Synchronism vs asynchronism in Boolean automata networks

Sylvain Sené

MOVE seminar

18th January 2018

Outline

Outline

BANs, non formally

- A discrete computational model of interaction systems.
- From a theoretical standpoint:
  - Simple setting and representation.
  - Able to capture dynamically a lot of behavioural intricacies and heterogeneities.
- From a more practical/applied standpoint:
  - Originate from neural theoretical modelling (McCulloch, Pitts, 1943).
  - Developed in the context of genetics (Kauffman, 1969; Thomas, 1973).
  - The most used mathematical objects for genetic regulation qualitative modelling.

The (a-)synchronicity problematic(s)

- The causality of events along time depends on the relation between automata updates and "time" but...
  - How to define this relation?
  - How to study the causal perturbations due to changes of this relation?
- Mathematical pertinence:
  - Neat problematic at the frontier of dynamical systems, combinatorics, complexity and computability.
- Biological pertinence:
  - Genetic expression and chromatin dynamics.
- A remaining question: does model synchronicity stand for modelled system simultaneity?

Outline

BANs and interaction graphs

A Boolean automata network (BAN) of size n is a function

$$f: \begin{array}{rcl} \mathbb{B}^n & \to & \mathbb{B}^n \\ x = (x_0, x_1, \ldots, x_{n-1}) & \mapsto & f(x) = (f_0(x), f_1(x), \ldots, f_{n-1}(x)) \end{array},$$

where $\forall i \in \{0, \ldots, n-1\}$, $x_i \in \mathbb{B}$ is the state of automaton i, and $\mathbb{B}^n$ is the set of configurations.

The interaction graph of f is the signed digraph $G(f) : (V, E \subseteq V \times V)$ where:

- $V = \{0, \ldots, n-1\}$;
- $(i,j) \in E$ is positive if $\exists x \in \mathbb{B}^n$ s.t. $f_j(x_0, \ldots, x_{i-1}, \mathbf{0}, x_{i+1}, \ldots, x_{n-1}) = 0$ and $f_j(x_0, \ldots, x_{i-1}, \mathbf{1}, x_{i+1}, \ldots, x_{n-1}) = 1$;
- $(i,j) \in E$ is negative if $\exists x \in \mathbb{B}^n$ s.t. $f_j(x_0, \ldots, x_{i-1}, \mathbf{0}, x_{i+1}, \ldots, x_{n-1}) = 1$ and $f_j(x_0, \ldots, x_{i-1}, \mathbf{1}, x_{i+1}, \ldots, x_{n-1}) = 0$.

BANs and interaction graphs

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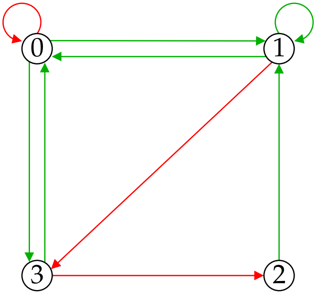
where $\forall i \in \{0, \ldots, n-1\}$, $x_i \in \mathbb{B}$ is the state of automaton i, and $\mathbb{B}^n$ is the set of configurations.

$$f \quad : \quad \mathbb{B}^4 \to \mathbb{B}^4$$

$$f \quad = \quad \begin{cases} f_0(x) = \neg x_0 \vee x_1 \wedge x_3 \\ f_1(x) = x_0 \wedge (x_1 \vee x_2) \\ f_2(x) = \neg x_3 \\ f_3(x) = x_0 \vee \neg x_1 \end{cases}$$



Automata updates

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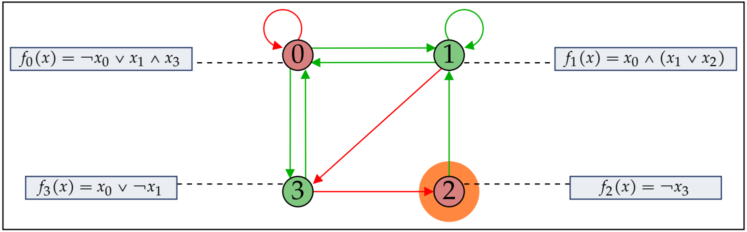

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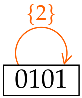

