Dynamics on Games: Simulation-Based Techniques and Applications to Routing

Benjamin Monmege (Aix-Marseille Université, France)
Thomas Brihaye Marion Hallet Bruno Quoitin (Mons, Belgium)
Gilles Geeraerts (Université libre de Bruxelles, Belgium)

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Two points of view on the prisoner dilemma

Two suspects are arrested by the police. The police, having separated both prisoners, visit each of them to offer the same deal.

- *If one testifies (Defects) for the prosecution against the other and the other remains silent (Cooperate), the betrayer goes free and the silent accomplice receives the full 10-years sentence.*
- *If both remain silent, both are sentenced to only 3-years in jail.*
- *If each betrays the other, each receives a 5-years sentence.*

How should the prisoners act?

The prisoner dilemma - the (matrix) game

The matrix associated with the prisoner dilemma:

	C	D
C	$(-3,-3)$	$(-10,0)$
D	$(0,-10)$	$(-5,-5)$

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D	$(0,-10)$	$(-5,-5)$

Equivalently (since only the relative order of payoffs matters):

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

The first point of view: strategic games

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

Rules of the game

- The game is played only once by two players
- The players choose simultaneously their actions (no communication)
- Each player receives his payoff depending of all the chosen actions
- The goal of each player is to maximise his own payoff

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Hypotheses made in strategic games

- The players are **intelligent** *(i.e. they reason perfectly and quickly)*
- The players are **rational** *(i.e. they want to maximise their payoff)*
- The players are **selfish** *(i.e. they only care for their own payoff)*

The first point of view: strategic games

	C	D
C	$(3,3)$	$(1,4)$
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(D, D) is the only rational choice!

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The second point of view: evolutionary games

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- We have a **large** population of individuals
- Individuals are repeatedly drawn at random to play the above game
- The payoffs are supposed to represent the gain in biological fitness or reproductive value

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- Each individual is **genetically programmed** to play either C or D
- The individuals are no more **intelligent**, nor **rational**, nor **selfish**

The second point of view: evolutionary games

	C	D
C	$(3, 3)$	$(1, 4)$
D	$(4, 1)$	$(2, 2)$

The strategy D is evolutionary stable, facing an invasion of the mutant strategy C.

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Outline

Strategic games

Definition

A *strategic game* G is a triple $\big(N, (A_i)_{i \in N}, (P_i)_{i \in N}\big)$ where:

- N is the finite and non empty set of players,
- A_i is the non empty set of actions of player i,
- $P_i : A_1 \times \cdots \times A_N \to \mathbb{R}$ is the payoff function of player i.

	C	D
C	$(3,3)$	$(1,4)$
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Nash equilibrium

Nash Equilibrium - Definition

Let (N, A_i, P_i) be a strategic game and $a = (a_i)_{i \in N}$ be a *strategy profile*.

We say that $a = (a_i)_{i \in N}$ is a *Nash equilibrium* iff

$$\forall i \in N \; \forall b_i \in A_i \quad P_i(b_i, a_{-i}) \leq P_i(a_i, a_{-i})$$

	C	D
C	$(3,3)$	$(1,4)$
D	$(4,1)$	$(2,2)$

(D,D) is the unique Nash equilibrium

Do all the finite matrix games have a Nash equilibrium?

Do all the finite matrix games have a Nash equilibrium?

No: matching pennies

	L	R
L	$(1,-1)$	$(-1,1)$
R	$(-1,1)$	$(1,-1)$

Mixed strategies

Notations

Given E, we denote $\Delta(E)$ the set of *probability distribution over E*.

Assuming $E = \{e_1, \ldots, e_n\}$, we have that:

$$\Delta(E) = \{(p_1, \ldots, p_n) \mid p_i \geq 0 \text{ and } p_1 + \ldots + p_n = 1\}.$$

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Mixed strategy

If A_i are strategies of player i, $\Delta(A_i)$ is his set of **mixed strategies**.

Expected payoff

Given $(N, (A_i)_i, (P_i)_i)$. Let $(\sigma_1, \ldots, \sigma_n)$ be a mixed strategies profile. The expected payoff of player i is

$$P_i(\sigma_1, \ldots, \sigma_n) = \sum_{(a_1,\ldots,a_N) \in A_1 \times \cdots \times A_N} \underbrace{\left(\prod_{i \in N} \sigma_i(a_i) \right)}_{\text{probability of } (a_1,\ldots,a_N)} P_i(a_1, \ldots, a_N)$$

Nash equilibria in mixed strategies

	L	R
L	$(1, -1)$	$(-1, 1)$
R	$(-1, 1)$	$(1, -1)$

The following profile is a *Nash equilibrium in mixed strategies*:

$$\sigma_1 = \begin{cases} \text{L} & \text{with proba } \frac{1}{2} \\ \text{R} & \text{with proba } \frac{1}{2} \end{cases} \quad \text{and} \quad \sigma_2 = \begin{cases} \text{L} & \text{with proba } \frac{1}{2} \\ \text{R} & \text{with proba } \frac{1}{2} \end{cases}$$

whose expected payoff is 0.

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Nash Theorem [1950]

Every finite game admits mixed Nash equilibria.

