aligned} \forall x = (x_0, \dots, x_{n-1}) \in \{0, 1\}^n: \\ f_i(x_0, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n-1}) \leq f_i(x_0, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_{n-1}), \end{aligned} \quad (2.3)$$

in which case the arc (j, i) is said to be *positive*, or

$$\begin{aligned} \forall x = (x_0, \dots, x_{n-1}) \in \{0, 1\}^n: \\ f_i(x_0, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n-1}) \geq f_i(x_0, \dots, x_{j-1}, 1, x_{j+1}, \dots, x_{n-1}), \end{aligned} \quad (2.4)$$

in which case the arc (j, i) is said to be *negative*.

Because the set of global states of a network N of size n is finite, all trajectories $x(0), x(1), x(2), \dots$ necessarily end up looping, *i.e.*, $\forall x(0) \in \{0, 1\}^n$, $\exists t, p \in \mathbb{N}: x(t+p) = x(t)$. We call *attractor* the orbit of $x(t)$, *i.e.*, the finite set $\{x(t+k) \mid k \in \mathbb{N}\}$. The *period* of this attractor is its cardinal, *i.e.*, the smallest p such that for any $k \in \mathbb{N}$, $x(t+k+p) = x(t+k)$. Elements belonging to an attractor of period 1 are usually called *fixed points* and attractors of period greater than one *limit cycles*. Generally, attractors of period $p \geq 1$ are called p -attractors. The set of configurations belonging to a p -attractor of N is denoted by

$$\mathbf{S}_p(N) = \{x \mid F^p(x) = x \text{ and } \forall d < p: F^d(x) \neq x\} \subseteq \{0, 1\}^n.$$

The number of p -attractors of N is denoted by

$$\mathbf{A}_p(N) = \frac{1}{p} \cdot |\mathbf{S}_p(N)|.$$

We call *iteration graph* of a network $N = (G, F)$ of size n the digraph whose set of nodes is the set $\{0, 1\}^n$ of configurations of N and whose set of arcs is $\{(x, F(x)) \mid x \in \{0, 1\}^n\}$.

Boolean automata circuits

A *circuit* of size n is a digraph that we denote here by \mathbb{C}_n . Its set of nodes $V = \{0, \dots, n-1\}$ is assimilated to the set of elements of $\mathbb{Z}/n\mathbb{Z}$ so that, considering two nodes i and j , $i+j$ designates the node $i+j \pmod n$. The set of arcs of the circuit is $A = \{(i, i+1) \mid i \in \mathbb{Z}/n\mathbb{Z}\}$. A *Boolean automata circuit* $C = (\mathbb{C}_n, F)$ of size n is a Boolean automata network whose interaction graph is \mathbb{C}_n . Because of (2.2), its local transition functions are all equal either to the identity function $id : a \in \{0, 1\} \mapsto a$ or to the negation function $neg : a \in \{0, 1\} \mapsto \neg a$ so that (2.1), in the case of a Boolean automata circuit, is written

$$\forall x \in \{0, 1\}^n, \forall i \in \mathbb{Z}/n\mathbb{Z}: F(x)_i = f_i(x_{i-1}),$$

where we have abused notations to simplify them and considered that local transition functions are of arity 1 since nodes have in-degree 1. According to (2.3) (*resp.* (2.4)), an arc $(i-1, i)$ whose endpoint is a node i such that $f_i = id$ (*resp.* $f_i = neg$) is positive (*resp.* negative). A (Boolean automata) circuit is said to be *positive* (*resp.* *negative*) if its number of negative arcs is even (*resp.* odd). In the sequel, we make substantial use of the following functions:

$$\forall i, j \in \mathbb{Z}/n\mathbb{Z}: F[j, i] = \begin{cases} f_j \circ f_{j-1} \circ \dots \circ f_i & \text{if } i \leq j, \\ f_j \circ f_{j-1} \circ \dots \circ f_0 \circ f_{n-1} \circ \dots \circ f_i & \text{otherwise.} \end{cases}$$

There are several things to note about these functions that will be exploited later. First, because $\forall k \in V: f_k \in \{id, neg\}$, $F[j, i]$ is injective. Second, if \mathbb{C}_n is positive (*resp.* negative) then $\forall j \in V: F[j, j+1] = id$ (*resp.* $F[j, j+1] = neg$).

Finally, $\forall t \in \mathbb{N}, \forall p = k \cdot n + d \in \mathbb{N}$ s.t. $d < n$, the following equation holds and conveys more concretely how the dynamics of a Boolean automata network takes place:

$$\begin{aligned}
\forall i \in \mathbb{Z}/n\mathbb{Z}: \\
x_i(t+p) &= f_i(x_{i-1}(t+p-1)) = f_i \circ f_{i-1}(x_{i-2}(t+p-2)) \\
&= \dots \\
&= F[i, 1]x_0(t+p-i) \\
&= \dots \\
&= F[i, i+1](x_i(t+p-n)) \\
&= \dots \\
&= F[i, i+1]^k(x_i(t+p-k \cdot n)) = F[i, i+1]^k(x_i(t+d)) \\
&= \dots \\
&= F[i, i+1]^k \circ F[i, i-d+1](x_{i-d}(t)).
\end{aligned} \tag{2.5}$$

The importance of (2.5) lies in that it relates the state of one automaton at a given time step to the state of another at a previous time step.

3 Dynamics of Boolean automata circuits

The main results of this section and of this paper are summed up in the following theorem. They characterise the dynamics of these special networks and are proven below. Figure 1 illustrates these results. It pictures three different circuits of size 4 (two positive and one negative) as well as their iteration graphs.

Theorem 1. *Let $C = (\mathbb{C}_n, F)$ be a Boolean automata circuit of size n . Then:*

- (i) *All configurations of C belong to an attractor.*
- (ii) *If C is positive, then the period of any attractor divides n .*
- (iii) *If C is positive, then the number of p -attractors for $p \in \mathbb{N}$ that divides n and the total number of attractors are given respectively by:*

$$\mathbf{A}_p^+ = \frac{1}{p} \cdot \sum_{d|p} \mu\left(\frac{p}{d}\right) \cdot 2^d \quad \text{and} \quad \mathbf{T}_n^+ = \frac{1}{n} \cdot \sum_{p|n} \psi\left(\frac{n}{p}\right) \cdot 2^p,$$

where μ is the Möbius function and ψ the Euler totient function (see below).

- (iv) *If C is negative, then the period p of any attractor divides $2n$ without dividing n , i.e., $\frac{2n}{p} \in \mathbb{N}$ is odd.*
- (v) *If C is negative, then the number of $2p$ -attractors for $2p \in \mathbb{N}$ satisfying (iv) and the total number of attractors are given respectively by:*

$$\mathbf{A}_{2p}^- = \frac{1}{2p} \cdot \sum_{\text{odd } d|p} \mu(d) \cdot 2^{\frac{p}{d}} \quad \text{and} \quad \mathbf{T}_n^- = \frac{1}{2n} \cdot \sum_{\text{odd } p|n} \psi(p) \cdot 2^{\frac{n}{p}}.$$

Thus, given the sign of C , the number of its p -attractors as well as the total number of its attractors depends neither on the number of negative arcs in C nor on their localisation.

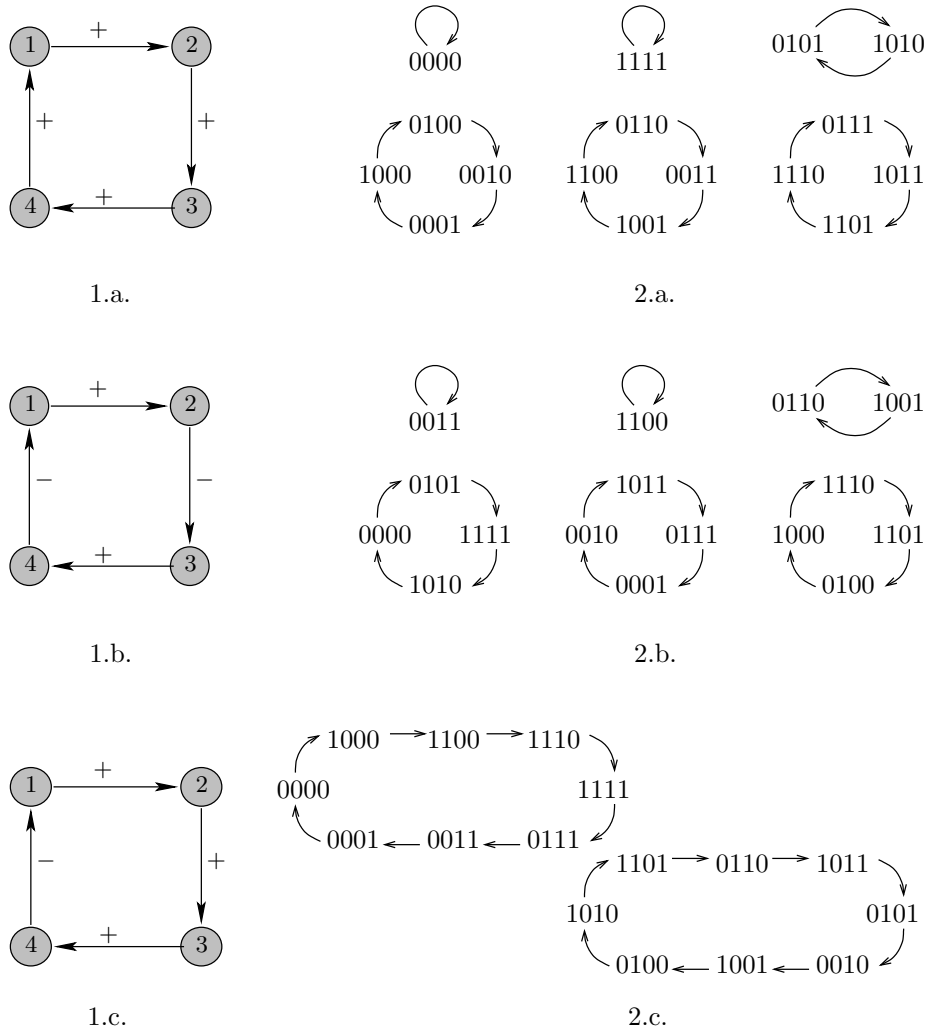


Fig. 1. Figures 1.a., 1.b. and 1.c. represent three different circuits of size $n = 4$. That of figures 1.a. and 1.b. are positive. That of figure 1.c. is negative. Figures 2.a., 2.b. and 2.c. picture respectively their iteration graphs. In all three cases here, all elements belong to an attractor. This is usually not the case with arbitrary Boolean automata networks which are not circuits.