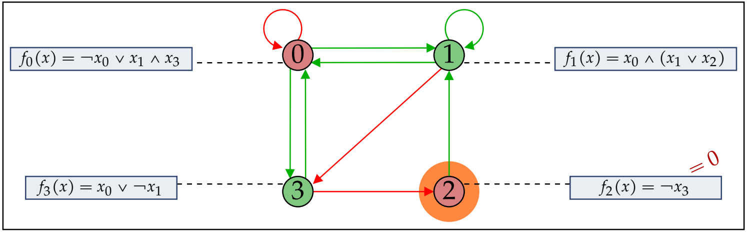

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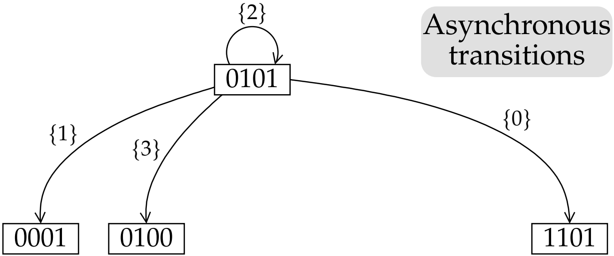

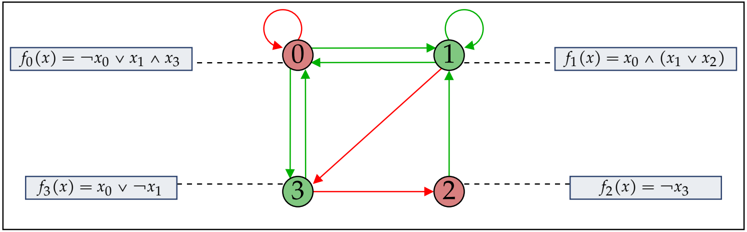

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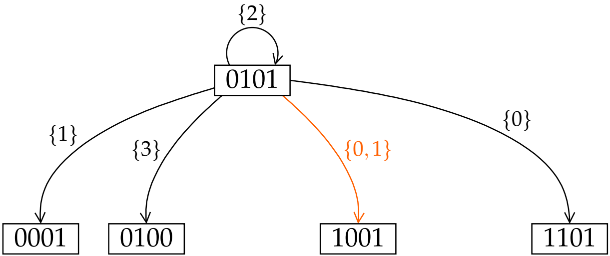

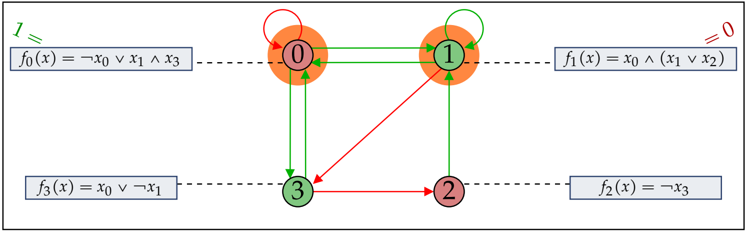

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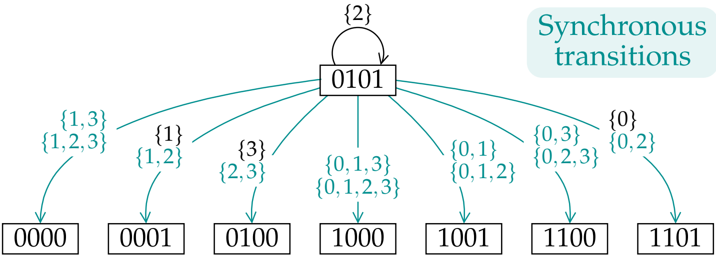
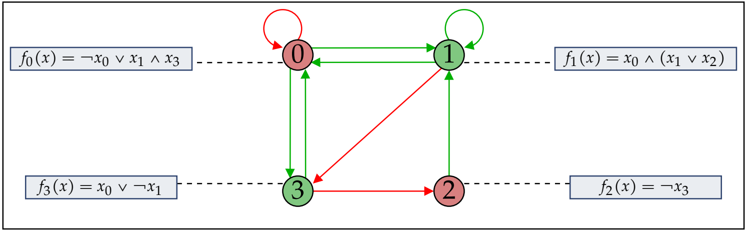


Synchronous transitions



Update modes and BAN behaviours

- An update mode is a way of organising the automata updates along time.
- It can be deterministic (periodic or not) or non-deterministic (stochastic or not).
- There exists an infinite number of update modes.