Symmetric games

	X	Y
X	(α, α)	(γ, δ)
Y	(δ, γ)	(β, β)

Symmetric games

A symmetric game is a game $\left(N, (A_i)_{i \in N}, (P_i)_{i \in N}\right)$ where:

- $A_1 = A_2 = \cdots = A_N$
- $\forall (a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N$, $\forall \pi$ permutations, $\forall k$, we have that $P_{\pi(k)}(a_1, \ldots, a_N) = P_k(a_{\pi(1)}, \ldots, a_{\pi(k)})$

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- Special case of 2-players: $\forall (a_1, a_2) \in A_1 \times A_2,\ P_2(a_1, a_2) = P_1(a_2, a_1)$

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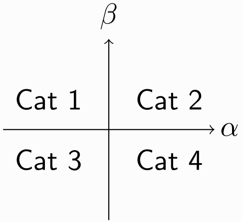
Symmetric Nash Equilibrium

A Nash equilibrium $(\sigma_1, \ldots, \sigma_N)$ is said symmetric when $\sigma_1 = \cdots = \sigma_N$.

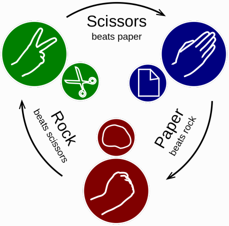
Example 1: 2×2 games - The 4 categories

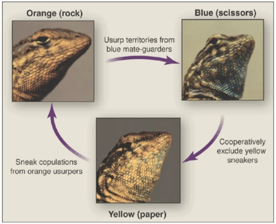
	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)



- Cat 1: $\alpha < 0$ et $\beta > 0$. NE$=\{(Y, Y)\}$
- Cat 2: α, $\beta > 0$. NE$=\{(X, X), (Y, Y), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$
- Cat 3: α, $\beta < 0$. NE$=\{(X, Y), (Y, X), (\sigma, \sigma)\}$ with $\sigma = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$
- Cat 4: $\alpha > 0$ et $\beta < 0$. NE$=\{(X, X)\}$

Example 2: The generalised Rock-Scissors-Paper Games





	R	S	P
R	$(1,1)$	$(2+a,0)$	$(0,2+a)$
S	$(0,2+a)$	$(1,1)$	$(2+a,0)$
P	$(2+a,0)$	$(0,2+a)$	$(1,1)$

(The original RPS game is obtained when $a=0$)

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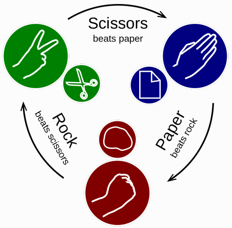

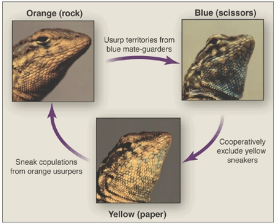

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(The original RPS game is obtained when $a=0$)

A unique Nash equilibrium (σ,σ,σ), where $\sigma=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$.

Some results on symmetric games

Theorem [Cheng et al, 2004]

Every 2-strategy symmetric game (i.e. $|A_i| = 2$) admits a (pure) Nash equilibrium. *But it might not be symmetric...*

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- no longer true if not "symmetric": Matching pennies

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L	$(1, -1)$	$(-1, 1)$
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	L	R
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- not necessarily symmetric: anti-coordination game

	X	Y
X	$(0, 0)$	$(1, 1)$
Y	$(1, 1)$	$(0, 0)$

Outline

Evolutionary game theory

We completely change the point of view !

Rules of the game

- We have a **large** population of individuals.
- Individuals are repeatedly drawn at random to play a symmetric game.
- The payoffs are supposed to represent the gain in biological fitness or reproductive value.

Hypotheses made in evolutionary games

- Each individual is **genitically programmed** to play a strategy.
- The individuals are no more **intelligent**, nor **rational**, nor **selfish**.

Can an existing population resist to the invasion of a mutant ?

Evolutionary Stable Strategy: robustness to mutations

Evolutionary Stable Strategy

We say that σ is an **evolutionary stable strategy (ESS)** if

- (σ, σ) is a Nash equilibrium
- $\forall \sigma' (\neq \sigma) \quad P(\sigma', \sigma) = P(\sigma, \sigma) \Longrightarrow P(\sigma', \sigma') < P(\sigma, \sigma')$

Thus if (σ, σ) is a **strict** Nash equilibrium, then σ is an ESS.

	A	B
A	$(1,1)$	$(1,1)$
B	$(1,1)$	$(2,2)$

	C	D
C	$(1,1)$	$(1,1)$
D	$(1,1)$	$(0,0)$

- (A,A), (B,B) and (C,C) are Nash equilibria.
- A is not an **ESS**.
- B and C are **ESS**.