The function ψ that appears in (iii) and (v) of Theorem 1 is the Euler totient: $\psi(n)$ counts the number of positive integers $k \leq n$ that share no positive factors other than 1 with n . The function $\mu : \mathbb{N}^* \rightarrow \{-1, 0, 1\}$ that appears in the same

sentences is the Möbius function [27]. It is defined as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not square-free,} \\ (-1)^k & \text{if } n = \prod_{i=0}^k p_i \text{ where the } p_i\text{'s} \\ & \text{are distinct positive primes.} \end{cases} \quad (3.1)$$

The importance of the μ function here lies however less in its definition than in the following property called the *Möbius inversion formula*: for any arithmetic functions F and G , it holds that [27]

$$F(n) = \sum_{d|n} G(d) \implies G(n) = \sum_{d|n} \mu(d) \cdot F\left(\frac{n}{d}\right). \quad (3.2)$$

Points (i), (ii) and (iv) of Theorem 1 above are an extension of a result presented in [28]. Let us prove them first.

Let $C = (\mathbb{C}_n, F)$ be a Boolean automata circuit of size n where F is defined by $\{f_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ and let $x(t) \in \{0, 1\}^n$ be an arbitrary configuration of C . Then, by (2.5), we have:

$$\forall i \in \mathbb{Z}/n\mathbb{Z}: x_i(t+n) = F[i, i+1](x_i(t)). \quad (3.3)$$

As a consequence, if C is positive, then $x_i(t+n) = x_i(t)$ holds so that $x(t)$ necessarily belongs to a p -attractor of C where p divides n . If C is negative, (3.3) is equivalent to $x_i(t+n) = \neg x_i(t)$ so that $x_i(t+2n) = x_i(t)$ holds and thus, $x(t)$ belongs to a p -attractor of C where p divides $2n$. In this latter case, suppose that p divides $n = q \cdot p$ as well. Then, $\forall i \in \mathbb{Z}/n\mathbb{Z}: x_i(t) = x_i(t+p) = x_i(t+q \times p) = x_i(t+n) = \neg x_i(t)$, which is a contradiction. Thus, if C is negative, then the period p of any configuration must divide $2n$ without dividing n . This means that p must be even and that $n = q \cdot \frac{p}{2}$ for some odd $q \in \mathbb{N}$. This proves points (i), (ii) and (iv) of Theorem 1. Let us add here that when C is negative, if $x(t) \in \mathbb{S}_p(C)$ where $p = 2n/q$ is even and $q = 2q'+1$ is odd, then $x(t+n) = x(t+q' \times p + \frac{p}{2}) = x(t+\frac{p}{2})$ belongs to the same attractor as $x(t)$ so that $x(t+n) \in \mathbb{S}_p(C)$ as well. As a consequence, it holds that:

$$\forall i \in \mathbb{Z}/n\mathbb{Z}: x_i(t) = \neg x_i(t+n) = \neg x_i(t + \frac{p}{2}). \quad (3.4)$$

In the sequel, we use the expression *possible attractor period* of a Boolean automata circuit C to refer to any integer that satisfies point (ii) or (iv) (according to the sign of C) of Theorem 1. Note that when the circuit is negative, from point (iv) we can derive that the only possible attractor period that is larger than the size of the circuit is $2n$, all other possible periods are strictly smaller than the size of the circuit.

Now, before proving the rest of Theorem 1, let us characterise configurations belonging to p -attractors of a Boolean automata circuit C of size n . Supposing that the composition of all local transition functions of C equals $F[0, 1] = S$, we use the following property \mathcal{P}_C defined on $\mathbb{N} \times \{0, 1\}^n$:

$$\begin{aligned} \forall p = k \cdot n + p' \in \mathbb{N} \text{ s.t. } p \equiv p' \pmod{n}, \forall x \in \{0, 1\}^n : \\ \mathcal{P}_C(p, x) \iff \forall i \in \mathbb{Z}/n\mathbb{Z}: x_i = S^k \circ F[i, i - p' + 1](x_{i-p'}). \end{aligned}$$

Informally, when all arcs of the circuit are positive (*i.e.*, all f_i s equal id), $\mathcal{P}_C(p, x)$ being satisfied means that the state x_i of any automata i of the circuit equals the state x_j of the automata j of the circuit that can be found by “counting p arcs back from i ”. The general case is similar except that one must compose all local transition functions f_i along the way (from i back to j) and apply the resulting function to x_j .

Under the same definitions of p, p' and k as above, by induction on $q \in \mathbb{N}$ defined by $i = q \cdot p' + r \equiv r \pmod{p'}$ ($i \in \mathbb{Z}/n\mathbb{Z}$), one can show that $\mathcal{P}_C(p, x)$ relates x_i and x_r as follows:

$$\begin{aligned} \mathcal{P}_C(p, x) \iff \\ \forall i \in \mathbb{Z}/n\mathbb{Z} \text{ s.t. } i = q \cdot p' + r \equiv r \pmod{p'}: x_i = S^{q \cdot k} \circ F[i, r + 1](x_r). \end{aligned} \quad (3.5)$$

Lemma 1. *For any possible attractor period $p \in \mathbb{N}$ of C :*

$$\mathbf{S}_p(C) = \{x \mid \mathcal{P}_C(p, x) \text{ and } \forall 0 < p' < p: \neg \mathcal{P}_C(p', x)\} \neq \emptyset. \quad (3.6)$$

Proof. Equation (2.5) implies that $\mathcal{P}_C(p, x(t)) \iff x(t) = x(t + p)$ holds. This yields the first equality of Lemma 1. Now, let us prove that for all *possible* attractor periods $p \in \mathbb{N}$, $\mathbf{S}_p(C) \neq \emptyset$. For any divisor k of the size n of C , let us define the configuration \tilde{x}^k of C as follows:

$$\begin{aligned} \tilde{x}_0^k &= 1 \text{ and} \\ \tilde{x}_i^k &= \begin{cases} F[i, 1](0) & \text{if } i \not\equiv 0 \pmod{k}, \\ F[i, 1](1) & \text{otherwise.} \end{cases} \end{aligned}$$

For any integer $d \leq n$ such that $\mathcal{P}_C(d, \tilde{x}^k)$ is satisfied, by (3.5), it holds that $\tilde{x}_d^k = F[d, 1](\tilde{x}_0^k) = F[d, 1](1)$. By the definition of \tilde{x}^k , this, in turn, implies that $d \equiv 0 \pmod{k}$. Now, suppose that p is a possible attractor period of C . First, suppose as well that C is positive so that p divides n . It can be checked that in this case, the configuration \tilde{x}^p of C satisfies $\mathcal{P}_C(p, \tilde{x}^p)$. In addition, there is no $0 < d < p$ such that $\mathcal{P}_C(d, \tilde{x}^p)$ is true (since $\mathcal{P}_C(d, \tilde{x}^p) \implies d \equiv 0 \pmod{p}$). Consequently, by the first part of (3.6), $\tilde{x}^p \in \mathbf{S}_p(C) \neq \emptyset$. Suppose now that C is negative. Then, the integer $m = p/2$ divides n and it may be checked that

$\mathcal{P}_C(p, \tilde{x}^m)$ is true. In addition, the only integer $0 < d < p$ possibly satisfying $\mathcal{P}_C(d, \tilde{x}^m)$ is $d = m$. But $\mathcal{P}_C(m, \tilde{x}^m)$ implies that $\tilde{x}^m = \tilde{x}^m(0) = \tilde{x}^m(m)$ and this is impossible by (3.4). Thus, we obtain that $\mathcal{P}_C(d, \tilde{x}^m)$ is satisfied for no $0 < d < p$ and $\tilde{x}^p \in \mathbf{S}_p(C) \neq \emptyset$. \square

Any *possible* attractor period p of a circuit C is thus effectively an attractor period of C . Possible attractor periods will now simply be called *attractor periods*.

Now, let $C = (\mathbb{C}_n, F)$ and $C' = (\mathbb{C}_m, H)$ be two Boolean automata circuits of same signs such that the size m of C' divides the size $n = m \cdot q$ of C and such that F and H are defined respectively by the sets $\{f_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ and $\{h_i \mid i \in \mathbb{Z}/m\mathbb{Z}\}$. Let $S = F[0, 1] = H[0, 1]$. In the case where both circuits are negative, we suppose that q is odd. This way, whatever the signs of the circuits are, any attractor period p of C' can be shown to be an attractor period of C . We are going to compare the dynamics of C and C' . More precisely, we are going to show that for integers p that are attractor periods of both circuits, the sets $\mathbf{S}_p(C)$ and $\mathbf{S}_p(C')$ are isomorphic. To do this, the idea is roughly to make C “mimic” or “bisimulate” the dynamical behaviour of C' in the following sense. If C' and C start respectively in the initial configurations $x(0)$ and $y(0)$, then, all along their trajectories, $(x(t))_{t \in \mathbb{N}}$ and $(y(t))_{t \in \mathbb{N}}$, C maintains constantly its node 0 in the same state as that of node 0 of C' (i.e., $\forall t \in \mathbb{N}: x_0(t) = y_0(t)$). In order for this to be possible, all other nodes must satisfy some precise relationships.

