On covering planar graphs with a fixed number of balls

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Abstract. In this note, we prove that there exists a constant ρ such that any planar graph G of diameter $\leq 2R$ can be covered with at most ρ balls of radius R, a result conjectured by C. Gavoille, D. Peleg, A. Raspaud, and E. Sopena in 2001.

1 Introduction

Let $G = (V, E)$ be a finite connected planar graph. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path. The largest distance D in G is called the *diameter* of G. For a graph G of diameter $D \leq 2R$, the *ball* of center v and radius R consists of all vertices of G at distance at most R from v and is denoted by $B(v)$. Let $\mathcal{B}(G) = \{B(v) : v \in V\}$ be the set of all balls of radius R of the graph G. Denote by $\rho(G) = \rho(\mathcal{B}(G))$ the minimum number of balls of radius R covering G. In this note, we prove the following conjecture communicated to us by C. Gavoille in 1999 and formulated by Gavoille, Peleg, Raspaud, and Sopena in [4].

Theorem 1. *There exists a constant ρ such that any planar graph G of diameter $\leq 2R$ can be covered with at most ρ balls of radius R (i.e., $\rho(G) \leq \rho$).*

The main ingredients in the proof of Theorem 1 are the fact that the VC-dimension of $\mathcal{B}(G)$ for planar graphs G is at most 4, the recent result of J. Matoušek [6] showing that every set system with dual VC-dimension $q - 1$ has the Hadwiger-Debrunner (p, q)-property, and the result of Pach, Shahrokhi, and Szegedy [8] and Agarwal, Aronov, Pach, Pollack, and Sharir [1] that any topological graph with n vertices and without k pairwise crossing edges has $O(n \log^{2k-6} n)$ edges (the last result is used to prove that $\mathcal{B}(G)$ satisfies the $(p, 5)$-property for a certain integer $p \geq 5$).

2 Definitions and preliminary results

In this section, we present a few definitions and notations on set systems and topological graphs (see [5, 7] for additional information) and formulate the results from [6] and [1, 8] used in the proof.

Let $\mathcal{F}$ be a set system on a finite ground set V. A subset $T \subseteq V$ is called a *transversal* of $\mathcal{F}$ if $T \cap F \neq \emptyset$ for any $F \in \mathcal{F}$. The *transversal number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, is the smallest cardinality of a transversal of $\mathcal{F}$. A subfamily $\mathcal{F}' \subseteq \mathcal{F}$ is called a *covering* if $\bigcup\{F \in \mathcal{F}'\} = V$. The *covering number* of $\mathcal{F}$, denoted by $\rho(\mathcal{F})$, is the minimum number of sets of $\mathcal{F}$ required to cover the ground set X. The *dual set system* to $(V, \mathcal{F})$ is the set system $(W, \mathcal{F}^*)$ defined as follows: The ground set is $W = \{w_F : F \in \mathcal{F}\}$, where the w_F are pairwise distinct points, and for each $v \in V$ we have the set $\{w_F : F \in \mathcal{F}, v \in F\}$ in $\mathcal{F}^*$. Notice the well-known equalities $\tau(\mathcal{F}) = \rho(\mathcal{F}^*)$ and $\rho(\mathcal{F}) = \tau(\mathcal{F}^*)$. In the particular case of balls of a graph G, we have $\mathcal{B}(G) = \mathcal{B}^*(G)$. Indeed, if for a ball $B(x)$ we set $w_{B(x)} = x$, then for any $v \in V$ we obtain

$$\{w_{B(x)} : v \in B(x)\} = \{x \in V : v \in B(x)\} = \{x \in V : x \in B(v)\} = B(v).$$

The *Vapnik-Chervonenkis dimension* (or *VC-dimension* for short) of a set system $\mathcal{F}$ on a ground set V is the maximum size of a set $X \subseteq V$ that is *shattered* by $\mathcal{F}$, meaning that $\{X \cap F : F \in \mathcal{F}\} = 2^X$. It is well-known [5] that the VC-dimension of the dual of a set system of VC-dimension k is at most 2^k. Let $p \geq q \geq 2$ be integers. A family of sets $\mathcal{F}$ satisfies the (p, q)*-property* if among every p sets of $\mathcal{F}$, some q have a point in common. The key step in our proof is to ensure the conditions of the following result of J. Matoušek:

Theorem A. [6] *Let $\mathcal{F}$ be a set system such that the VC-dimension of $\mathcal{F}^*$ equals $q-1$, and let $p \geq q$. Then there exists a constant ρ such that if $\mathcal{F}$ satisfies the (p, q)-property, then $\tau(\mathcal{F}) = \rho(\mathcal{F}^*) \leq \rho$; that is, there exists a ρ-point set intersecting all sets of $\mathcal{F}$.*

The proof of this result is a generalization to set systems of the Alon and Kleitman' solution method [2] of the old piercing problem of Hadwiger and Debrunner for convex sets. Namely, [6] first establishes a fractional Helly property for sets systems of bounded VC-dimension, then LP duality and the existence of ϵ-nets for systems of bounded VC-dimension imply the result as in the Alon-Kleitman proof.

Since $\mathcal{B}(G) = \mathcal{B}^*(G)$, in order to derive Theorem 1 from Theorem A it suffices to show that the VC-dimension of $\mathcal{B}(G)$ is bounded by some constant $q-1$ and that $\mathcal{B}(G)$ satisfies the (p, q)-property for some $p \geq q$. In fact, we show that the VC-dimension of $\mathcal{B}(G)$ is at most 4. In order to establish the existence of a p such that $\mathcal{B}(G)$ satisfies the $(p, 5)$-property, we employ a result from geometrical graph theory which we recall next.

A *topological graph* is a graph drawn in the plane in such a way that its edges are represented by simple Jordan arcs so that (T1) no edge passes through any vertex other than its endpoints, (T2) no three edges cross at the same point, and (T3) no two arcs meet in more than one point (two edges *cross* each other if they have an interior point in common).

Theorem B. [8, 1] *Let $k \geq 4$ be an integer, and let Γ be a topological graph with n vertices and with no k pairwise crossing edges. Then the number of edges of Γ is $O(n \log^{2k-6} n)$.*

3 Proof of Theorem 1

Throughout this section, we suppose that the input planar graph $G = (V, E)$ of diameter $D \leq 2R$ is embedded in the plane as a topological plane graph.

Proposition 1. *The VC-dimension of $\mathcal{B}(G)$ is at most 4.*

Proof. Suppose by way of contradiction that there exists a 5-point set $A = \{a_1, \ldots, a_5\}$ of G that is shattered by $\mathcal{B}(G)$. For each pair of distinct vertices a_i, a_j of A, denote by c_{ij} the center of a ball of radius R such that $B(c_{ij}) \cap A = \{a_i, a_j\}$. Let b_{ij} be a furthest from c_{ij} vertex of G which lies simultaneously on a shortest path between c_{ij} and a_i, and on a shortest path between c_{ij} and a_j. Denote by P_{ij} an (a_i, a_j)-path consisting of a shortest path between a_i and b_{ij} and a shortest path between b_{ij} and a_j.

Claim 1. For any four distinct vertices $a_i, a_j, a_{i'}, a_{j'} \in A$, the paths P_{ij} and $P_{i'j'}$ are disjoint.

Proof of Claim 1: Suppose that P_{ij} and $P_{i'j'}$ share a common vertex, say a vertex x which belongs to the subpath of P_{ij} comprised between b_{ij} and a_i and to the subpath of $P_{i'j'}$ comprised between $b_{i'j'}$ and $a_{i'}$. Now, if $d(x, a_i) \leq d(x, a_{i'})$, then

$$\begin{aligned} d(c_{i'j'}, a_i) &\leq d(c_{i'j'}, x) + d(x, a_i) \\ &\leq d(c_{i'j'}, x) + d(x, a_{i'}) = d(c_{i'j'}, a_{i'}) \leq R, \end{aligned}$$

yielding $a_i \in B(c_{i'j'})$, which is impossible because $B(c_{i'j'}) \cap A = \{a_{i'}, a_{j'}\}$.

Claim 2. For any three distinct vertices $a_i, a_j, a_r \in A$, the intersection $P_{ij} \cap P_{ir} \cap P_{jr}$ is empty.

Proof of Claim 2: Suppose not, and let $x \in P_{ij} \cap P_{ir} \cap P_{jr}$. Now, if $d(x, a_i) \leq d(x, a_j) \leq d(x, a_r)$, then a_i belongs to the ball $B(c_{jr})$, because if, say, b_{jr} belongs to the subpath of P_{rj} comprised between x and a_j, then

$$\begin{aligned} d(c_{jr}, a_i) &\leq d(c_{jr}, b_{jr}) + d(b_{jr}, x) + d(x, a_i) \\ &\leq d(c_{jr}, b_{jr}) + d(b_{jr}, x) + d(x, a_r) = d(c_{jr}, a_r) \leq R. \end{aligned}$$

This contradiction completes the proof of the claim.