Evolutionary Stable Strategy - Alternative definition

- Imagine a population composed of a unique species σ
- A small proportion ϵ of the population mutes to a new species σ'
- The new population is thus $\epsilon\sigma' + (1-\epsilon)\sigma$

Proposition

A strategy σ is an **ESS** iff $\forall \sigma' (\neq \sigma)\ \exists \epsilon_0 \in (0,1)\ \forall \epsilon \in (0, \epsilon_0)$

$$P(\sigma, \epsilon\sigma' + (1-\epsilon)\sigma) > P(\sigma', \epsilon\sigma' + (1-\epsilon)\sigma)$$

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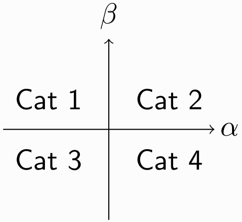
Static concept: it suffices to study the one-shot game

Evolutionary Stable Strategy - 2×2 games

	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)



$$
\begin{aligned}
&\textit{Cat } 1 : \text{NE} = \{(Y, Y)\} && \text{ESS} = \{Y\} \\
&\textit{Cat } 2 : \text{NE} = \{(X, X), (Y, Y), (\sigma, \sigma)\} && \text{ESS} = \{X, Y\} \\
&\textit{Cat } 3 : \text{NE} = \{(X, Y), (Y, X), (\sigma, \sigma)\} && \text{ESS} = \{\sigma\} \\
&\textit{Cat } 4 : \text{NE} = \{(X, X)\} && \text{ESS} = \{X\}
\end{aligned}
$$

The evolution of a population - intuitively

Population composed of several species

Variation of popu. the species $=$ Popu. of the species $\times$ Advantage of the species

Advantage of the species $=$ Fitness of the species $-$ Average fitness of all species

The evolution of a population - more formally (1)

- We consider a population where individuals are divided into n species. Individuals of species i are programmed to play the pure strategy a_i.
- We denote by $p_i(t)$ the number of individuals of species i at time t.
- The **total population at time t** is given by

$$p(t) = p_1(t) + \cdots + p_n(t)$$

- The **population state at time t** is given by

$$\sigma(t) = (\sigma_1(t), \ldots, \sigma_n(t)) \quad \text{where} \quad \sigma_i(t) = \frac{p_i(t)}{p(t)}$$

The evolution of a population - more formally (2)

The evolution of the state of the population is given by:

The replicator dynamics (RD)

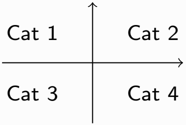
$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_i(t) = (P(a_i, \sigma(t)) - P(\sigma(t), \sigma(t))) \cdot \sigma_i(t)$$

Theorem

Given any initial condition $\sigma(0) \in \Delta(A)$, the above system of differential equations always admits a unique solution.

The replicator dynamics - 2×2 games

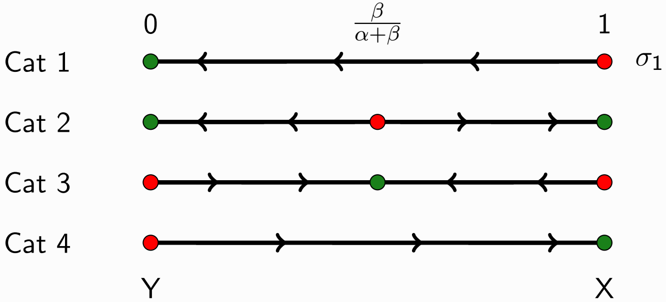
	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)



$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_1(t) = (\alpha\sigma_1(t) - \beta\sigma_2(t)) \cdot \sigma_1(t)\sigma_2(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\sigma_2(t) = (\beta\sigma_2(t) - \alpha\sigma_1(t)) \cdot \sigma_1(t)\sigma_2(t) \end{cases}$$

$\Delta(A) = \{(\sigma_1, \sigma_2) \in [0,1]^2 \mid \sigma_1 + \sigma_2 = 1\} \simeq [0,1]$, where σ_1 is the proportion of X

The solutions $(\sigma_1(t), 1 - \sigma_1(t))$ of the (RD) behave as follows:



Various concept of stability

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth enough and consider:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t)).$$

Let $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a maximal solution of the above equation.

Let $x_0 \in \mathbb{R}^n$, we say that

- x_0 is a **stationary point** iff $\forall t \in \mathbb{R} \quad \varphi(x_0, t) = x_0$
- x_0 is **Lyapunov stable** iff

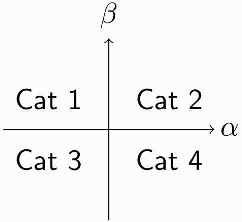
$$\forall U(x_0) \subseteq \mathbb{R}^n \quad \exists V(x_0) \subseteq \mathbb{R}^n \quad \forall x \in V(x_0) \quad \forall t \in \mathbb{R} \quad \varphi(x, t) \in U(x_0)$$

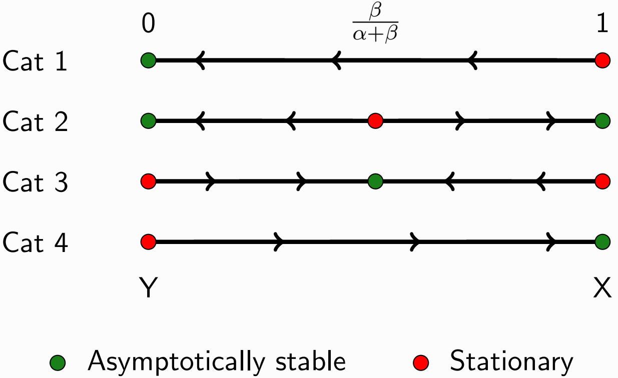
- x_0 is **asymptotically stable** iff x_0 is a Lyapunov stable point and

$$\exists W(x_0) \quad \forall x \in W(x_0) \quad \lim_{t \to +\infty} \varphi(x, t) = x_0$$