Formally, we define the injective function $\mathcal{Q}_H^F : \{0, 1\}^m \rightarrow \{0, 1\}^n$ that maps configurations of C' to configurations of C such that:

$$\forall i \in \mathbb{Z}/n\mathbb{Z}, i = k \cdot m + r \equiv r \pmod{m} : \\ \mathcal{Q}_H^F(x)_i = S^{q-1-k} \circ F[0, i+1] \circ H[0, r+1](x_r). \quad (3.7)$$

Note that because of all the hypotheses we have made, the equality $\mathcal{Q}_H^F(x)_0 = x_0$ holds indeed for any configuration $x \in \{0, 1\}^m$, and Lemma 2 follows:

Lemma 2. *The map \mathcal{Q}_H^F satisfies the following:*

- (i) $\mathcal{Q}_H^F \circ H = F \circ \mathcal{Q}_H^F$.
- (ii) $\forall x(t) \in \{0, 1\}^m, y(t) = \mathcal{Q}_H^F(x(t)), \forall k \in \mathbb{N}:$
 $x(t) = x(t+k) \iff y(t) = y(t+k).$
- (iii) $\forall x \in \{0, 1\}^m: x \in \mathbf{S}_p(C') \iff \mathcal{Q}_H^F(x) \in \mathbf{S}_p(C).$

Proof. (i) Let $x(t) \in \{0, 1\}^m, y(t) = \mathcal{Q}_H^F(x(t))$ and $z(t+1) = \mathcal{Q}_H^F(x(t+1))$. Let us show that $y(t+1) = z(t+1)$. Let $i = k \cdot m + r \in \mathbb{Z}/n\mathbb{Z}$ be such that $i \equiv r \pmod{m}$. There are three different cases to consider. First, suppose that

$r > 0$. Then:

$$\begin{aligned}
y_i(t+1) &= f_i(y_{i-1}(t)) \\
&= f_i \circ S^{q-1-k} \circ F[0, i] \circ H[0, r](x_{r-1}(t)) \\
&= f_i \circ S^{q-1-k} \circ F[0, i+1] \circ f_i \circ H[0, r+1] \circ h_r(x_{r-1}(t)) \\
&= S^{q-1-k} \circ F[0, i+1] \circ H[0, r+1](x_r(t+1)) \\
&= z_i(t+1).
\end{aligned}$$

Second, suppose that $r = 0$ and that $k > 0$. Then:

$$\begin{aligned}
y_{k \cdot m}(t+1) &= f_{k \cdot m}(y_{k \cdot m-1}(t)) \\
&= f_{k \cdot m} \circ S^{q-1-(k-1)} \circ F[0, k \cdot m] \circ H[0, 0](x_{m-1}(t)) \\
&= S^{q-1-k} \circ F[0, k \cdot m+1] \circ S \circ h_0(x_{m-1}(t)) \\
&= S^{q-1-k} \circ F[0, k \cdot m+1] \circ H[0, 1](x_0(t+1)) \\
&= z_{k \cdot m}(t+1).
\end{aligned}$$

Finally, suppose that $i = 0$. Then:

$$\begin{aligned}
y_0(t+1) &= f_0(y_{(q-1) \cdot m+m-1}(t)) \\
&= f_0 \circ S^0 \circ F[0, 0] \circ H[0, 0](x_{m-1}(t)) \\
&= h_0(x_{m-1}(t)) \\
&= x_0(t+1) \\
&= z_0(t+1).
\end{aligned}$$

Thus, $\forall i \in \mathbb{Z}/n\mathbb{Z}$: $y_i(t+1) = z_i(t+1)$.

(ii) follows from the injectivity of \mathcal{Q}_H^F and from (i).

(iii) follows from (ii). □

Note that from the injectivity of \mathcal{Q}_H^F and from Lemma 2 (iii) follows directly that for any attractor period of C' :

$$p \times \mathbf{A}_p(C') = |\mathbf{S}_p(C')| \leq |\mathbf{S}_p(C)| = p \times \mathbf{A}_p(C). \quad (3.8)$$

Now, let us define the new function $\mathcal{Q}_F^H : \{0, 1\}^n \rightarrow \{0, 1\}^m$ that maps configurations of C to configurations of C' such that:

$$\forall r \in \mathbb{Z}/m\mathbb{Z}: \mathcal{Q}_F^H(y)_r = F[0, r+1] \circ H[0, r+1](y_r). \quad (3.9)$$

The map \mathcal{Q}_F^H satisfies the following lemma:

Lemma 3. *If $y \in \mathbf{S}_p(C)$ for a certain attractor period p of C and of C' , then $y = \mathcal{Q}_H^F(\mathcal{Q}_F^H(y))$.*

Proof. Let $y \in \{0, 1\}^n$ be an arbitrary configuration of C . Then, $\forall i = k \cdot m + r \in \mathbb{Z}/n\mathbb{Z}$, the following holds:

$$\begin{aligned}
& \mathcal{Q}_H^F(\mathcal{Q}_F^H(y))_i \\
&= S^{q-1-k} \circ F[0, i+1] \circ H[0, r+1](\mathcal{Q}_F^H(y)_r) \\
&= S^{q-1-k} \circ F[0, i+1] \circ H[0, r+1] \circ F[0, r+1] \circ H[0, r+1](y_r) \\
&= S^{q-1-k} \circ F[i, r+1](y_r) \\
&= \begin{cases} F[i, r+1](y_r) & \text{if } S = id \text{ or } k \text{ is even,} \\ \neg F[i, r+1](y_r) & \text{if } S = neg \text{ and } k \text{ is odd.} \end{cases} \tag{3.10}
\end{aligned}$$

Now, suppose that p is an attractor period of C' and of C and that $y \in \mathbf{S}_p(C)$. Let $a = r \bmod p$. By (3.5), since $\mathcal{P}_C(p, y)$ is true, $y_r = F[r, a+1](y_a)$, and, equivalently $y_a = F[r, a+1](y_r)$. If both circuits are positive or if both are negative and k is even, it can be verified that $i \bmod p = a$. Thus, $\mathcal{P}_C(p, y)$ and (3.5) imply that

$$\begin{aligned}
y_i &= F[i, a+1](y_a) \\
&= F[i, a+1] \circ F[r, a+1](y_r) \\
&= F[i, r+1](y_r) \\
&= \mathcal{Q}_H^F(\mathcal{Q}_F^H(y))_i.
\end{aligned}$$

If both circuits are negative and k is odd, then (2.5) and (3.4) yield $y_i = \neg F[i, i - \frac{p}{2} + 1](y_{i - \frac{p}{2}})$. Since p is an attractor period of C' , there is an odd integer q' such that $m = q' \cdot \frac{p}{2}$. Thus, $i = k \cdot m + r = k \cdot q' \cdot \frac{p}{2} + r$ and $i - \frac{p}{2} = (k \cdot q' - 1) \cdot \frac{p}{2} + r$. Because k and q' are both odd, $k \cdot q' - 1$ is even and $i - \frac{p}{2} \bmod p = r \bmod p = a$. Consequently $\mathcal{P}_C(p, y)$ and (3.5) imply that

$$\begin{aligned}
y_i &= \neg F[i, i - \frac{p}{2} + 1](y_{i - \frac{p}{2}}) \\
&= \neg F[i, i - \frac{p}{2} + 1] \circ F[i - \frac{p}{2}, a+1](y_a) \\
&= \neg F[i, a+1] \circ F[r, a+1](y_r) \\
&= \neg F[i, r+1](y_r) \\
&= \mathcal{Q}_H^F(\mathcal{Q}_F^H(y))_i.
\end{aligned}$$

As a result, in all cases, $y_i = \mathcal{Q}_H^F(\mathcal{Q}_F^H(y))_i$. □

Now by Lemma 2, if P is the set of attractor periods of C' , then the map

$$\mathcal{Q}_H^F: \{0, 1\}^m = \bigcup_{p \in P} \mathbf{S}_p(C') \rightarrow \bigcup_{p \in P} \mathbf{S}_p(C)$$

is a bijection whose inverse is $\mathcal{Q}_F^H: \bigcup_{p \in P} \mathbf{S}_p(C) \rightarrow \{0, 1\}^m$. As a consequence, the following holds for any $p \in P$:

$$\mathbf{A}_p(C') = \mathbf{A}_p(C) = \mathbf{A}_p^s, \tag{3.11}$$

where s is the sign of both circuits ($s = -$ if C and C' are both negative and $s = +$ if they are both positive) and \mathbf{A}_p^s is the number of p -attractors of any

Boolean automata circuit of sign s that has $p \in \mathbb{N}$ as an attractor period. The first two examples of circuits in Figure 1 are an illustration of (3.11) in which $n = m = 4$ and $s = +$. Equation (3.11) yields the following result:

Lemma 4. *For any integer n that can be written $n = 2^e m$ for some $e \in \mathbb{N}$ and odd $m \in \mathbb{N}$:*

$$2^n = \sum_{p|n} \mathbf{A}_p^+ \times p \quad \text{and} \quad 2^n = \sum_{\substack{\text{odd } q|n}} \mathbf{A}_{\frac{2n}{q}}^- \times \frac{2n}{q} = \sum_{d|m} \mathbf{A}_{2^{e+1}d}^- \times 2^{e+1}d.$$

Proof. Point (i) of Theorem 1 implies that

$$\{0, 1\}^n = \bigcup_{\substack{p \text{ attractor} \\ \text{period of } C}} \mathbf{S}_p(C)$$

and by (3.11), if p is an attractor period of C of sign $s \in \{+, -\}$, then $|\mathbf{S}_p(C)| = \mathbf{A}_p^s \times p$. Moreover, as mentioned above, any attractor period p of a negative circuit of size n divides $2n$ without dividing n . As a result, it satisfies $n = q \cdot p/2$ for some odd divisor q of n and can be written $p = \frac{2n}{q} = 2^{e+1}d$ for some divisor $d = \frac{m}{q}$ of m . \square

Now, the first part of Theorem 1 (iii) follows directly by applying the Möbius inversion formula ((3.2) on page 7) to the first equality in Lemma 4 (taking $F(n) = 2^n$ and $G(d) = \mathbf{A}_d^+ \times d$). Similarly, the Möbius inversion formula can be applied to the second expression of Lemma 4 taking $F(m) = 2^{2^e m}$ and $G(d) = \mathbf{A}_{2^{e+1}d}^- \times 2^{e+1}d$. This yields:

$$\mathbf{A}_{2n}^- \times 2n = \sum_{d|m} \mu(d) \cdot 2^{n/d},$$

which in turn yields the first part of Theorem 1 (v).