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- The update mode defines the network behaviour.
- The behaviour of a BAN f is described by a transition graph

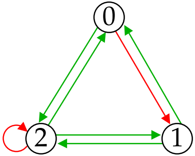
$$\mathscr{G}_{\blacklozenge}(f) = (\mathbb{B}^n, T \subseteq \mathbb{B}^n \times (\mathscr{P}(V) \backslash \varnothing) \times \{0,1\}^n),$$

where $\blacklozenge$ represents a given “fair” update mode.

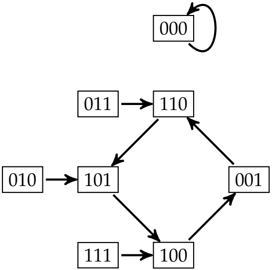
Some examples

$$f \quad : \quad \mathbb{B}^3 \to \mathbb{B}^3$$

$$f \quad = \quad \begin{cases} f_0(x) = x_1 \vee x_2 \\ f_1(x) = \neg x_0 \wedge x_2 \\ f_2(x) = \neg x_2 \wedge (x_0 \vee x_1) \end{cases}$$



Parallel evolution



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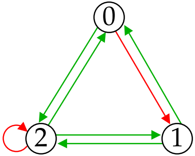

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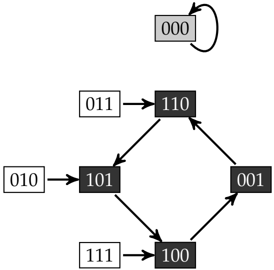

- An attractor of $(f, \blacklozenge)$ is a terminal SCC of $\mathcal{G}_{\blacklozenge}(f)$.
- A fixed point (stable configuration) is a trivial attractor.
- A limit cycle (stable oscillation) is a non-trivial attractor.

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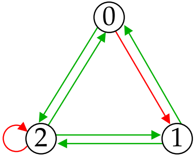
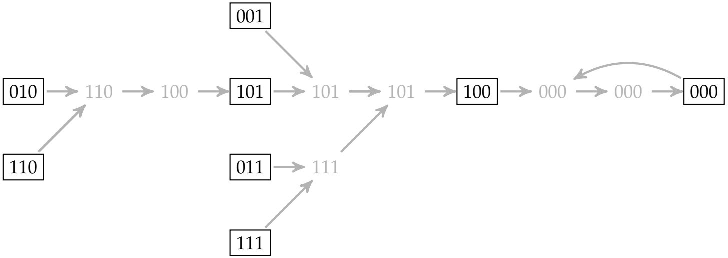


$(\{0\},\{1\},\{2\})$-sequential evolution



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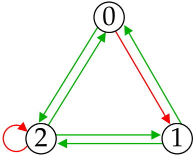

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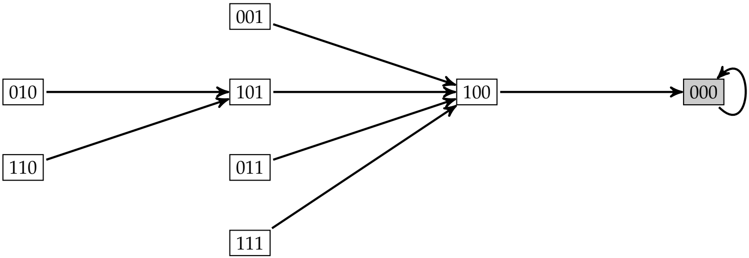

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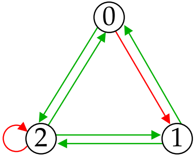
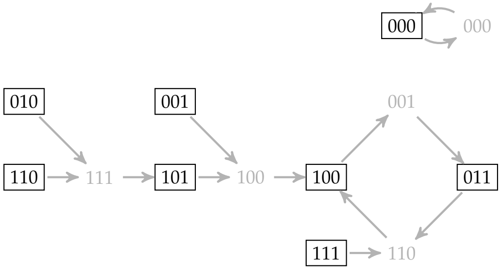


$(\{0,2\},\{1\})$-block-sequential evolution



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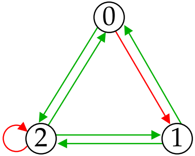