2×2 games - Stability

	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)

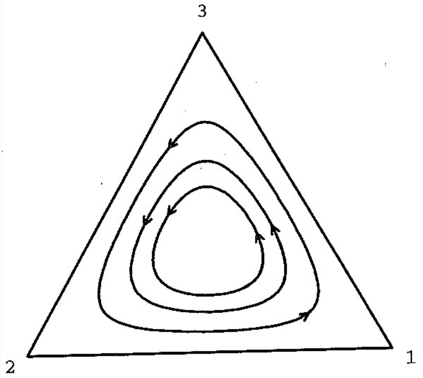




Rock-Scissors-Paper

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is Lyapunov stable but not asymptotically stable.

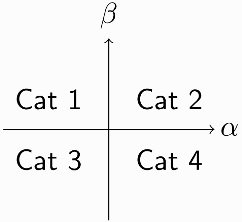
	R	S	P
R	$(1,1)$	$(2,0)$	$(0,2)$
S	$(0,2)$	$(1,1)$	$(2,0)$
P	$(2,0)$	$(0,2)$	$(1,1)$

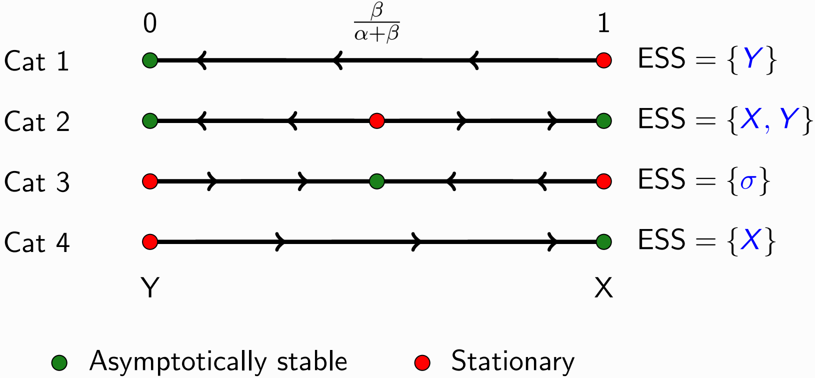


The picture is taken from *Evolutionnary game theory* by J.W. Weibull.

2×2 games - RD Vs ESS

	X	Y
X	(α, α)	$(0, 0)$
Y	$(0, 0)$	(β, β)



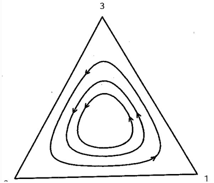


The generalised Rock-Scissors-Paper Games

$a = 0$

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not an ESS

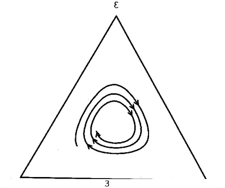
	R	S	P
R	$(1,1)$	$(2,0)$	$(0,2)$
S	$(0,2)$	$(1,1)$	$(2,0)$
P	$(2,0)$	$(0,2)$	$(1,1)$



$a > 0$

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is an ESS

	R	S	P
R	$(1,1)$	$(3,0)$	$(0,3)$
S	$(0,3)$	$(1,1)$	$(3,0)$
P	$(3,0)$	$(0,3)$	$(1,1)$



$a < 0$

$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is not an ESS

	R	S	P
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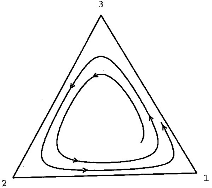
S	$(0,1)$	$(1,1)$	$(1,0)$
P	$(1,0)$	$(0,1)$	$(1,1)$



The pictures are taken from *Evolutionnary game theory* by J.W. Weibull.

Results

There are several results relating various notions of “static” stability:

- Nash equilibrium,
- Evolutionary Stable Strategy,
- Neutrally Stable Strategy...

with various notions of “dynamic” stability:

- stationary points,
- Lyapunov stable points,
- asymptotically stable point ...

Theorems

- If $\sigma \in \Delta$ is Lyapunov stable, then σ is a NE.
- If $\sigma \in \Delta$ is an ESS, then σ is asymptotically stable.

An alternative dynamics

Replicator dynamics

Variation of popu. the species = Popu. of the species $\times$ Advantage of the species

Advantage of the species = Fitness of the species $-$ Average fitness of all species

An alternative dynamics

Replicator dynamics

Variation of popu. the species $=$ Popu. of the species $\times$ Advantage of the species

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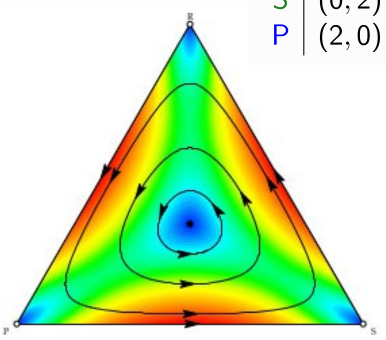
Alternative hypothesis: offspring react **smartly** to the mixture of past strategies played by the opponents, by playing a **best-reply strategy** to this mixture