The second parts of Theorem 1 (iii) and (v) come from the fact that $\psi(m) = \sum_{r|m} (m/r) \cdot \mu(r)$. We show how below in the positive case (the negative case can be dealt with similarly):

$$\begin{aligned} \mathbf{T}_n^+ &= \sum_{p|n} \mathbf{A}_p^+ = \sum_{p|n} \sum_{d|p} \frac{1}{p} \cdot \mu\left(\frac{p}{d}\right) \cdot 2^d \\ &= \frac{1}{n} \cdot \sum_{p|n} \sum_{d|p} 2^d \cdot \frac{n}{p} \cdot \mu\left(\frac{p}{d}\right) = \frac{1}{n} \cdot \sum_{p|n} \sum_{d|p} 2^d \cdot \frac{n}{(p/d) \cdot d} \cdot \mu\left(\frac{p}{d}\right) \\ &= \frac{1}{n} \cdot \sum_{d|n} 2^d \sum_{k|n/d} \frac{n}{k \cdot d} \cdot \mu(k) = \frac{1}{n} \cdot \sum_{d|n} \psi\left(\frac{n}{d}\right) \cdot 2^d. \end{aligned}$$

$p \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	21	22
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	-	1	-	1	-	1	-	1	-	1	-	1	-	1
3	-	-	2	-	-	2	-	-	2	-	-	2	2	-
4	-	-	-	3	-	-	-	3	-	-	-	3	-	-
5	-	-	-	-	6	-	-	-	-	6	-	-	-	-
6	-	-	-	-	-	9	-	-	-	-	-	9	-	-
7	-	-	-	-	-	-	18	-	-	-	-	-	18	-
8	-	-	-	-	-	-	-	30	-	-	-	-	-	-
9	-	-	-	-	-	-	-	-	56	-	-	-	-	-
10	-	-	-	-	-	-	-	-	-	99	-	-	-	-
11	-	-	-	-	-	-	-	-	-	-	186	-	-	186
12	-	-	-	-	-	-	-	-	-	-	-	335	-	-
21	-	-	-	-	-	-	-	-	-	-	-	-	99858	-
22	-	-	-	-	-	-	-	-	-	-	-	-	-	190557
T_n^+	2	3	4	6	8	14	20	36	60	108	188	352	99880	190746

a.

$p \backslash n$	1	2	3	4	5	6	7	8	15	16	17	18	21	22
2	1	-	1	-	1	-	1	-	1	-	1	-	1	-
4	-	1	-	-	-	1	-	-	-	-	-	1	-	1
6	-	-	1	-	-	-	-	-	1	-	-	-	1	-
8	-	-	-	2	-	-	-	-	-	-	-	-	-	-
10	-	-	-	-	3	-	-	-	3	-	-	-	-	-
12	-	-	-	-	-	5	-	-	-	-	-	5	-	-
14	-	-	-	-	-	-	9	-	-	-	-	-	9	-
16	-	-	-	-	-	-	-	16	-	-	-	-	-	-
30	-	-	-	-	-	-	-	-	1091	-	-	-	-	-
32	-	-	-	-	-	-	-	-	-	2048	-	-	-	-
34	-	-	-	-	-	-	-	-	-	-	3855	-	-	-
36	-	-	-	-	-	-	-	-	-	-	-	7280	-	-
42	-	-	-	-	-	-	-	-	-	-	-	-	49929	-
44	-	-	-	-	-	-	-	-	-	-	-	-	-	95325
T_n^-	1	1	2	2	4	6	10	16	1096	2048	3856	7286	49940	95326

b.

Table 1. Number of p -attractors of positive (a.) and negative (b.) Boolean automata circuits of size n (the number in cell (p, n) is $A_p(C)$ where C is a Boolean automata circuit of size n).

We performed computer simulations of the dynamical behaviour of Boolean automata circuits of different sizes that confirmed Theorem 1. Table 1 shows some of the results we obtained with these simulations. Notice that in these tables, as (3.11) predicts, all numbers appearing on one line are identical. In particular, the first line of Table 1.a. indicates that all positive circuits have two fixed points (whereas negative circuits have none). Indeed, 1 is a divisor of all $n \in \mathbb{N}$ and the only positive Boolean automata circuit of size 1 has trivially two fixed points. Thus, so do every other positive Boolean automata circuits by (3.11). This recalls the results of [10] that characterised positive circuits by this property and from which the authors derived that *arbitrary* networks containing only negative circuits have no fixed points. Other particular cases of couples (p, n) may be pointed out. When $n = 2^k$, for instance, since 1 is the only odd divisor of n , $A_{2n}^- = T_n^- = 2^{n-k-1}$ (see cells (16, 8) and (32, 8) of Table 1.b.). Also, if n is prime then, because $\mu(n) = -1$, we have:

$$T_n^+ = 2 + A_n^+ \quad \text{and} \quad A_n^+ = \frac{1}{n} \cdot (\mu(n) \cdot 2 + \mu(1) \cdot 2^n) = \frac{2^n - 2}{n}.$$

4 Related problems

Let us first note, in this section, that [13] also study the dynamics of isolated circuits updated in parallel. However, rather than focussing on the states x_i of each automaton i of the network, in the current configuration x , the authors focus on the automata that *call for a change* in x , that is, the automata i satisfying $f_i(x_{i-1}) \neq x_i$. This results in a different description of the iteration graphs of isolated circuits that decomposes the set of configurations into attractors corresponding to a given, fixed number of automata that call for a change.

Now, thanks to the results of the previous section, it is possible to define and concentrate on one *canonical* circuit of each sign and size. Indeed, since A_p^s does not depend on the distribution nor on the number of negative arcs in the circuit as long as this number has the required parity, it is possible to choose a circuit $C_n^s = (C_n, F^s)$ as the representative of all circuits of sign s and size n . Then, from the previous section, it can be derived that there exists a permutation $\sigma : \{0, 1\}^n \rightarrow \{0, 1\}^n$ of configurations ($\sigma = \mathcal{Q}_{F^s}^F$) such that every transition $(x, F^s(x))$ of C_n^s is mapped bijectively to the transition $(\sigma(x), \sigma(F^s(x)))$ of C making the iteration graphs of both circuits isomorph. Choosing a canonical *positive* circuit is straightforward: C_n^+ can be the circuit of size n that has no negative arcs, *i.e.*, F^+ is defined by the set of local transition functions $\{f_i^+ = id \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ and acts as a rotation of the components of vectors in $\{0, 1\}^n$:

$$\forall (x_0, \dots, x_{n-1}) \in \{0, 1\}^n : F^+(x_0, \dots, x_{n-1}) = (x_{n-1}, x_0, \dots, x_{n-2}). \quad (4.1)$$

Choosing a canonical *negative* circuit is less obvious because a circuit with only negative arcs is negative only if its size is odd. The choice of this circuit must therefore depend on the use we want to make of it. Generally, we simply choose as canonical negative circuit a circuit that has one unique negative arc.

Now, sequences $(A_n^+)_{n \in \mathbb{N}}$, $(T_n^+)_{n \in \mathbb{N}}$, $(A_{2n}^-)_{n \in \mathbb{N}}$ and $(T_n^-)_{n \in \mathbb{N}}$ defined in Theorem 1 happen to correspond precisely and respectively to the integer sequences A1037, A31, A48 and A16 of the OEIS [29] in which these sequences are defined by different combinatorial problems. Defining canonical Boolean automata circuits allowed us to study how some of them relate to the problem of counting the number of attractors (of given period or in total) of Boolean automata circuits. In particular we focused on those problems related to *binary necklaces* and *Lyndon words* [30,31,32]⁶, to binary shift register sequences [33,34] and to cycles in a digraph under $x^2 \bmod q$ where $q = 2^{n+1} - 1$ is a Mersenne prime [35]. The relationships found with these problems provided interesting different ways of formalising the dynamics of Boolean automata circuits updated synchronously. For instance, the work presented in [33,34] allowed us to formalise the dynamics of positive circuits in terms of the action of a permutation group on the set of global states and corroborated the formulas for T_n^+ and T_n^- using the Burnside Lemma [36]. Drawing inspiration from [35], we also derived a new expression for $S_p(C_n^+)$:

$$S_p(C_n^+) = \{x \in G \mid \text{ord}(x) = d \text{ and } p = \text{ord}_d(2)\}$$

where $d = \text{ord}(x) = \min\{k \mid x \cdot k \equiv 0 \bmod (2^n - 1)\}$ is the order of x in the cyclic additive group $G = \mathbb{Z}/(2^n - 1)\mathbb{Z}$ and $\text{ord}_d(2)$ is the order of 2 in the group $(\mathbb{Z}/d\mathbb{Z})^*$. And since $\psi(d)$ counts the number of elements of order d in $\mathbb{Z}/(2^n - 1)\mathbb{Z}$, we thus obtain another way of explaining the presence of the Euler totient function in the formula of T_n^+ .

