Rational word functions

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April 11, 2016

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Rational Languages:

- Finite sets
- Closure under: union $\cup$, concatenation $\cdot$, Kleene star $*$

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- 1DFA
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Defined by MSO formulas:

$\varphi ::= \exists X\ \varphi \mid \exists x\ \varphi \mid \varphi \wedge \varphi \mid \neg\varphi \mid$
$x \in X \mid \sigma(x) \mid x < y$

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Algebraic recognizability:

- Congruences of finite index
- Finite Monoids

First order logic (FO)

FO formula

First order variables: $x, y, \ldots$

$$\varphi ::= \exists x\ \varphi \mid \varphi \wedge \varphi \mid \neg\varphi \mid \sigma(x) \mid x < y \mid (\varphi)$$

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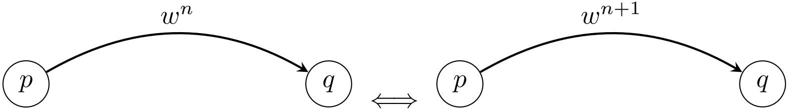
Theorem [Sch65, MP71]

Defined by an FO formula $\Leftrightarrow$ Recognized by a finite *aperiodic* automaton.

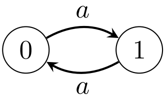
Aperiodicity

Aperiodic automaton

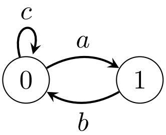
$\exists n$ integer, $\forall p, q$ states, $\forall w$ word:



Example

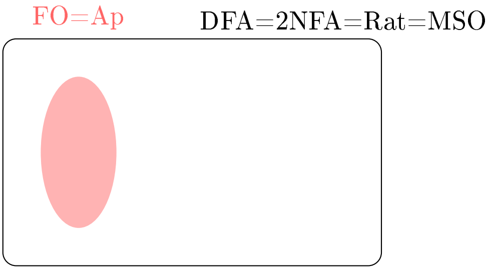


Automaton P



Automaton A

Language classes



Minimal automaton

Property

Recognized by aperiodic automaton $\Leftrightarrow$ Aperiodic minimal automaton
$\rightarrow$ decision procedure

Functions and relations

Relation

Relation: subset of $\Sigma^* \times \Sigma^*$.
Function: functional relation (this talk).

Rational relation

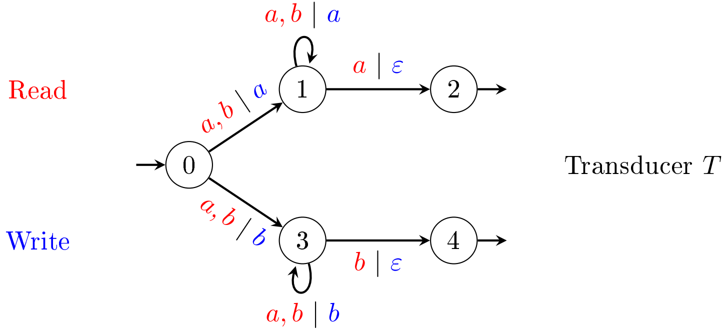
- Finite sets.
- Closed under union.
- Closed under concatenation (component wise).
- Closed under star.

Example

$\Sigma = \{a, b\}$.
$f = \left\{(w, a^{|w|})\right\} = ((a, a) + (b, a))^*$.