Best-reply dynamics

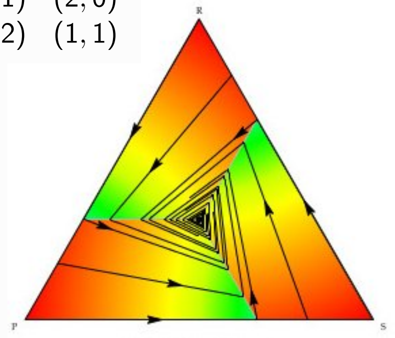
Variation of Strategy Mixture $=$ Best-Reply Strategy $-$ Current Strategy Mixture

Replicator Vs Best-reply

	R	S	P
R	$(1,1)$	$(2,0)$	$(0,2)$
S	$(0,2)$	$(1,1)$	$(2,0)$
P	$(2,0)$	$(0,2)$	$(1,1)$



Replicator dynamics



Best-reply dynamics

Pictures taken from *Evolutionnary game theory* by W. H. Sandholm

Other dynamics

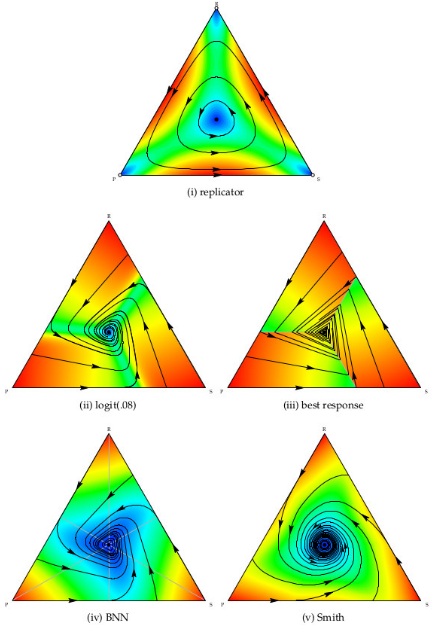
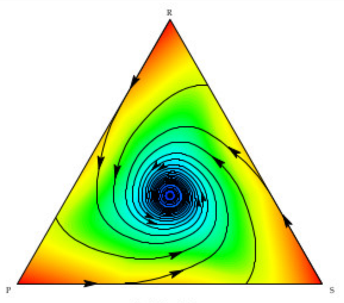


Figure 1: Five basic deterministic dynamics in standard Rock-Paper-Scissors. Colors represent speeds: red

Static vs dynamic approach

Static approach **Dynamic approach**

Equilibria ⟷ Stable Points



Picture taken from *Evolutionnary game theory* by W. H. Sandholm

Static vs dynamic approach

Static approach		**Dynamic approach**
Equilibria	↭	Stable Points

If we discover a new game

- Find immediately a good strategy is concretely impossible

Static vs dynamic approach

Static approach | **Dynamic approach**

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- If so, will this strategy be a *good* strategy?

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- With enough different plays, will we eventually stabilize?
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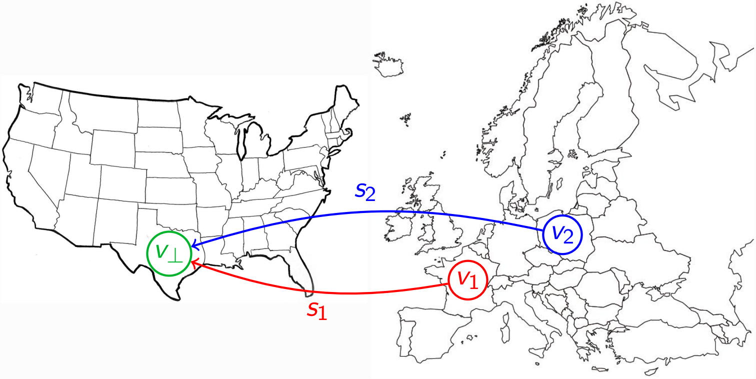
Our Goal

- Apply this idea of improvement/mutation on games played on graphs
- Prove stabilisation via reduction/minor of games
- Show some links with interdomain routing

Interdomain routing problem

Two service providers: v_1 and v_2 want to route packets to $v_\perp$.



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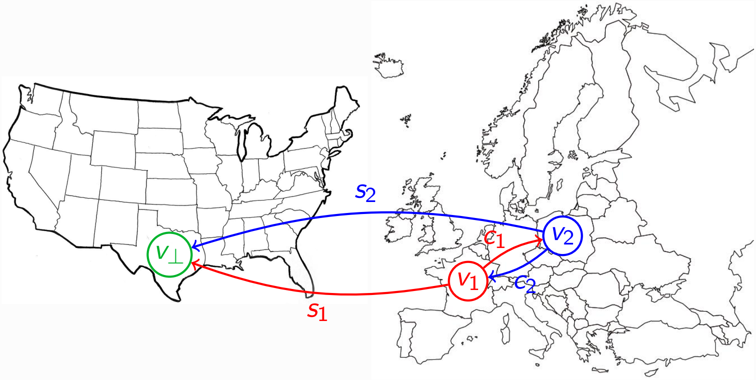