5 Dynamics of intersected Boolean automata circuits

Now, in order to gain some intuition about how intersected circuits work, in this section, we give some additional results concerning the dynamics of one simple instance of such networks: networks whose interaction graphs are what we will call *double circuits*, namely, graphs composed of two circuits that share a node. We do not give any proofs here⁷ as they use techniques that are very similar to that used for the case of simple circuits in Section 3 and because the idea is not to detail another very particular case but just to make one first step towards an understanding of how circuits interact together. Our hope is that the intuition we gain this way on the additional complexity of the dynamics that is induced by a simple composition of two circuits will lead us later to be able to analyse the dynamics of arbitrary networks directly without needing to focus on all possible simple patterns they may contain.

⁶ A *binary necklace* of length n is a circular string of length n on the alphabet $\{0, 1\}$. Two strings that are equal up to a rotation are considered as the same necklace (compare with (4.1)). Binary necklaces of length n are in bijection with the attractors of a positive Boolean automata circuit of size n . *Lyndon words* are aperiodic binary necklaces. Lyndon words of size p are in bijection with the p -attractors of positive Boolean automata circuits for which p is a possible attractor period.

⁷ They can however be found in the appendix of this document.

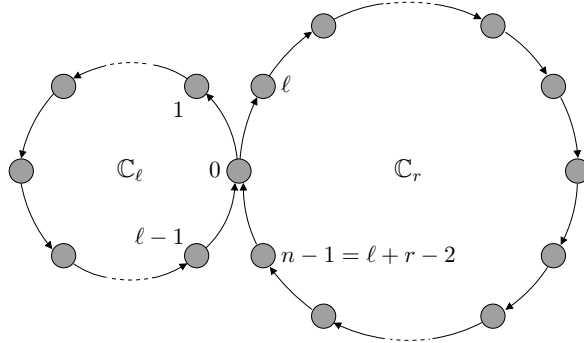


Fig. 2. Double circuit $\mathbb{D}_{\ell,r}$.

A double circuit of *left-size* $\ell \in \mathbb{N}$ and of *right-size* $r \in \mathbb{N}$ is a graph that we denote by $\mathbb{D}_{\ell,r}$. It has $\ell + r - 1$ nodes and is composed of two subgraphs which are both circuits, called the *side-circuits* of $\mathbb{D}_{\ell,r}$, of respective sizes ℓ and r that intersect in node 0. Node 0 is the only node with in- and out-degree 2. All other nodes have in- and out-degree 1. The *left-circuit* and *right-circuit* of $\mathbb{D}_{\ell,r}$ are respectively isomorphic to \mathbb{C}_ℓ and \mathbb{C}_r (see Section 2).

Networks $D = (\mathbb{D}_{\ell,r}, F)$ whose interaction graphs are double circuits will be referred to as *double Boolean automata circuits* or DBACs for short. In order for all local transition functions to be locally monotone, the function f_0 is defined as suggested in the following definition of F :

$$\forall x \in \{0, 1\}^{\ell+r-1}, \quad \begin{cases} F(x)_i = f_i(x_{i-1}), \quad \forall i \notin \{0, \ell\}, \\ F(x)_0 = f_0(x_{\ell-1}, x_{\ell+r-2}) = f_0^L(x_{\ell-1}) \star f_0^R(x_{\ell+r-2}), \\ F(x)_\ell = f_\ell(x_0) \end{cases} \quad (5.1)$$

where $\star \in \{\wedge, \vee\}$ and all the local transition functions f_i , $i \neq 0$ as well as f_0^L and f_0^R equal either *id* or *neg*.

It can be shown again that the number and the distribution of negative arcs along each side-circuit do not impact on the combinatorics of the dynamics of DBACs. The same goes for the local transition function of the intersection. The parameters that do impact are the sizes and signs of each side-circuit:

Lemma 5. *Let $D = (\mathbb{D}_{\ell,r}, F)$ be a DBAC. Independently of whether \star is replaced by \wedge or \vee in the definition of f_0 (see (5.1) above) the dynamics of D is isomorphic to the dynamics of any other DBAC with the same left and right sizes and the same left and right signs.*

One consequence of Lemma 5 is to allow us to focus on *canonical* DBACs in the proofs of the following results. Lemma 6 below lists some results describing possible attractor periods of DBACs with arbitrary side signs and sizes:

Lemma 6. *Let $D = (\mathbb{D}_{\ell,r}, F)$ be a DBAC and $p \in \mathbb{N}$ the period of an attractor of D . Then:*

- (i) *p divides the sizes of the positive side-circuits of D (if it has any).*
- (ii) *Unless $p = 1$, p does not divide the size of any negative side-circuit of D (if it has any).*
- (iii) *If the left and right signs are equal, p divides the sum $\ell + r$.*
- (iv) *Unless $p = \ell + r$ (which is only possible if both side circuits are negative), p is no greater than the size of the largest side-circuit of D .*

Going further than points (i) and (ii) above, we can show that:

Proposition 1. *A DBAC has as many fixed points as it has positive side-circuits.*

In addition, point (i) of Lemma 6 happens to be a characterisation of the possible attractor periods of doubly positive DBACs from which we derive the last result of this section:

Proposition 2. *Let $D = (\mathbb{D}_{\ell,r}, F)$ be a DBAC with positive left and right signs. For any $p \in \mathbb{N}$, the number of p -attractors of D equals A_p^+ if p divides $\gcd(\ell, r)$ and 0 otherwise. In other words, D behaves asymptotically as an isolated positive Boolean automata circuit of size $\gcd(\ell, r)$.*

Some additional information on the dynamics of DBACs can be derived from the previous results. First, it can be shown that, in terms of combinatorics, a DBAC $D = (\mathbb{D}_{\ell,r}, F)$ whose side-circuits have identical signs and sizes behaves as an isolated Boolean automata circuit of size $\ell = r$ and same sign as the side-circuits (see Table 2 (b) for the case of doubly negative DBACs). On the other hand, if both side circuits of D have same sizes but different signs then, obviously (by Lemma 6 (i) and (ii) and Proposition 1), the only attractor of D is a fixed point.

Also, a DBAC $D = (\mathbb{D}_{\ell,r}, F)$ has no more *recurrent configurations* (i.e. configurations belonging to attractors) than an isolated circuit of size that of its largest side-circuit. Indeed, if $\ell \geq r$, any recurrent configuration $x(t)$ satisfies $\forall 0 < i < r$, $x_{(\ell-1)+i}(t) = x_0(t-i) = x_i(t)$. Consequently, any recurrent configuration can be completely defined by the sole knowledge of the states, in this configuration, of automata belonging to the left-circuit. Further, comparing Lemma 6 (i) and (iii) and the results obtained in Section 3, we find that D does not have larger *attractor periods* than an isolated circuit of size that of its largest side-circuit. As for the *number* of attractors, Proposition 2 gives a precise answer in the case of doubly positive DBACs. In both other cases, characterisations of configurations belonging to p -attractors have also been derived (they involve properties that are similar to property \mathcal{P}_C defined in Section 3) but are significantly more intricate than those obtained for the case of isolated circuits. These characterisations do not appear here because they do not yield additional insight on the dynamical behaviours of interacting circuits. Thus, we now conclude the actual study with

$\ell \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	2	3	3	5	5	8	10	15	19	31	41	64
2	1	1	2	3	3	4	5	8	10	14	19	31	41	63
3	1	2	1	3	3	6	5	8	8	15	19	33	41	64
4	1	1	2	1	3	4	5	11	10	14	19	24	41	63
5	1	2	2	3	1	5	5	8	10	26	19	31	41	64
6	1	1	1	3	3	1	5	8	8	14	19	63	41	63
7	1	2	2	3	3	5	1	8	10	15	19	31	41	158
8	1	1	2	1	3	4	5	1	10	14	19	24	41	63
9	1	2	1	3	3	6	5	8	1	15	19	33	41	64
10	1	1	2	3	1	4	5	8	10	1	19	31	41	63
11	1	2	2	3	3	5	5	8	10	15	1	31	41	64
12	1	1	1	1	3	1	5	11	8	14	19	1	41	63
13	1	2	2	3	3	5	5	8	10	15	19	31	1	64
14	1	1	2	3	3	4	1	8	10	14	19	31	41	1

a.

$\ell \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	–	–	–	–	–	–	–	–	–	–	–	–	–
2	1	1	–	–	–	–	–	–	–	–	–	–	–	–
3	1	1	2	–	–	–	–	–	–	–	–	–	–	–
4	1	2	1	2	–	–	–	–	–	–	–	–	–	–
5	2	1	2	2	4	–	–	–	–	–	–	–	–	–
6	1	1	3	3	2	6	–	–	–	–	–	–	–	–
7	2	2	3	2	4	3	10	–	–	–	–	–	–	–
8	2	3	2	8	3	4	6	16	–	–	–	–	–	–
9	3	2	2	3	5	9	7	7	30	–	–	–	–	–
10	2	4	3	4	17	7	7	10	11	52	–	–	–	–
11	4	3	5	6	7	7	11	11	16	19	94	–	–	–
12	3	4	9	2	7	42	11	33	17	23	28	172	–	–
13	5	6	7	7	11	11	16	19	24	28	39	46	316	–
14	6	7	7	10	11	17	105	23	28	38	46	60	75	586

b.