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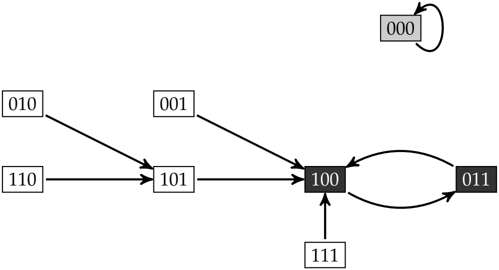

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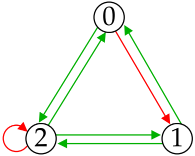

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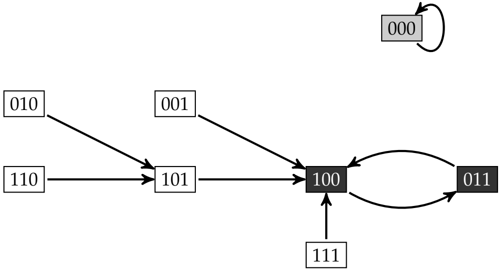

Number of ordered partitions:

$$\mathscr{B}_n^{\text{ord}} = \sum_{k=0}^{n-1} \binom{n}{k} \mathscr{B}_k^{\text{ord}},$$

with $\mathscr{B}_0^{\text{ord}} = 1$.

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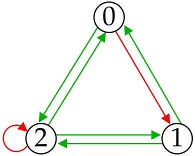
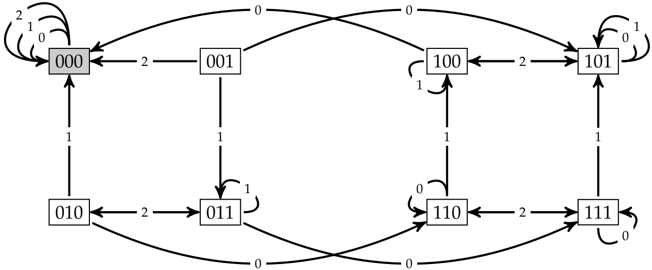


Asynchronous evolution



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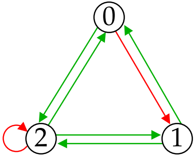
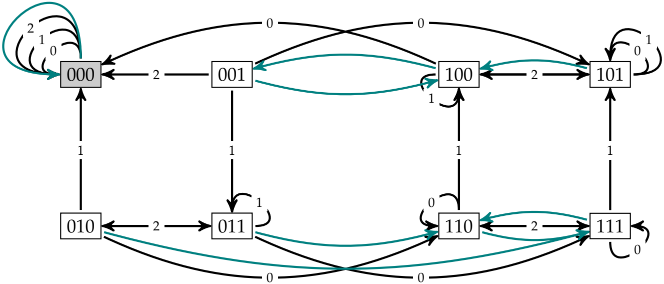


Asynchronous evolution + $\{0,2\}$-synchronous transitions



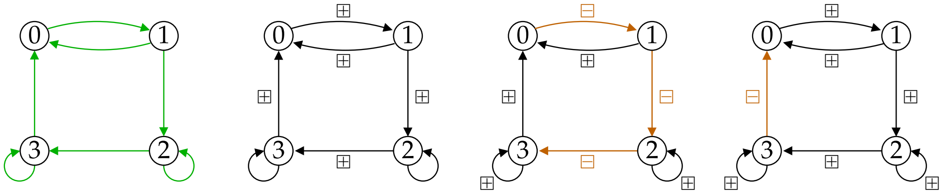
Outline

Update graphs

Given an interaction graph $G = (V, E)$, a **labelled graph** is a graph (G, lab), with $\text{lab} : E \to \{\boxplus, \boxminus\}$.

A labelled graph (G, lab) is an **update graph** if there exist $s : V \to \{1, \dots, n\}$ s.t.

$$\forall (i,j) \in E,\ \text{lab}(i,j) = \begin{cases} \boxplus & \text{if } s(i) \geqslant s(j) \\ \boxminus & \text{if } s(i) < s(j) \end{cases}.$$



$(\{0,1,2,3\})$ $(\{0\},\{1\},\{2\},\{3\})$ $(\{2,3\},\{0,1\})$

Update graphs and dynamics

Let f be a BAN and $G(f) = (V, E)$ its interaction graph, let π be the parallel update mode, and let $s \neq s'$ be two distinct block-sequential modes different from π.