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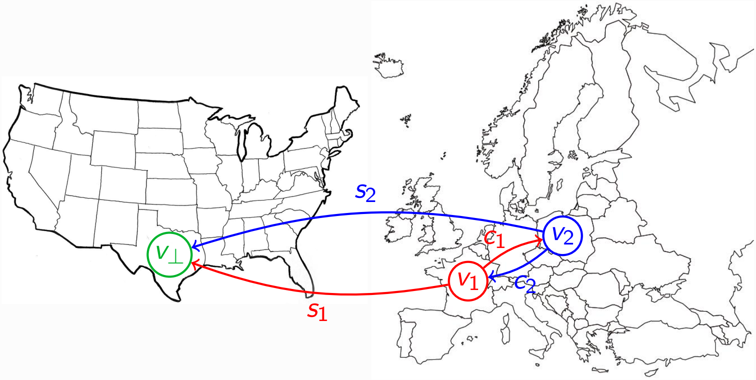
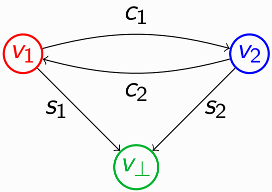


v_1 prefers the route $v_1 v_2 v_\perp$ to the route $v_1 v_\perp$ (preferred to $(v_1 v_2)^\omega$)

v_2 prefers the route $v_2 v_1 v_\perp$ to the route $v_2 v_\perp$ (preferred to $(v_2 v_1)^\omega$)

Interdomain routing problem as a game played on a graph

Two service providers: v_1 and v_2 want to route packets to $v_\perp$.



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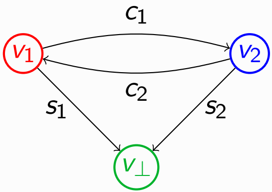
v_2 prefers the route $v_2 v_1 v_\perp$ to the route $v_2 v_\perp$ (preferred to $(v_2 v_1)^\omega$)

$$v_1 v_\perp \prec_1 v_1 v_2 v_\perp \quad \text{and} \quad v_2 v_\perp \prec_2 v_2 v_1 v_\perp$$

Games played on a graph – The strategic game approach

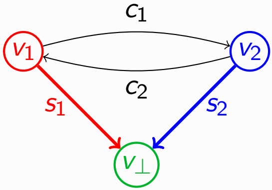


	c_2	s_2
c_1	$(0,0)$	$(2,1)$
s_1	$(1,2)$	$(1,1)$

2 Nash equilibria: (c_1, s_2) and (s_1, c_2)

Static vision of the game: players are perfectly informed and supposed to be **intelligent**, **rational** and **selfish**

Games played on a graph – The evolutionnary approach



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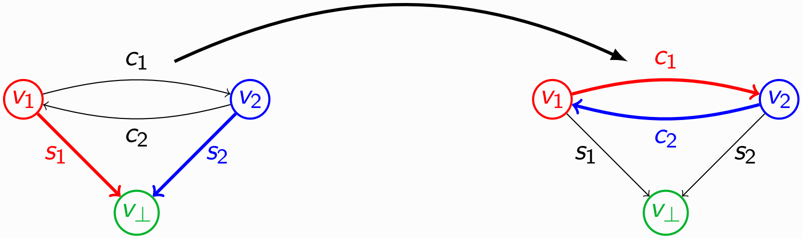

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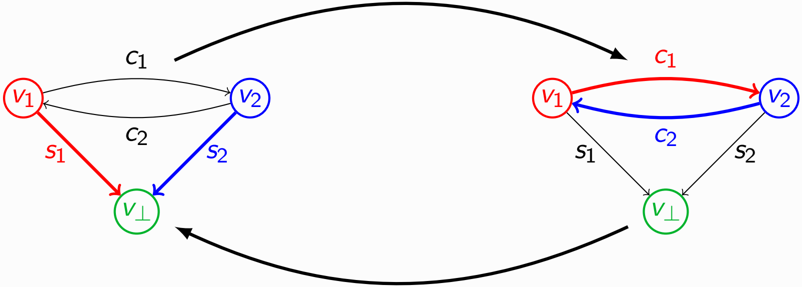
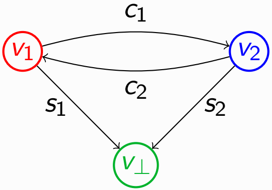
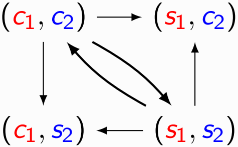


Asynchronous nature of the network could block the packets in an undesirable cycle...

Interdomain routing problem - open problem



The game G



The graph of the dynamics: $G\langle\rightarrow\rangle$

Identify necessary and sufficient conditions on G such that $G\langle\rightarrow\rangle$ has no cycle

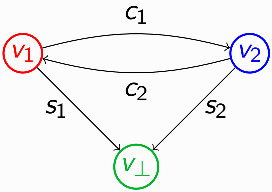
Ideally, the conditions should be algorithmically simple, locally testable...