For a vertex $a_i \in A$, let $S_i := \bigcup \{P_{ij} \cap P_{ij'} : a_j, a_{j'} \in A, a_j \neq a_{j'}\}$.

Claim 3. The sets $S_1, \ldots, S_5$ are pairwise disjoint.

Proof of Claim 3: Suppose by way of contradiction that for distinct vertices a_i, a_j, the sets S_i and S_j share a common vertex x, say $x \in (P_{ir} \cap P_{ir'}) \cap (P_{jt} \cap P_{jt'})$. If $\{i, r, r'\} = \{j, t, t'\}$, say, $i = t', j = r'$, and $r = t$, then we get a contradiction with Claim 2. In all other cases, we obtain a contradiction with Claim 1. Indeed, if $j \in \{r, r'\}$, say $j = r'$, then $x \in P_{ir} \cap P_{jt}$ if $r \neq t$ and $x \in P_{ir} \cap P_{jt'}$ if $r \neq t'$ (the case $i \in \{t, t'\}$ is analogous). Finally, if $i \notin \{t, t'\}$ and $j \notin \{r, r'\}$, then, since $r \neq r'$ and $t \neq t'$, we can pick two different vertices, one in $\{r, r'\}$ and another one in $\{t, t'\}$, say $r \neq t$, yielding $x \in P_{ir} \cap P_{jt}$. This concludes the proof of Claim 3.

Thus the sets $S_1, \ldots, S_5$ induce pairwise disjoint connected subgraphs of G. Contracting each of these subgraphs to a vertex and replacing the non-contracted part of each path P_{ij} by an edge, in view of Claim 1 we will obtain the complete graph K_5 on 5 vertices. Hence, G has a K_5-minor, in contradiction with the planarity of G. $\square$

According to Theorem B, for any integer k there exists a constant ε_k such that any topological graph Γ with n vertices and with no k pairwise crossing edges has at most $\varepsilon_k n \log^{2k-6} n$ edges. Therefore, there exists an integer p such that any topological drawing of the complete graph with p vertices contains at least 7 pairwise crossing edges.

We assert that $\mathcal{B}(G)$ satisfies the $(p,5)$-property, i.e., among any set $\mathcal{B}$ of p balls of radius R of G there are some 5 that intersect. Let $C = \{c_1, \ldots, c_p\}$ be the set of centers of the balls from $\mathcal{B}$. For each pair of balls $B(c_i), B(c_j)$ denote by P_{ij} a shortest path between their centers c_i and c_j. We can select these shortest paths in such a way that every intersection $P_{ij} \cap P_{rt}$ is either empty or a path: if not, it suffices to replace every subpath of P_{rt} between two consecutive common vertices of P_{ij} and P_{rt} by the subpath of P_{ij} between the same vertices. Since $d(c_i, c_j) \leq D = 2R$, on the path P_{ij} there exists a vertex which belongs to both balls $B(c_i)$ and $B(c_j)$. Denote this vertex by b_{ij}.

Notice that if C contains 8 distinct vertices $c_{i_1}, c_{j_1}, c_{i_2}, c_{j_2}, c_{i_3}, c_{j_3}, c_{i_4}, c_{j_4}$ such that $\bigcap_{k=1}^{4} P_{i_k j_k} \neq \emptyset$, then among the 8 balls centered at these vertices there are 5 that intersect. Indeed, let $x \in \bigcap_{k=1}^{4} P_{i_k j_k}$, and assume without loss of generality that for any $k \in \{1,2,3,4\}$ the vertex $b_{i_k j_k}$ belongs to the subpath of $P_{i_k j_k}$ comprised between x and c_{j_k}. Now, if c_{i_4} is a furthest from x vertex among the centers $c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}$, then the vertex $b_{i_4 j_4}$ belongs to the balls $B(c_{i_1}), B(c_{i_2}), B(c_{i_3}), B(c_{i_4})$, and $B(c_{j_4})$. Thus, further we can assume that $\bigcap_{k=1}^{4} P_{i_k j_k} = \emptyset$ for any 8 distinct centers of C.