Theorem 1 (Aracena et al., 2009)

If $G(f, \text{lab}_s) = G(f, \text{lab}_{s'})$ then $\mathscr{G}_s(f) = \mathscr{G}_{s'}(f)$.

Theorem 2 (Tchuente, 1988; Aracena et al., 2009)

If s is defined as $\forall j \in \{0, \ldots, n-1\}, \forall i$ s.t. $(i, j) \in E$, $s(i) \geqslant s(j)$ then $\mathscr{G}_s(f) = \mathscr{G}_\pi(f)$.

Theorem 3 (Aracena et al., 2009)

Consider s and f s.t. all the loops in $G(f)$ are positive. Then there exists s' such that $\mathscr{G}_s(f)$ and $\mathscr{G}_{s'}(f)$ do not have any common limit cycle.

Update graphs and dynamics

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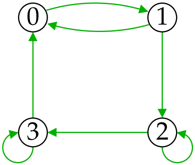

$$s_1 \equiv (\{1\}, \{0\}, \{2\}, \{3\})$$
$$s_2 \equiv (\{1\}, \{2\}, \{0\}, \{3\})$$
$$s_3 \equiv (\{1\}, \{2\}, \{0, 3\})$$

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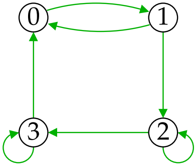

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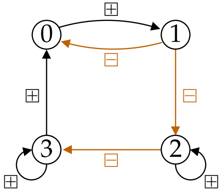
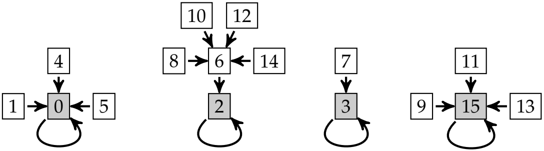
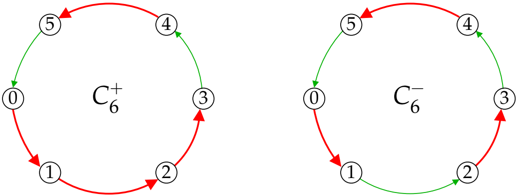




Interaction cycles

2 types of interaction cycles, the positive and the negative ones:

an even number of negative arcs



an odd number of negative arcs

Seminal results:

Theorem 4 (Robert, 1986)

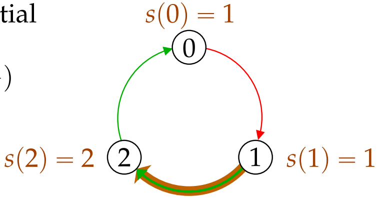
If $G(f)$ is acyclic, then f admits a unique attractor which is a fixed point.

Theorem 5 (Thomas, 1981; Richard, Comet, 2007)

If there are no positive cycles in $G(f)$, f admits no more than one fixed point.

Impact of update modes on cycles

Block-sequential mode
$s \equiv (\{0,1\},\{2\})$



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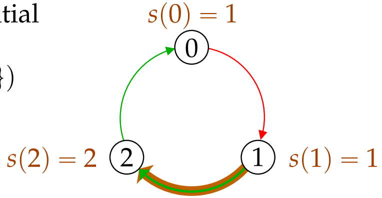

$x_0(t+1) = f_0(x_2(t))$

$x_1(t+1) = f_1(\neg x_0(t))$

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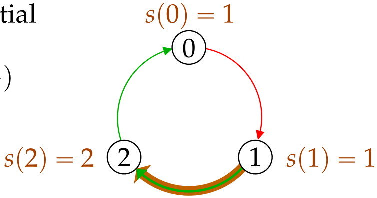

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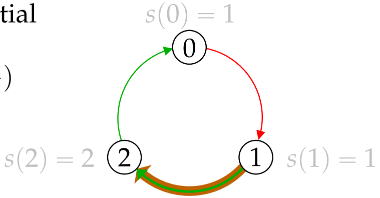

$x_0(t+1) = f_0(x_2(t))$

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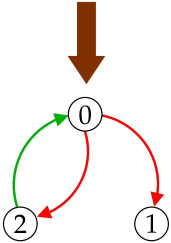
$x_2(t+1) = f_2(x_1(t+1))$
$= f_2(f_1(\neg x_0(t)))$

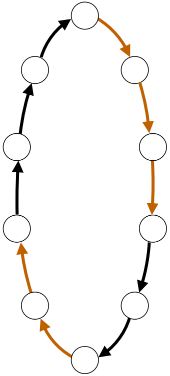


Interaction graph $G(f,s) = (V, E(s))$

Each arc $(i,j) \in E(s)$ represents the dependence of $x_j(t+1)$ on $x_i(t)$.