Table 2. Total number of attractors of a DBAC $D = (\mathbb{D}_{\ell,r}, F)$ where (a.) the left-circuit is negative and the right-circuit is positive and (b.) both side-circuits are negative.

observations on the results of the simulations we performed which are reported in Table 2. From these simulations of the dynamical behaviours of DBACs we draw that as our characterisations of recurrent configurations strongly suggest,

the total number of attractors of a DBAC is never greater (and often significantly smaller) than that of its largest side-circuit when it is isolated. Note that in Table 2 there are many regularities⁸. However, although these regularities are very intriguing, we leave the problem of determining the number of attractors of DBACs containing negative side-circuits as an open problem. As we mentioned earlier, a more indepth study of these particular networks would lead us, we believe, too far away from our original problem which is to understand the dynamics of arbitrary Boolean automata networks.

6 Conclusion

In this article, we have described exhaustively the combinatorics of Boolean automata circuits dynamics and we have given some initiatory insight on the behaviour of such networks when they interact with one another. From these results several questions arise. The first and most obvious of them is now that we know the dynamics of circuits *synchronously* updated, how do these results translate into other update schedules. This question has been solved in [37] in the case of update schedules defined as integer ordered partitions as in [?]. It seems relevant, however, to go further and focus on more general update schedules.

Also, besides its obvious need for an extension towards more general update schedules, our work of course calls for an extension towards more general networks such as networks that are acyclic except in some places where they contain a circuit, feedforward networks and eventually small-world networks. From Section 5, we may draw the conclusion that circuits that find themselves closely linked in a network probably interact in a way that unallows considering them as isolated when attempting to estimate the number of attractors of the whole network as it has been done in the past.

Further, we project to compare our work with other related studies and the results they produced. For instance, in [2], [10] and in [38], experimental and theoretical results suggest and prove that the networks in question have only very little different asymptotic dynamical behaviours ($\mathcal{O}(\sqrt{n})$ in the case of connectivity 2 networks considered in [2] and [10], one or two in the case of the small networks studied in [38]). This seems, at first sight, to be in contradiction with the exponential number of attractors of Boolean automata circuits that we found above. It would be interesting to connect the two sources of results in order to lift the contradiction. A partial answer can however already be given. Indeed, first, it has been shown [39] that random networks only contain very little circuits. So, the total number of attractors they are responsible for remains small compared to the sizes of such networks. Second, Section 5 implies that the number and sizes of attractors seem to fall significantly as soon as circuits are no longer isolated. Going further and in the same direction as [2] and [10], we conjecture that real regulation networks only have a number of attractors that is

⁸ Notably, in the values appearing in each column of Table 2 (a.) that correspond to keeping constant the size of the positive circuit, and in the diagonals of Table 2 (b.) that correspond to keeping constant the sum of the side-circuits sizes $\ell + r$.

polynomial in the size of the network. Indeed, on the first hand, [40] emphasised (using statistical physics methods) that regulation networks are mostly small-world networks and one of the significant features of these networks is to have a high clustering coefficient which implies many interacting circuits. On the other hand, Section 5 suggests that the more there are interacting circuits in a network, the more there are constraints that hinder their dynamics and thus cut the total number of attractors down.

In any case, relating the dynamics of a network with that of its embedded circuits now seems to be a next natural and essential step towards the comprehension of the dynamics of ordinary Boolean automata networks.

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Appendix

In this appendix we give some supplementary notations concerning double Boolean automata circuits (DBACs) as well as proofs of results that appear in Section 5 of the main article.

A Additional definitions and notations

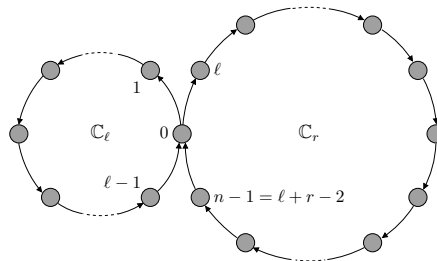


Fig. 3. Double circuit $\mathbb{D}_{l,r}$.

Given a DBAC $D = (\mathbb{D}_{\ell,r}, F)$, we call C^L and C^R the Boolean automata circuits whose interaction graphs are respectively the left and right circuits of $\mathbb{D}_{\ell,r}$ and such that the local transition function of node 0 is f_0^L in C^L and f_0^R in C^R . Given a configuration $x \in \{0, 1\}^n$ of D , we use the following notation:

$$L(x) = (x_0, \dots, x_{\ell-1}) \in \{0, 1\}^\ell \quad \text{and} \quad R(x) = (x_0, x_\ell, \dots, x_{n-1}) \in \{0, 1\}^r,$$

so that $L(x)$ is a configuration of C^L and $R(x)$ is a configuration of C^R . In the sequel, we suppose that $n = \ell + r - 1$ and we use the following notations:

$$\begin{aligned} V_L &= \{0, \dots, \ell - 1\}, \\ V_R &= \{0\} \cup \{(\ell - 1) + 1, \dots, (\ell - 1) + r - 1\}, \\ V &= \{0, \dots, n - 1\} = V_L \cup V_R. \end{aligned}$$

For any two nodes $i, j \in V_S$, $S \in \{L, R\}$ of a DBAC D that belong to the same side, we define the following function:

$$F^S[j, i] = \begin{cases} f_j \circ f_{j-1} \circ \dots \circ f_i & \text{if } j \geq i > 0, \\ f_j \circ f_{j-1} \circ \dots \circ f_0^S & \text{if } i = 0, \\ f_j \circ f_{j-1} \circ \dots \circ f_0^S \circ f_{\max\{i \in V_S\}} \circ \dots \circ f_i & \text{if } i > j. \end{cases}$$

Note that $\forall i \neq 0$: $F^S[i, i + 1]$ equals the sign of the side-circuit containing i and that $F^S[i, i] = f_i$ (this will be used later). Unless $i = j = 0$, there is no ambiguity as to which side S is being considered. In this case we will drop the superscript S and just write $F[j, i]$.

The iteration graph (see page 4) of a DBAC D is denoted by $\mathcal{I}(D)$.

B Isomorph dynamics (proof of Lemma 5)

The first step in our study of DBACs is to show Lemma 5 which states that in terms of combinatorics (*i.e.*, in terms of the number of attractors of each type), the dynamics of these networks depend only on the signs and sizes of each side circuit and not on the number of negative arcs, nor on their localisation, nor on the definition of the function f_0 (as long as it is locally monotone).

The first result of this section states that the dynamics of DBACs do not depend on the localisation nor the number of negative arcs.

lemma B1 *Let $D = (\mathbb{D}_{\ell,r}, F)$ and $D' = (\mathbb{D}_{\ell,r}, H)$ be two DBACs with the same left and right sizes, the same left and right signs and such that*

$$f_0(a, b) = f_0^L(a) \star f_0^R(b) \quad \text{and} \quad h_0(a, b) = h_0^L(a) \star h_0^R(b),$$

where \star is either \wedge in both cases or \vee in both cases. Then, their iteration graphs $\mathcal{I}(D)$ and $\mathcal{I}(D')$ are isomorphic.

Proof. We define the following bijective map from the set of states of one DBAC to that of the other:

$$\sigma: \begin{cases} \{0, 1\}^n \rightarrow \{0, 1\}^n \\ x \mapsto \sigma(x) = (\sigma_0(x_0), \sigma_1(x_1), \dots, \sigma_{n-1}(x_{n-1})) \end{cases},$$

such that:

$$\begin{aligned} \sigma_0 &= id, \\ \forall i \neq 0: \sigma_i(x_i) &= \begin{cases} x_i & \text{if } F[0, i+1] = H[0, i+1], \\ \neg x_i & \text{otherwise.} \end{cases} \end{aligned}$$

Then, we have:

$$\begin{aligned} \forall i \neq 0: H(\sigma(x))_i &= h_i(\sigma_{i-1}(x_{i-1})) \\ &= \begin{cases} h_i(x_{i-1}) & \text{if } F[0, i] = H[0, i] \\ h_i(\neg x_{i-1}) & \text{otherwise} \end{cases} \\ &= \begin{cases} f_i(x_{i-1}) & \text{if } F[0, i+1] = H[0, i+1] \\ \neg f_i(x_{i-1}) & \text{otherwise} \end{cases}, \end{aligned}$$

where the last equality holds because $F[0, i+1] = H[0, i+1] \iff (F[0, i] = H[0, i] \wedge f_i = h_i) \vee (F[0, i] \neq H[0, i] \wedge f_i \neq h_i)$, and:

$$\begin{aligned} \forall i \neq 0: \sigma(F(x))_i &= \sigma_i(f_i(x_{i-1})) \\ &= \begin{cases} f_i(x_{i-1}) & \text{if } F[0, i+1] = H[0, i+1] \\ \neg f_i(x_{i-1}) & \text{otherwise} \end{cases} \\ &= H(\sigma(x))_i. \end{aligned}$$

As for node 0, using the fact that $F^S[0, 0] = f_0^S$, $S \in \{L, R\}$, and similarly for H , we obtain that:

$$\begin{aligned} h_0^L(\sigma_{\ell-1}(x_{\ell-1})) &= \begin{cases} h_0^L(x_{\ell-1}) & \text{if } F^L[0, 0] = H^L[0, 0] \\ \neg h_0^L(x_{\ell-1}) & \text{otherwise} \end{cases} \\ &= f_0^L(x_{\ell-1}), \end{aligned}$$

and similarly:

$$h_0^R(\sigma_{n-1}(x_{n-1})) = f_0^R(x_{n-1}).$$

Thus:

$$\begin{aligned} \sigma(F(x))_0 &= \sigma_0(F(x)_0) = F(x)_0 = f_0^L(x_{\ell-1}) \star f_0^R(x_{n-1}) \\ &= h_0^L(\sigma_{\ell-1}(x_{\ell-1})) \star h_0^R(\sigma_{n-1}(x_{n-1})) = H(\sigma(x))_0, \end{aligned}$$

and, as a result:

$$\sigma(F(x)) = H(\sigma(x)).$$

□

The following result states that the dynamics of DBACs do not depend on the definition of the function f_0 .

lemma B2 *Let $D = (\mathbb{D}_{\ell,r}, F)$ and $D' = (\mathbb{D}_{\ell,r}, H)$ be two DBACs that differ only by the local transition function of node 0:*

$$\forall i \neq 0: h_i = f_i,$$

and:

$$\forall a, b \in \{0, 1\}: \begin{cases} f_0: a, b \mapsto f_0^L(a) \vee f_0^R(b), \\ h_0: a, b \mapsto f_0^L(a) \wedge f_0^R(b). \end{cases}$$

Then the iteration graphs $\mathcal{I}(D)$ and $\mathcal{I}(D')$ are isomorphic.