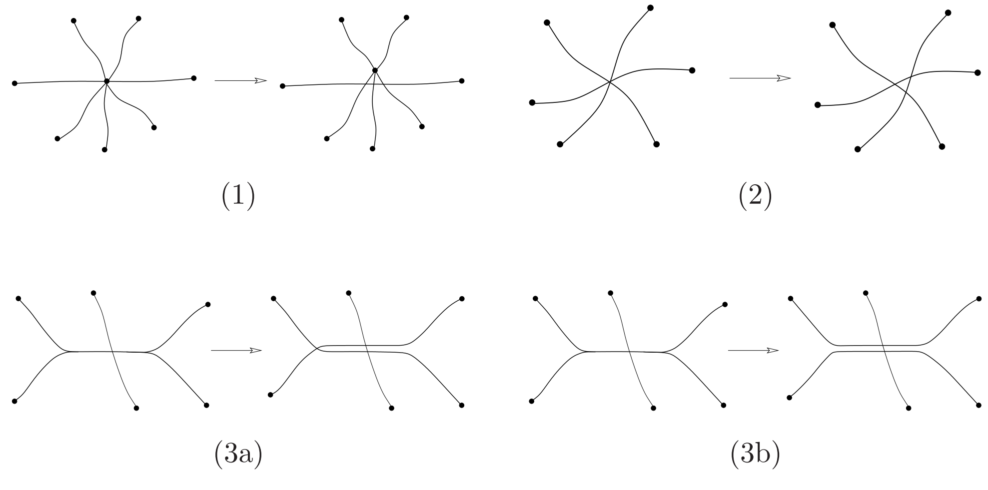

Figure 1.

Consider the drawing Γ of the complete graph on p vertices having C as the vertex set and the Jordan arcs P_{ij}, $c_i, c_j \in C$, as the set of edges. Γ is not a topological graph, however it can be transformed into a topological graph Γ^t having the same set of vertices C and such that any pair of crossing arcs $Q_{ij}, Q_{rt}, \{i,j\} \cap \{r,t\} = \emptyset$, in Γ^t corresponds to a pair of intersecting paths P_{ij}, P_{rt} in Γ. For this, we use the standard operations that are indicated in Fig. 1. To ensure the condition (T1), we use a sequence of operations type (1), (T2) is obtained by using the standard transformation (2) and the fact that no four arcs of Γ share a common point, and, finally, the condition (T3) is satisfied by employing a sequence of transformations (3a) and (3b). From Theorem B and the definition of p we conclude that the topological graph Γ^t contains 7 pairwise crossing edges, which give rise to 7 pairwise intersecting paths $P_{1\bar{1}}, \ldots P_{7\bar{7}}$ of the initial graph G. The end-vertices $c_1, c_{\bar{1}} \ldots, c_7, c_{\bar{7}}$ of these paths are centers of 14 different balls from $\mathcal{B}$.

For two paths $P_{i\bar{i}}$ and $P_{j\bar{j}}$ intersecting in a vertex x consider the pair of end-vertices c_α, c_β, $\alpha \in \{i, \bar{i}\}$ and $\beta \in \{j, \bar{j}\}$, such that the subpath of $P_{i\bar{i}}$ comprised between x and c_α minus x does not contain the vertex $b_{i\bar{i}}$ and the subpath of $P_{j\bar{j}}$ comprised between x and c_β minus x does not contain the vertex $b_{j\bar{j}}$. We call this pair of vertices $\Phi(i,j) = \{c_\alpha, c_\beta\}$ a *Φ-pair* for the paths $P_{i\bar{i}}$ and $P_{j\bar{j}}$; see Fig. 2 for an illustration. Notice that if $x \notin \{b_{i\bar{i}}, b_{j\bar{j}}\}$ then among the four pairs $\{c_i, c_j\}, \{c_i, c_{\bar{j}}\}, \{c_{\bar{i}}, c_j\}, \{c_{\bar{i}}, c_{\bar{j}}\}$ exactly one constitutes a Φ-pair. If $x = b_{i\bar{i}}$ or $x = b_{j\bar{j}}$, then we have two Φ-pairs, and if $b_{i\bar{i}} = x = b_{j\bar{j}}$, then all four pairs are Φ-pairs. In last two cases we take as $\Phi(i,j)$ anyone of these Φ-pairs. Given $\Phi(i,j) = \{c_\alpha, c_\beta\}$ and $c_\alpha \in P_{i\bar{i}}, c_\beta \in P_{j\bar{j}}$, define the binary relation $\prec$ by setting $i \prec j$ if and only if $d(c_\alpha, x) \le d(c_\beta, x)$. Note that if $i \prec j$, then the vertex $b_{j\bar{j}}$ belongs to the ball $B(c_\alpha)$ because

$$\begin{aligned} d(c_\alpha, b_{j\bar{j}}) &\le d(c_\alpha, x) + d(x, b_{j\bar{j}}) \\ &\le d(c_\beta, x) + d(x, b_{j\bar{j}}) = d(c_j, b_{j\bar{j}}) \le R. \end{aligned}$$