Transducers

Example of a transducer

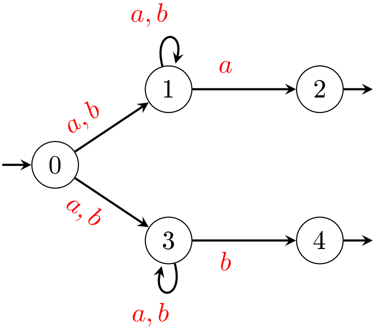


Function realized by T: $[\![T]\!] : \begin{array}{l} wa \mapsto a^{|w|} \\ wb \mapsto b^{|w|} \end{array}, w \in \Sigma^+.$

Transducers

Example of a transducer

Read



Underlying automaton of T

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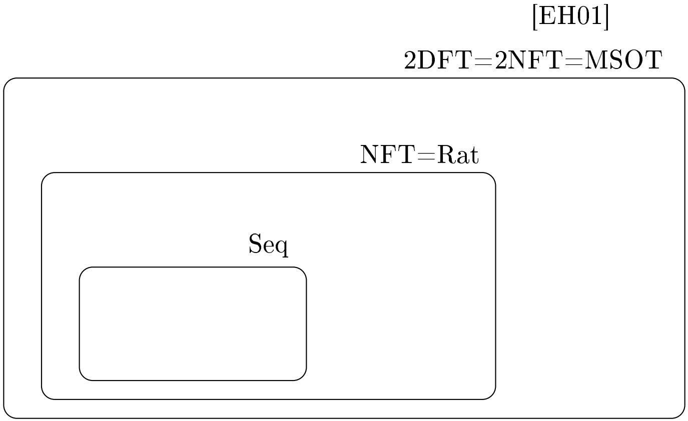
Function classes



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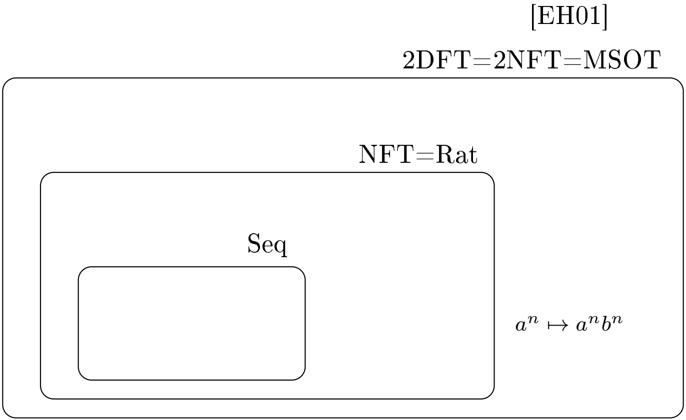

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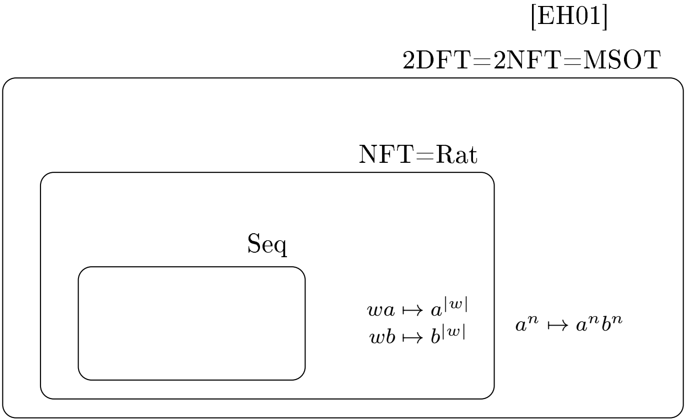

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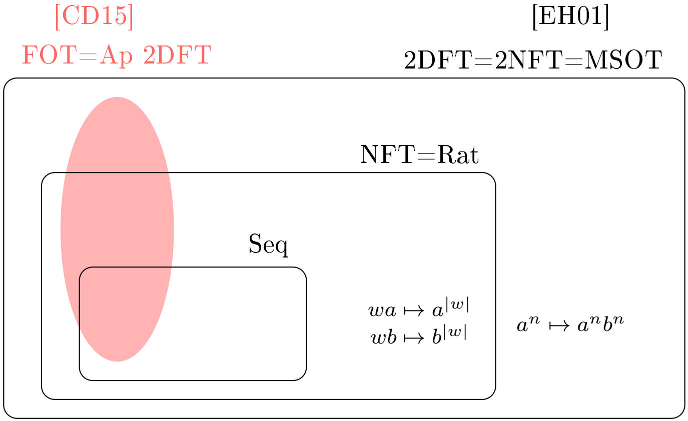


Sequential case

For sequential functions, there exists a minimal sequential transducer [Cho03].