Impact of update modes on cycles

$\mathrm{inv}(s)$
$= \{(i, i+1) \mid s(i) < s(i+1)\}$



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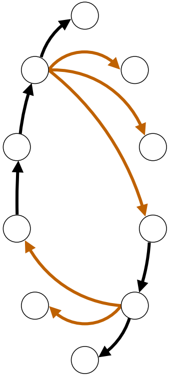
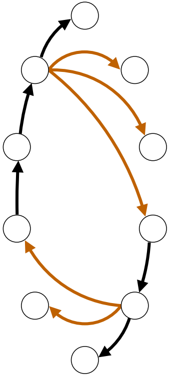
$\mathrm{inv}(s)$
$= \{(i, i+1) \mid s(i) < s(i+1)\}$



Impact of update modes on cycles

$\mathrm{inv}(s)$
$= \{(i, i+1) \mid s(i) < s(i+1)\}$



Theorems (Goles, Noual, 2010)

- The dynamics induced by two update modes s and s' are equal iff $\mathrm{inv}(s) = \mathrm{inv}(s')$.
  - ⇝ Given a cycle of size n, the total number of distinct dynamics induced by block-sequential update modes is:

$$\sum_{k=0}^{n-1} \binom{n}{k} = 2^n - 1.$$

- $\mathrm{inv}(s) \neq \mathrm{inv}(s') \implies$ no common limit cycles.
- Iterating a cycle of size n with an update mode s with $|\mathrm{inv}(s)| = k$ corresponds to iterating a cycle of same sign and of size $n - k$ in parallel.

Impact of update modes on cycles

Theorem 6 (Goles, Noual, 2010)

$\mathrm{inv}(s) \neq \mathrm{inv}(s') \implies$ no common limit cycles.

Proof

First, let us note that $\forall i,j \in V,\ f[j,i] : \begin{cases} f_j \circ f_{j-1} \circ \cdots \circ f_i & \text{if } i \leqslant j \\ f_j \circ f_{j-1} \circ \cdots \circ f_0 \circ f_{n-1} \circ \cdots \circ f_i & \text{if } i > j \end{cases}.$

Suppose that $(i,i+1) \in \mathrm{inv}(s) \setminus \mathrm{inv}(s')$ and that $\exists x = x^s(t) = x^{s'}(t)$ s.t. $x^s(t+1) = x^{s'}(t+1)$. Then:

$$x^s_{i+1}(t+2) = f_{i+1}(x^s_i(t+2)) = f[i+1, i^\star+1](x^s_{i^\star}(t+1)),$$

and $$x^{s'}_{i+1}(t+2) = f_{i+1}(x^{s'}_i(t+1)) = f_{i+1}(x^s_i(t+1)) = f[i+1, i^\star+1](x^s_{i^\star}(t)),$$

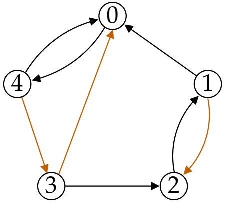
where $i^\star = \max(\{k < i \mid s(k) \geqslant s(k+1)\})$.

By the injectivity of $f[i+1, i^\star+1]$, if $x^s(t+2) = x^{s'}(t+2)$ then $x_{i^\star}(t+1) = x_{i^\star}(t)$. Now, if x belongs to an attractor that is induced identically by both s and s', then $x^s(t) = x^{s'}(t)$ $\forall t$. As result, in this case, $\forall t,\ x^s_{i^\star}(t+1) = x^{s'}_{i^\star}(t) = x^s_{i^\star}(t)$. In other terms, the state of node $i^\star$ is fixed in the attractor. Hence the states of all nodes are fixed in the attractor which therefore is a fixed point. □

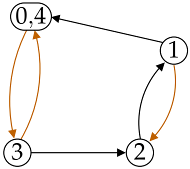
Update graphs other related results

$\mathcal{Q}$: Is a labelled graph an update graph?