Proof. For any $x \in \{0, 1\}^n$, let $\neg x = (\neg x_0, \dots, \neg x_{n-1})$. Then it holds that $\neg H(x) = F(\neg x)$. Indeed:

$$\forall i \neq 0: (\neg H(x))_i = \neg H(x)_i = \neg f_i(x_{i-1}) = f_i(\neg x_{i-1}) = F(\neg x)_i,$$

and:

$$\begin{aligned} (\neg H(x))_0 &= \neg H(x)_0 \\ &= \neg[f_0^L(x_{\ell-1}) \wedge f_0^R(x_{n-1})] = \neg f_0^L(x_{\ell-1}) \vee \neg f_0^R(x_{n-1}) \\ &= f_0^L(\neg x_{\ell-1}) \vee f_0^R(\neg x_{n-1}) = F(\neg x)_0. \end{aligned}$$

□

As a consequence of Lemmas B1 and B2, we may now focus on *canonical* instances of each type of DBAC, *i.e.*, one for each assignment of the left and right signs, of the left and right sizes and of the f_0 function. Indeed, if $D = (\mathbb{D}_{\ell,r}, F)$ is a canonical DBAC then all DBACs $D' = (\mathbb{D}_{\ell,r}, H)$ with the same left and right signs will have an iteration graph $\mathcal{I}(D')$ isomorph to $\mathcal{I}(D)$. In other words, there will exist a permutation σ of $\{0, 1\}^n$ such that $\forall x \in \{0, 1\}^n: H(x) = F(\sigma(x))$. Now, we define canonical DBACs (see Figure 4) $D = (\mathbb{D}_{\ell,r}, F)$ the following way.

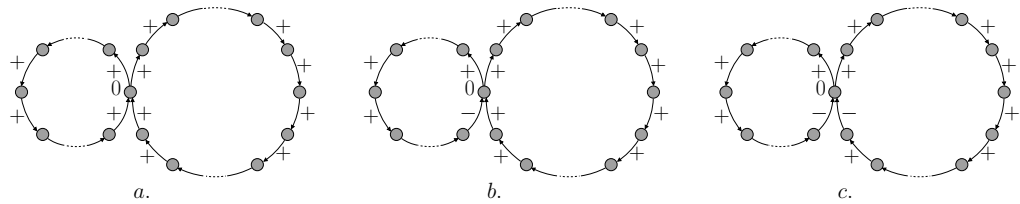


Fig. 4. Canonical double circuits $\mathbb{D}_{\ell,r}$: a. a doubly positive canonical DBAC, b. a mixed canonical DBAC and c. a doubly negative canonical DBAC.

All local transition functions of a canonical DBAC, except possibly f_0^L and f_0^R , equal the identity:

$$\forall i \neq 0: f_i = id.$$

If the left-sign (resp. right-sign) of D is positive then f_0^L (resp. f_0^R) also equals the identity. If the left-sign (resp. right-sign) of D is negative then $f_0^L = neg$ (resp. $f_0^R = neg$). In every case, $\forall a, b \in \{0, 1\}: f_0(a, b) = f_0^L(a) \vee f_0^R(b)$. Thus, from now, all proofs focus on *canonical* DBACs only.

C Possible attractor periods (proof of Lemma 6 and of Proposition 1)

We are now going to characterise what conditions an integer p must satisfy in order to be a possible attractor period of a (canonical) DBAC. First we show that almost as soon as the state of node 0 of a DBAC has started looping then, the entire DBAC starts looping:

lemma C1 *Let $D = (\mathbb{D}_{\ell, r}, F)$ be a DBAC and let $x(0) \in \{0, 1\}^n$ be a configuration of D . Then:*

$$\begin{aligned} \exists d \in \mathbb{N}, \forall t, k \in \mathbb{N}: x_0(t) &= x_0(t + k \cdot d) \\ \implies (\forall t \geq \max\{\ell, r\}, \forall k \in \mathbb{N}: x(t) &= x(t + k \cdot d)). \end{aligned}$$

Proof. It suffices to remark that:

$$\forall i \in V_L, \forall t \geq i: x_i(t + k \cdot d) = x_0(t + k \cdot d - i) = x_0(t - i) = x_i(t),$$

and similarly for nodes $i \in V_R$. □

In the proofs of the next lemmas, we will need the following equation which is true for any canonical DBAC D (remember that all local transition functions of such DBACs, except possibly f_0^L and f_0^R , equal *id*):

$$\begin{aligned} \forall x(0) \in \{0, 1\}^n, \forall t \geq \max\{\ell, r\}: \\ x_0(t) &= f_0^L(x_{\ell-1}(t-1)) \vee f_0^R(x_{n-1}(t-1)) \\ &= f_0^L(x_{\ell-k}(t-k)) \vee f_0^R(x_{n-k'}(t-k')) \quad (\text{C.1}) \\ &= f_0^L(x_0(t-\ell)) \vee f_0^R(x_0(t-r)) \end{aligned}$$

Proposition C2 (Lemma 6 (i)) *Any attractor period of a DBAC D divides the sizes of its positive side-circuits (if it has any).*

Proof. Suppose the left-circuit of $D = (\mathbb{D}_{\ell, r}, F)$ is positive. Let $x(0)$ belong to a p -attractor of D . If $p = 1$, we are done : $p|\ell$. If not, $\exists t_1 \in \mathbb{N}: x_0(t_1) = 1$ and

$\exists t_2 \in \mathbb{N}$: $x_0(t_2) = 0$ (otherwise following Lemma C1, $x(0)$ still is a fixed point).
 Suppose that $x_0(t) = 1$. Then (see (C.1)):

$$\begin{aligned} x_0(t + \ell) &= x_0(t) \vee x_{n-1}(t + \ell - 1) \\ &= 1 \vee x_{n-1}(t + \ell - 1) \\ &= 1 \\ &= x_0(t). \end{aligned}$$

Suppose now that $x_0(t) = 0$ and $x_0(t + \ell) = 1$. By induction on $k \in \mathbb{N}$, it is easy to show that this and the equality above implies that:

$$\forall k \in \mathbb{N} \text{ s.t. } k > 0: x_0(t + k \cdot \ell) = 1.$$

But then because $x(t)$ belongs to an attractor, for $k = p$, it holds that $0 = x_0(t) = x_0(t + p \cdot \ell) = x_0(t + \ell) = 1$ which is a contradiction so if $x_0(t) = 0$ then, $x_0(t + \ell) = 0 = x_0(t)$. Thus, in all cases, $x_0(t + \ell) = x_0(t)$ and by Lemma C1, $x(t + \ell) = x(t)$. As a consequence, p must divide ℓ . \square

Proposition C3 (Lemma 6 (ii) and Proposition 1) *Let $D = (\mathbb{D}_{\ell,r}, F)$ be a DBAC with negative left-sign. Then there are no p -attractors such that $p \neq 1$ and p divides ℓ and D has one unique fixed point if its right-sign is positive, otherwise it has none.*

Proof. Let $x(0)$ belong to a p -attractor of the canonical DBAC $D = (\mathbb{D}_{\ell,r}, F)$ whose left sign is negative. Suppose that p divides $\ell = q \cdot p$ and that $\exists t \in \mathbb{N}$, $\forall k \in \mathbb{N}$: $x_0(t) = 0 = x_0(t + k \cdot p)$. Then:

$$\begin{aligned} 0 &= x_0(t) \\ &= x_0(t + (k + 1) \cdot q \cdot p) \\ &= x_0(t + (k + 1) \cdot \ell) \\ &= \neg x_0(t + k \cdot \ell) \vee f_0^R(x_{n-1}(t + (k + 1) \cdot \ell - 1)) \\ &= \neg 0 \vee f_0^R(x_{n-1}(t + (k + 1) \cdot \ell - 1)) \\ &= 1, \end{aligned}$$

which is obviously a contradiction. Thus, either p does not divide ℓ , or $\forall t \in \mathbb{N}$: $x_0(t) = 1$. In the latter case, Lemma C1 implies that $x(0)$ is a fixed point. Further, by an inductive argument, we can easily show that $\forall i \neq 0$: $x_i(0) = x_i(1) = f_i(x_{i-1}(0)) = x_{i-1}(0) = 1$. Then, because $x_0(0) = 1 = \neg x_{\ell-1}(0) \vee f_0^R(x_{n-1}(0)) = \neg 1 \vee f_0^R(1) = f_0^R(1)$, we must have $f_0^R = id$, i.e., the right-sign of D must be positive. \square

Proposition C4 (Lemma 6 (iii)) *If the left and right signs of a DBAC $D = (\mathbb{D}_{\ell,r}, F)$ are equal, then any attractor period divides the sum $\ell + r$.*