Theorem

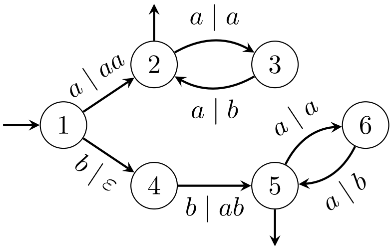
Realizable by an aperiodic sequential transducer $\Leftrightarrow$ The minimal sequential transducer is aperiodic.

Minimization

Sequential function.

Syntactic congruence

- $\widehat{f}(u) = \text{prefix}\,\{f(uw) \mid uw \in \text{dom}(f)\}$
- $u \sim_f v$ if $(\widehat{f}(u))^{-1} f(uw) = (\widehat{f}(v))^{-1} f(vw)$



Example

$[\varepsilon]$, $[b]$, $[a+bb]$, $[(a+bb)(a^2)^*]$
$[(a+bb)a(a^2)^*]$

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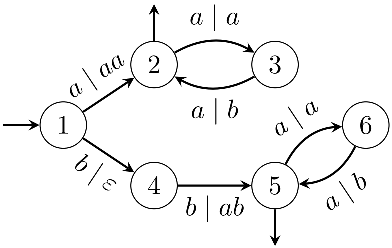

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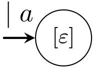

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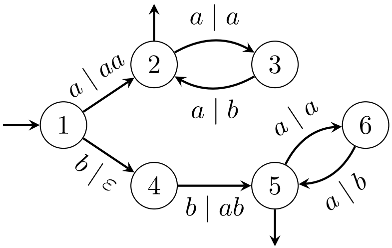

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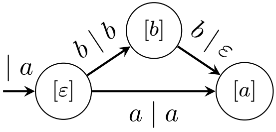

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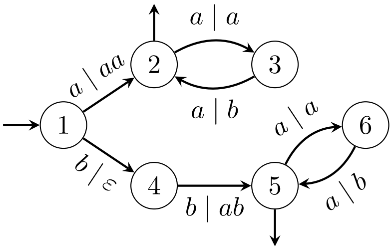

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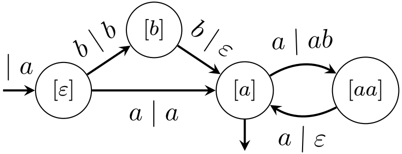


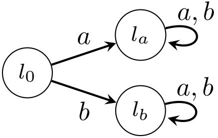
Non-deterministic case

- No minimization.

Non-deterministic case

- No minimization.
- Canonical machine [RS91], through bimachines [Sch61].

Bimachines



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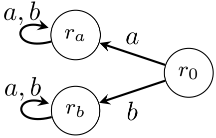


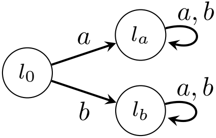
run of L

a b a a Input

run of R

Output

Bimachines



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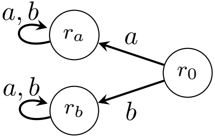

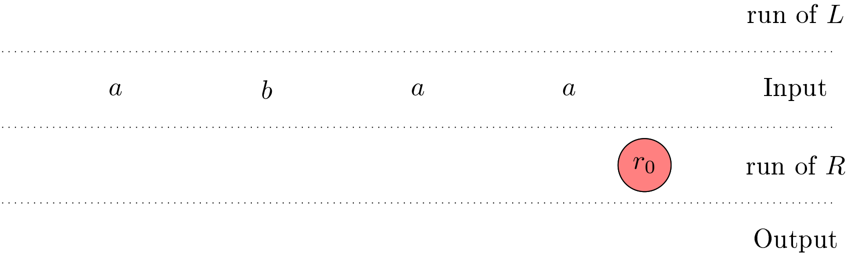

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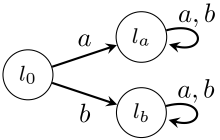