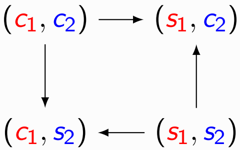
Numerous interesting partial solutions proposed in the literature

Daggitt, Gurney, Griffin. Asynchronous convergence of policy-rich distributed Bellman-Ford routing protocols. 2018

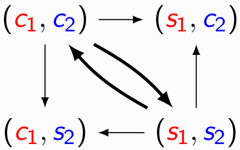
Games played on a graph – The evolutionnary approach

Different dynamics





D_1 with no cycle



D_2 with a cycle

Outline

Positional 1-step dynamics $\xrightarrow{\text{P1}}$

$$\text{profile}_1 \xrightarrow{\text{P1}} \text{profile}_2$$

if:

- a single player changes **at a single node**
- this player improves his own outcome

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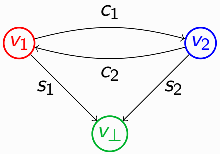
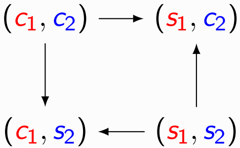


$G\langle \xrightarrow{\text{P1}} \rangle$:



Positional Concurrent Dynamics $\xrightarrow{\text{PC}}$

$$\text{profile}_1 \xrightarrow{\text{PC}} \text{profile}_2$$

if

- one or several players change **at a single node**
- all players that change **intend** to improve their outcome
- but synchronous changes may result in worst outcomes...

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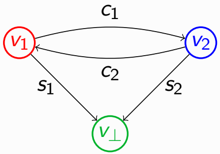

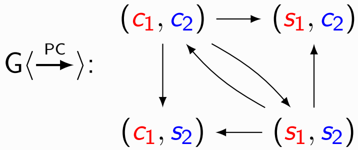

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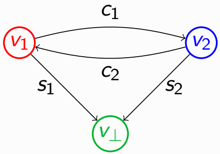

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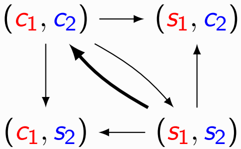

both players intend *to reach their best outcome ($v_1 v_\perp \prec_1 v_1 v_2 v_\perp$ and $v_2 v_\perp \prec_2 v_2 v_1 v_\perp$), even if they do not manage to do it (as the reached outcome is $(v_1 v_2)^\omega$ and $(v_2 v_1)^\omega$)*

Questions

What condition G should satisfy to ensure that

$$\mathsf{G}\langle \rightarrowtail \rangle \text{ has no cycles,} \qquad \text{i.e. dynamics } \rightarrowtail \text{ terminates on } \mathsf{G}?$$

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What relations $\rightarrowtriangle_1$ and $\rightarrowtriangle_2$ should satisfy to ensure that

$$\mathsf{G}\langle \rightarrowtriangle_1 \rangle \text{ has no cycles} \;\text{ if and only if }\; \mathsf{G}\langle \rightarrowtriangle_2 \rangle \text{ has no cycles?}$$

Questions

What condition G should satisfy to ensure that

$$G\langle \rightarrowtail \rangle \text{ has no cycles,} \qquad \text{i.e. dynamics } \rightarrowtail \text{ terminates on } G?$$

What relations $\rightarrowtail_1$ and $\rightarrowtail_2$ should satisfy to ensure that

$$G\langle \rightarrowtail_1 \rangle \text{ has no cycles } \text{ if and only if } G\langle \rightarrowtail_2 \rangle \text{ has no cycles?}$$

What should G_1 and G_2 have in common to ensure that

$$G_1\langle \rightarrowtail \rangle \text{ has no cycles } \text{ if and only if } G_2\langle \rightarrowtail \rangle \text{ has no cycles?}$$

Simulation relation on dynamics graphs

G simulates G' ($G' \sqsubseteq G$) if **all that G' can do, G can do it too**.

$$
\begin{array}{ccc}
\forall\, \text{profile}'_1 & \longrightarrow & \forall\, \text{profile}'_2 \\
\sqcap\!| & & \sqcap\!| \\
\forall\, \text{profile}_1 & &
\end{array}
$$

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Folklore

If $\mathrm{G}_1\langle \twoheadrightarrow_1 \rangle$ simulates $\mathrm{G}_2\langle \twoheadrightarrow_2 \rangle$ and the dynamics $\twoheadrightarrow_1$ terminates on G_1, then the dynamics $\twoheadrightarrow_2$ terminates on G_2.

Relation between games

G' is a minor of G if it is obtained by a succession of operations:

- deletion of an edge (and all the corresponding outcomes);
- deletion of an isolated node;
- deletion of a node v with a single edge $v \to v'$ and no predecessor $u \to v$ such that $u \to v'$.

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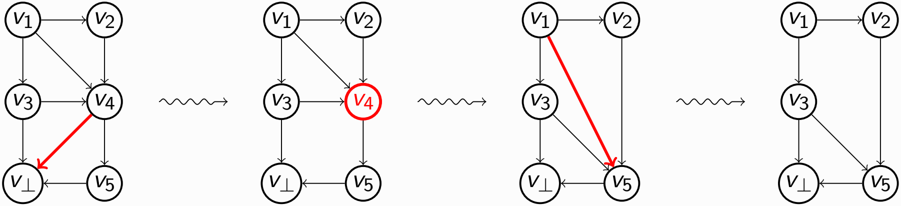

Relation between simulation and minor

Theorem

If G' is a minor of G, then $G\langle \xrightarrow{P1} \rangle$ simulates $G'\langle \xrightarrow{P1} \rangle$. In particular, if $\xrightarrow{P1}$ terminates for G, it terminates for G' too.