Proof. Let p be an attractor period of D . If both side signs of D are positive, then by Proposition C2, p divides ℓ and r and thus divides $\ell + r$ as well. Suppose

both side-signs of D are negative. Let $x(0)$ be a configuration of D belonging to a p -attractor. Since $p \neq 1$, $\exists t \in \mathbb{N}: x_0(t) = 1$ and $\exists t \in \mathbb{N}: x_0(t) = 0$. Suppose that $x_0(t) = 0$. Then (see (C.1)):

$$\begin{aligned}
x_0(t + \ell + r) &= \neg x_0(t + r) \vee \neg x_0(t + \ell) \\
&= \neg[\neg x_0(t + r - \ell) \vee \neg x_0(t)] \vee \neg[\neg x_0(t) \vee \neg x_0(t + \ell - r)] \\
&= x_0(t) \wedge [x_0(t + r - \ell) \vee x_0(t + \ell - r)] \\
&= 0 \wedge [x_0(t + r - \ell) \vee x_0(t + \ell - r)] \\
&= 0.
\end{aligned}$$

Now suppose $x_0(t) = 1$ and $x_0(t + \ell + r) = 0$. By induction on $k, k' \in \mathbb{N}$, it can be shown that this implies:

$$\forall k, k' \in \mathbb{N} \text{ s.t. } k > 0: x_0(t + k \cdot \ell + k' \cdot r) = \begin{cases} 0 & \text{if } k + k' \text{ is even,} \\ 1 & \text{if } k + k' \text{ otherwise.} \end{cases}$$

Then: $1 = x_0(t) = x_0(t + (\ell + r) \cdot p) = x_0(t + p \cdot \ell + p \cdot r) = 0$ (since $p + p = 2 \cdot p$ is even) which is a contradiction so $x_0(t) = 1 \Rightarrow x_0(t + \ell + r) = 1 = x_0(t)$ so in all cases, $x_0(t + \ell + r) = x_0(t)$ and, by Lemma C1, $x(t + \ell + r) = x(t)$. As a consequence, p must divide $\ell + r$. \square

Proposition C5 (Lemma 6 (iv)) *If p is an attractor period of a DBAC $D = (\mathbb{D}_{\ell, r}, F)$ then either $p = \ell + r$ (which is only possible if both side circuits are negative) or p is no greater than the size of the largest side-circuit of D .*

Proof. Let p be the period of an attractor of D . If one of the side-circuits of D is positive then p divides (and thus is no greater than) its size by Proposition C2. Suppose both side-circuits are negative and $p > \ell \geq r$. By Proposition C4, $\exists k \in \mathbb{N}^*: 2 \cdot \ell \geq \ell + r = k \cdot p > k \cdot \ell$. This implies that $k = 1$. \square

Corollary C6 *The period p of an attractor of a DBAC D is no greater than the largest attractor period of an isolated circuit of size that of the largest side-circuit of D .*

D Characterisation of configurations belonging to attractors

In the sequel, we use the following notation (see Figure 5). Given a DBAC $D = (\mathbb{D}_{\ell, r}, F)$, we have:

$$\forall i \in V, \forall p \in \mathbb{N}: \text{mod}_p(i) = \begin{cases} i \bmod p & \text{if } i \in V_L, \\ (i - \ell + 1) \bmod p & \text{if } i \in V_R, \end{cases}$$

and we write:

$$\forall i \in V, \forall p \in \mathbb{N}: \begin{cases} i_p = \text{mod}_p(i) & \text{if } i \in V_L \\ i_p = \ell - 1 + \text{mod}_p(i) & \text{if } i \in V_R \end{cases} \quad (\text{D.1})$$

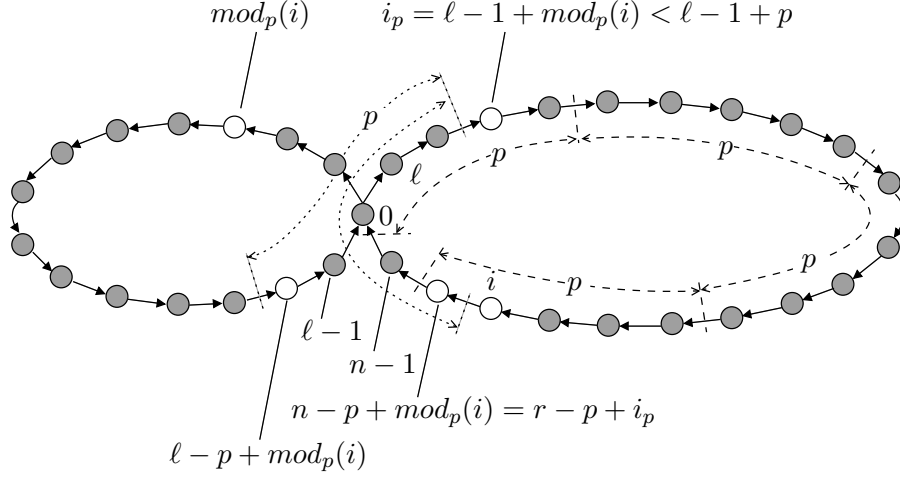


Fig. 5. Double circuit $\mathbb{D}_{\ell,r}$.

To characterise configurations belonging to attractors of DBACs, we will also use property \mathcal{P}'_D relative to a *canonical* DBAC $D = (\mathbb{D}_{\ell,r}, F)$ of size n such that $\ell \geq r$, to an integer $p \leq \ell$ and to a configuration $x \in \{0, 1\}^n$ of D :

$$\begin{aligned} \mathcal{P}'_D(p, x) &\iff \\ \left\{ \begin{array}{l} (i) \quad \forall i \in V: x_i = x_{\text{mod}_p(i)}, \\ (ii) \quad \forall i \in V \text{ s.t. } i = \text{mod}_p(i): x_i = f_0^L(x_{(\ell+i) \bmod p}) \vee f_0^R(x_{(r+i) \bmod p}). \end{array} \right. & \quad (\text{D.2}) \end{aligned}$$

In particular, (D.2) (i) above implies that $\forall i \in V$ s.t. $i \neq i_p$: $x_i = x_{i_p} = x_{i-p} = x_{\text{mod}_p(i)}$.

lemma D1 *Let $D = (\mathbb{D}_{\ell,r}, F)$ be a DBAC such that $\ell \geq r$, let $x(t)$ be a configuration of D and let $p \leq \ell$. Then:*

$$\forall k \in \mathbb{N}: x(t + k \cdot p) = x(t) \iff \mathcal{P}'_D(p, x(t)),$$

so that $\mathbf{S}_p(D) = \{x \in \{0, 1\}^n \mid (\mathcal{P}'_D(p, x)) \wedge (\forall d < p: \neg \mathcal{P}'_D(d, x))\}$.

Proof. Since the DBAC is canonical, $\forall 0 < i < j < \ell \in V_L: F[j, i] = id$, and similarly for nodes in V_R . Thus, on the first hand, (2.5) yields the following:

- 1) $i > i_p \implies x_i(t+p) = x_{i-p}(t)$ and

2) $i = i_p \implies$

$$\begin{aligned}
x_i(t+p) &= x_0(t+p - \text{mod}_p(i)) \\
&= f_0^L(x_{\ell-1}(t+p - \text{mod}_p(i) - 1)) \vee \\
&\quad f_0^R(x_{n-1}(t+p - \text{mod}_p(i) - 1)) \\
&= f_0^L(x_{\ell-p+\text{mod}_p(i)}(t)) \vee f_0^R(x_{n-p+\text{mod}_p(i)}(t))
\end{aligned}$$

Suppose that $\mathcal{P}'_D(p, x(t))$ is satisfied. Then, for all $i > i_p$, it holds that $\text{mod}_p(i) = \text{mod}_p(i-p)$ so that $x_i(t) = x_{\text{mod}_p(i)}(t) = x_{i-p}(t)$ and by 1) above, $x_i(t) = x_i(t+p)$. In addition, note that the following is true:

$$\begin{aligned}
\text{mod}_p(\ell - p + \text{mod}_p(i)) &= (\ell - p + \text{mod}_p(i)) \bmod p \\
&= (\ell + \text{mod}_p(i)) \bmod p,
\end{aligned} \tag{D.3}$$

and:

$$\begin{aligned}
\text{mod}_p(n - p + \text{mod}_p(i)) &= (n - p + \text{mod}_p(i)) \bmod p \\
&= (n + \text{mod}_p(i)) \bmod p.
\end{aligned} \tag{D.4}$$

Therefore, 2) above and $\mathcal{P}'_D(p, x(t))$ yield:

$\forall i = i_p$:

$$\begin{aligned}
x_i(t+p) &= f_0^L(x_{\text{mod}_p(\ell-p+\text{mod}_p(i))}(t)) \vee f_0^R(x_{\text{mod}_p(n-p+\text{mod}_p(i))}(t)) \\
&= f_0^L(x_{(\ell+\text{mod}_p(i)) \bmod p}(t)) \vee f_0^R(x_{(n+\text{mod}_p(i)) \bmod p}(t)) \\
&= x_{\text{mod}_p(i)}(t) \\
&= x_i(t).
\end{aligned}$$

Conversely, suppose that $x(t) = x(t+p)$. By induction, it can be shown that $\forall i > i_p$: $x_i(t) = x_{\text{mod}_p(i)}(t)$ holds. $\mathcal{P}'_D(p, x(t))$ then follows from 2) and from (D.3) and (D.4). \square

Doubly positive DBACs (proof of Proposition 2)

lemma D2 *Let $D = (\mathbb{D}_{\ell,r}, F)$ be a DBAC whose both sides are positive. Then:*

$$\mathcal{P}'_D(p, x) \Leftrightarrow \begin{cases} (i) & \mathcal{P}_{CL}(p, L(x)), \\ (ii) & \forall i \in V_R, x_i = x_{\text{mod}_p(i)}. \end{cases}$$

Proof. Equation (D.2) (i) and Lemma C2 suffice to conclude. \square

Proposition 2 then follows naturally from Lemma D2.