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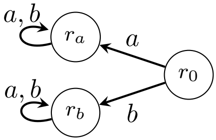

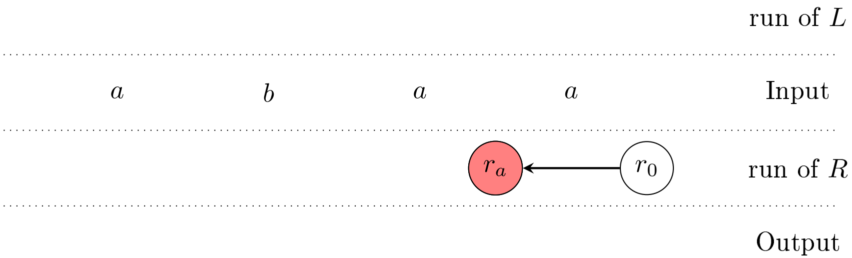

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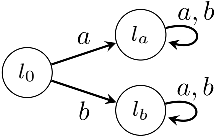

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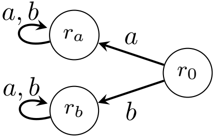

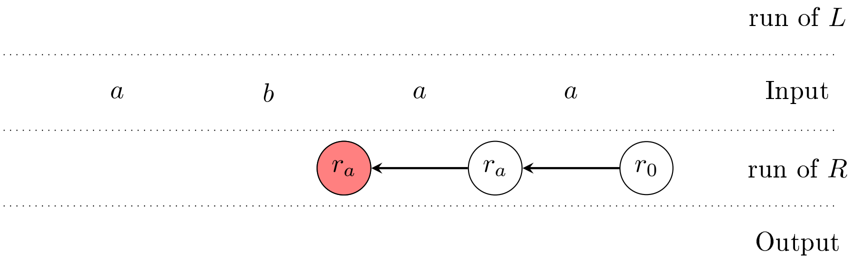

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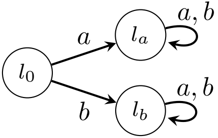

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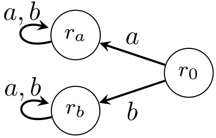

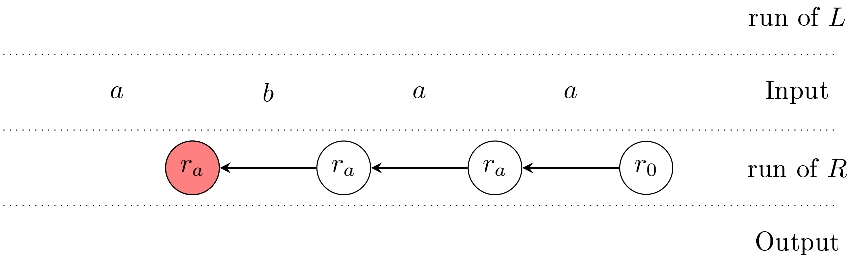

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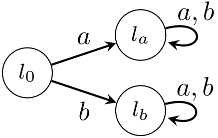

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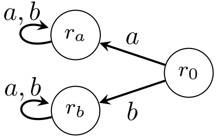