For example, in Fig. 2 we have $\Phi(i,j) = \{c_i, c_{\bar{j}}\}$ and $i \prec j$, yielding $b_{j\bar{j}} \in B(c_i)$. Notice also that if $c_j \in P_{i\bar{i}}$, then $x = c_j$ and therefore $j \prec i$ holds independently of the choice of the vertex $c_{\bar{j}}$ and $b_{i\bar{i}}$ belongs to the ball $B(c_j)$.

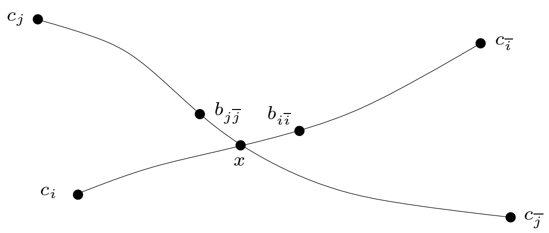

Figure 2. $\Phi(i,j) = \{c_i, c_{\bar{j}}\}$ and $i \prec j$.

Define a tournament $\mathcal{T}$ with 7 vertices $v_1, \ldots, v_7$, in which we draw an arc from vertex v_i to vertex v_j if and only if $i \prec j$. Then $\mathcal{T}$ necessarily contains a vertex with in-degree larger or equal to 3. To see this, pick a vertex v_i of $\mathcal{T}$. If its in-degree is ≥ 3, then we are done, otherwise consider the sub-tournament $\mathcal{T}'$ induced by the out-neighborhood of v_i. Since $\mathcal{T}'$ contains at least 4 vertices, it contains a vertex v_j with in-degree ≥ 2. Then the in-degree of v_j in the tournament $\mathcal{T}$ is at least 3 because v_i belongs to the in-neighborhood of v_j.

Let v_7 be a vertex of $\mathcal{T}$ whose in-degree is at least 3, and let v_1, v_2, and v_3 be the vertices having outgoing arcs to v_7. Consider the paths $P_{1\bar{1}}, P_{2\bar{2}}, P_{3\bar{3}}$, and the path $P_{7\bar{7}}$ of the graph G. Suppose without loss of generality that $c_1 \in \Phi(1,7), c_2 \in \Phi(2,7)$, and $c_3 \in \Phi(3,7)$. From the definition of the vertex $b_{7\bar{7}}$ we know that it belongs to the balls $B(c_7)$ and $B(c_{\bar{7}})$. We assert that $b_{7\bar{7}}$ also belongs to the balls $B(c_1), B(c_2)$, and $B(c_3)$. Indeed, since $1 \prec 7, 2 \prec 7$, and $3 \prec 7$, each of the vertices c_1, c_2, and c_3 is closer to or at equal distance from x and $b_{7\bar{7}}$ than its companion from the respective Φ-pair $\Phi(1,7), \Phi(2,7), \Phi(3,7)$. For example, if $\Phi(1,7) = \{c_1, c_7\}$, then $d(c_1, b_{7\bar{7}}) \leq d(c_7, b_{7\bar{7}}) \leq R$. This shows that $b_{7\bar{7}}$ belongs to 5 balls of $\mathcal{B}$, thus establishing the $(p,5)$-property, completing the proof of Theorem 1.

Remark 1. It is known that $\rho(G) = 2$ if G is outerplanar [4] or if G is a plane triangulation with all inner vertices of degree at least 6 [3]. On the other hand, [4] presents for any $k \geq 8$ a planar graph G_k of diameter $2k$ for which $\rho(G_k) \geq 4$.

Remark 2. From Theorem 8 of [4] and Theorem 1 follows that any planar graph of diameter D has an interval routing scheme with dilation at most $\lceil 1.5D \rceil$ and a constant number of intervals per edge.

Remark 3. Proposition 1 generalizes to graphs G without K_{r+1}-minors: in that case, the VC-dimension of $\mathcal{B}(G)$ is at most r.

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