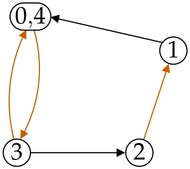
Labelled graph
(G, lab)



Reduced labelled graph
$(G, \text{lab})^{\boxplus}$



Reversed labelled graph
$(G, \text{lab})^{\boxplus}_{R}$



Update graphs other related results

$\mathcal{Q}$: Is a labelled graph an update graph?

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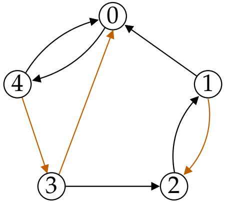
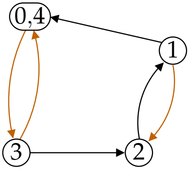
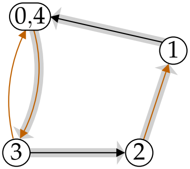


Reduced labelled graph
$(G, \mathrm{lab})^{\boxplus}$



Reversed labelled graph
$(G, \mathrm{lab})^{\boxplus}_{R}$



Theorem 7 (Aracena et al., 2011)

A labelled digraph (G, lab) is an update graph iff $(G, \mathrm{lab})^{\boxplus}_{R}$ does not contain any forbidden cycle.

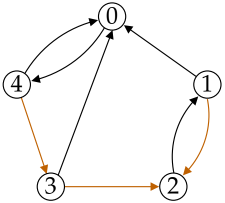
Idea

$$\begin{array}{l} s(1) < s(2) \\ s(1) \geqslant (s(0) = s(4)) \end{array} \quad \wedge \quad \begin{array}{l} s(3) < (s(0) = s(4)) \\ s(3) \geqslant s(2) \end{array} \quad \Longrightarrow \quad \begin{array}{l} s(1) < s(3) \\ s(3) < s(1) \end{array}$$

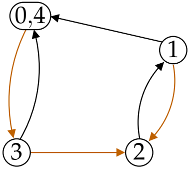
Update graphs other related results

$\mathscr{Q}$: How to find the most compact update mode on (G, lab)?

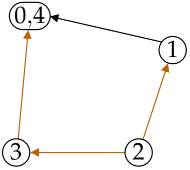
(G', lab)



$(G', \text{lab})^{\boxplus}$



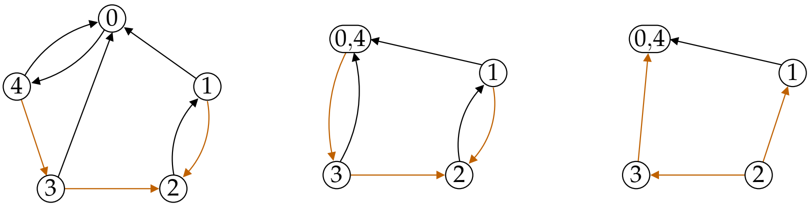
$(G', \text{lab})^{\boxplus}_{R}$



Update graphs other related results

$\mathcal{Q}$: How to find the most compact update mode on (G, lab)?

(G', lab) $\quad$ $(G', \text{lab})^{\boxplus}$ $\quad$ $(G', \text{lab})^{\boxplus}_{R}$



$s \equiv (\{0,4\}, \{1,3\}, \{2\})$

Algorithm Init. Take $G' := (G, \text{lab})^{\boxplus}_{R}$ and $t := 1$.

(1) Compute the paths $P_{\boxminus} = \{P \mid \#(\boxminus \in P) \text{ is max.}\}$ on G'. If $P_{\boxminus} = \emptyset$, goto (4).

(2) The targets T of the last negative arc of each P of $P_{\boxminus}$, and their successors $S(T)$ are scheduled at time step t. $\quad t := t+1$.

(3) Remove T, $S(T)$ and all their incoming arcs from G', and go back to (1).

(4) All the remaining nodes are scheduled all at once, at time step t.