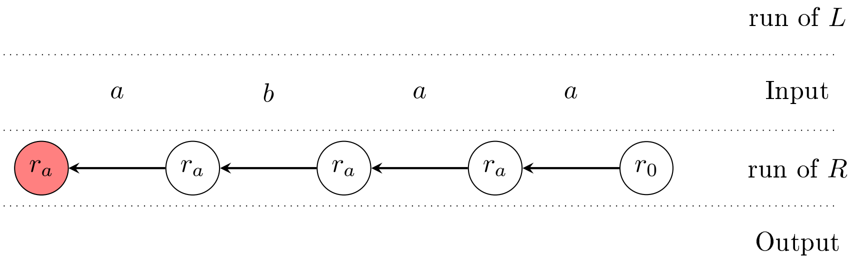

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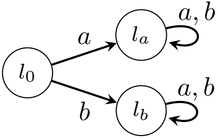

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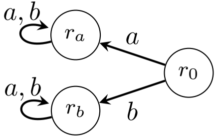

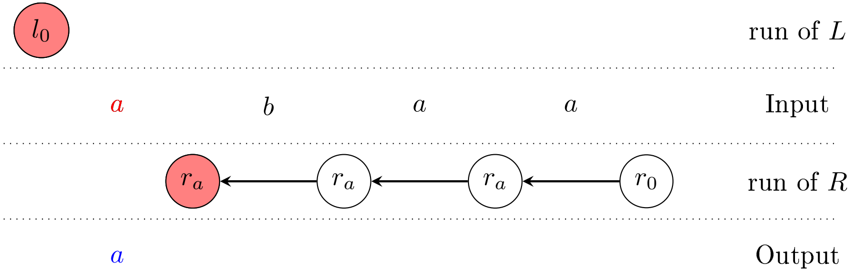

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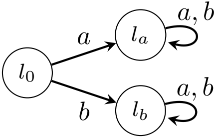

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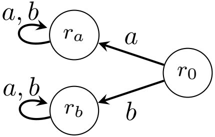

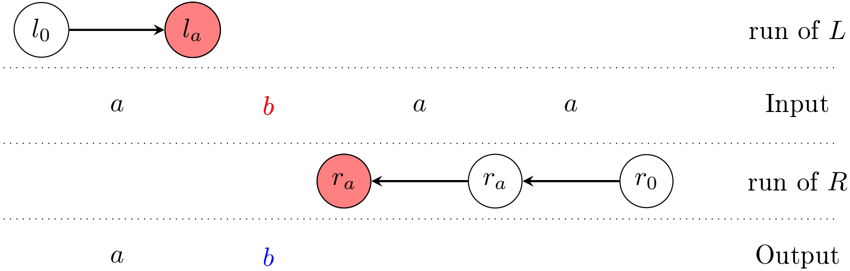

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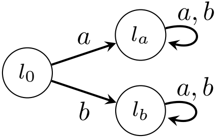

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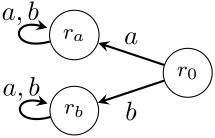

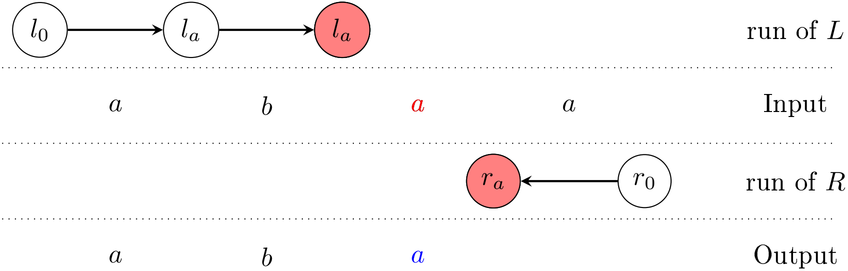

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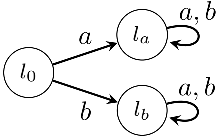

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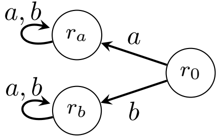