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Theorem

If G' is a minor of G, then $G\langle\xrightarrow{PC}\rangle$ simulates $G'\langle\xrightarrow{PC}\rangle$. In particular, if $\xrightarrow{PC}$ terminates for G, it terminates for G' too.

Remark: $G\langle\xrightarrow{P1}\rangle \sqsubseteq G\langle\xrightarrow{PC}\rangle$

More realistic conditions

Adding fairness

- Termination might be too strong to ask in interdomain routing...
- Every router that wants to change its decision will have the opportunity to do it in the future...
- Study of *fair termination*

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More realistic dynamics

Consider *best reply* variants $\xrightarrow{\text{bP1}}$ and $\xrightarrow{\text{bPC}}$ of the two dynamics, where each player that modifies its strategy changes in the best possible way

What results?

Previous theorem

If G' is a minor of G, then $G\langle \xrightarrow{PC} \rangle$ simulates $G'\langle \xrightarrow{PC} \rangle$. In particular, if $\xrightarrow{PC}$ terminates for G, it terminates for G' too.

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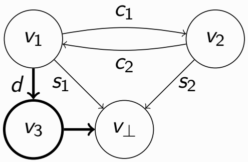

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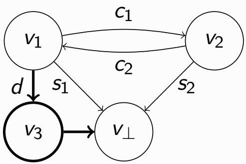
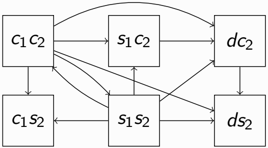
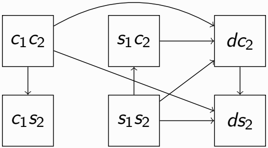




$G\langle\xrightarrow{PC}\rangle$



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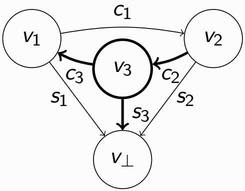

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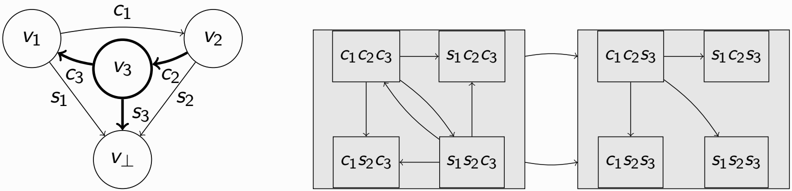

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- The reciprocal does not hold...

Theorem

If G' is a *dominant minor* of G, then $\xrightarrow{bPC}/\xrightarrow{bP1}$ fairly terminates for G if and only if it fairly terminates for G'.

What results?

Previous theorem

If G' is a minor of G, then $G\langle \xrightarrow{PC} \rangle$ simulates $G'\langle \xrightarrow{PC} \rangle$. In particular, if $\xrightarrow{PC}$ terminates for G, it terminates for G' too.

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Theorem

If G' is a *dominant minor* of G, then $\xrightarrow{bPC} / \xrightarrow{bP1}$ fairly terminates for G if and only if it fairly terminates for G'.

- Use of simulations that are partially invertible...

Interdomain routing

- Particular case of game with one target for all players (reachability game) and players owning a single node (router)

Theorem [Sami, Shapira, Zohar, 2009]

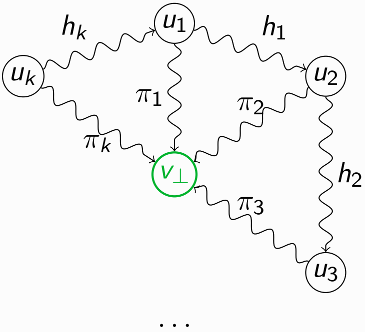
If G is a one-target game for which $\xrightarrow{\text{bPC}}$ fairly terminates, then it has exactly one *equilibrium*.

Interdomain routing

- Particular case of game with one target for all players (reachability game) and players owning a single node (router)

Theorem [Griffin, Shepherd, Wilfong, 2002]

There exists a pattern, called *dispute wheel* such that if G is a one-target game that has no dispute wheels, then $\xrightarrow{\text{bPC}}$ fairly terminates.



$$\forall 1 \leq i \leq k \quad \pi_i \prec_{u_i} h_i \pi_{i+1}$$

Reciprocal?

Theorem

There exists a stronger pattern, called *strong dispute wheel*, such that if $\xrightarrow{\text{PC}}$ terminates for G, then G has no strong dispute wheel.

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Theorem

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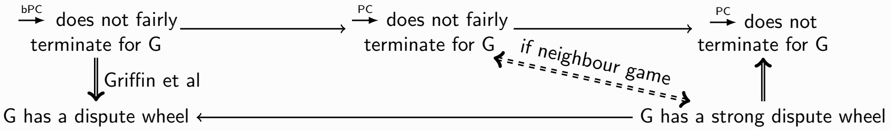

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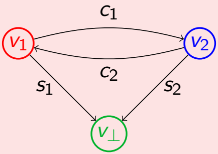
Theorem

Finding a strong dispute wheel in G can be tested by searching whether G contains the following game as a minor:



Summary

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- Different possible dynamics
- Conditions for (fair) termination
- Use of game minors and graph simulations
- In the article, non-positional strategies are also considered

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Thank you! Questions?