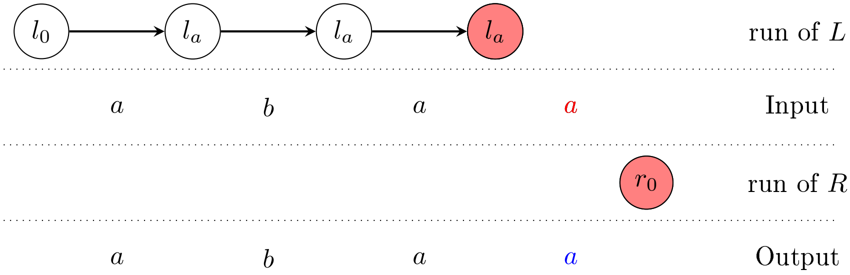

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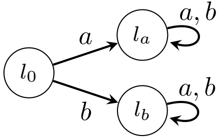

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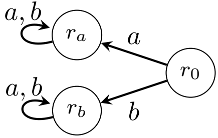

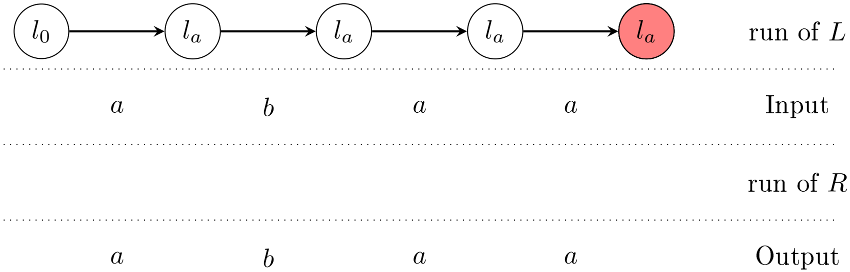

Canonical bimachine

Canonical bimachine [RS91]

- canonical right automaton
- Left minimization **w.r.t.** a fixed right automaton

Canonical bimachine

Canonical bimachine [RS91]

- canonical right automaton
- Left minimization **w.r.t.** a fixed right automaton

Results

Theorem

Realizable by an aperiodic transducer $\Leftrightarrow$ aperiodic canonical bimachine

Results

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Realizable by an aperiodic transducer $\Leftrightarrow$ aperiodic canonical bimachine

Problem

Generalization to other algebraic varieties.

Results

Theorem
Realizable by an aperiodic transducer $\Leftrightarrow$ aperiodic canonical bimachine

Problem
Generalization to other algebraic varieties.

Theorem
For a given variety $\mathbf{V}$:
Realizable by a $\mathbf{V}$-NFT $\Leftrightarrow$ one of the minimal bimachines is a $\mathbf{V}$-bimachine

Problems & Prospects

- other logics
- infinite words
- 2DFT (sweeping)

Thanks !

Julius Richard Büchi.
Weak second-order arithmetic and finite automata.
Mathematical Logic Quarterly, 6(1-6):66–92, 1960.

Olivier Carton and Luc Dartois.
Aperiodic two-way transducers and fo-transductions.
In *24th EACSL Annual Conference on Computer Science Logic, CSL 2015, September 7-10, 2015, Berlin, Germany*, pages 160–174, 2015.

Christian Choffrut.
Minimizing subsequential transducers: a survey.
Theor. Comput. Sci., 292(1):131–143, 2003.

Joost Engelfriet and Hendrik Jan Hoogeboom.
MSO definable string transductions and two-way finite-state transducers.
ACM Trans. Comput. Log., 2(2):216–254, 2001.

Robert McNaughton and Seymour Papert.
Counter-free automata.
M.I.T. Press Cambridge, Mass, 1971.

Christophe Reutenauer and Marcel-Paul Schützenberger.

Minimization of rational word functions.
SIAM J. Comput., 20(4):669–685, 1991.

Marcel-Paul Schützenberger.
A remark on finite transducers.
Information and Control, 4(2-3):185–196, 1961.

Marcel-Paul Schützenberger.
On finite monoids having only trivial subgroups.
Information and Control, 8(2):190–194, 1965.