Outline

Basic definitions and notations

$$\forall x = (x_0, \ldots, x_{n-1}) \in \mathbb{B}^n, \forall i \in V,\ \overline{x}^i = (x_0, \ldots, x_{i-1}, \neg x_i, x_{i+1}, \ldots, x_{n-1})$$

$$\forall x \in \mathbb{B}^n, \forall W = W' \uplus \{i\} \subseteq V,\ \overline{x}^W = \overline{(\overline{x}^i)}^{W'} = \overline{(\overline{x}^{W'})}^i$$

The sign of an influence of i on j in x is

$$\mathrm{sign}_x(i,j) = \frac{f_j(x) - f_j(\overline{x}^i)}{x_i - \overline{x}^i_i} = \mathsf{s}(x_i) \cdot (f_j(x) - f_j(\overline{x}^i)),$$

where $\mathsf{s} : b \in \mathbb{B} \mapsto b - \neg b \in \{-1, 1\}$.

Given $x, y \in \mathbb{B}^n$, $\mathrm{D}(x,y) = \{i \in V \mid x_i \neq y_i\}$ and $d(x,y) = |\mathrm{D}(x,y)|$.

$E(x) = \{(i,j) \in V \times V \mid \mathrm{sign}_x(i,j) \neq 0\}$ represents the set of effective influences of $G(f)$ in x, which formally means that

$$\forall i,j \in V, \exists x \in \mathbb{B}^n,\ f_j(x) \neq f_j(\overline{x}^i) \iff (i,j) \in E.$$

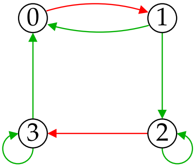
Monotonicity, unstabilities and frustrations

A local function f_i is locally monotonic in j if either:

$$\forall x, f_i(x_0,\dots,x_{j-1},0,x_{j+1},\dots,x_{n-1}) \leqslant f_i(x_0,\dots,x_{j-1},1,x_{j+1},\dots,x_{n-1})$$

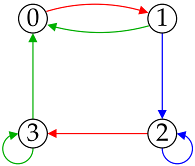
or: $$\forall x, f_i(x_0,\dots,x_{j-1},0,x_{j+1},\dots,x_{n-1}) \geqslant f_i(x_0,\dots,x_{j-1},1,x_{j+1},\dots,x_{n-1}) \quad .$$

$$f = \begin{cases} f_0(x) = x_1 \wedge x_3 \\ f_1(x) = \neg x_0 \\ f_2(x) = x_1 \vee x_2 \\ f_3(x) = \neg x_2 \vee x_3 \end{cases}$$



is monotonic.

$$g = \begin{cases} g_0(x) = x_1 \wedge x_3 \\ g_1(x) = \neg x_0 \\ g_2(x) = x_1 \oplus x_2 \\ g_3(x) = \neg x_2 \vee x_3 \end{cases}$$



is not.

Monotonicity, unstabilities and frustrations

A local function f_i is locally monotonic in j if either:

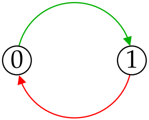
$$\forall x, f_i(x_0,\dots,x_{j-1},0,x_{j+1},\dots,x_{n-1}) \leqslant f_i(x_0,\dots,x_{j-1},1,x_{j+1},\dots,x_{n-1})$$

or: $$\forall x, f_i(x_0,\dots,x_{j-1},0,x_{j+1},\dots,x_{n-1}) \geqslant f_i(x_0,\dots,x_{j-1},1,x_{j+1},\dots,x_{n-1}) \quad .$$

An automaton $i \in V$ is unstable (resp. stable) in $x \in \mathbb{B}^n$ if it belongs to the set

$$\mathrm{U}(x) = \{i \in V \mid f_i(x) \neq x_i\} \quad (\text{resp. } \overline{\mathrm{U}}(x) = V \setminus \mathrm{U}(x)).$$

$$f = \begin{cases} f_0(x) = \neg x_1 \\ f_1(x) = x_0 \end{cases}$$



x	$f_0(x)$	$f_1(x)$	$\mathrm{U}(x)$
$(0,0)$	1	0	$\{0\}$
$(0,1)$	0	0	$\{1\}$
$(1,0)$	1	1	$\{1\}$
$(1,1)$	0	1	$\{0\}$

Monotonicity, unstabilities and frustrations

A local function f_i is locally monotonic in j if either:

$$\forall x, f_i(x_0,\dots,x_{j-1},0,x_{j+1},\dots,x_{n-1}) \leqslant f_i(x_0,\dots,x_{j-1},1,x_{j+1},\dots,x_{n-1})$$

or: $$\forall x, f_i(x_0,\dots,x_{j-1},0,x_{j+1},\dots,x_{n-1}) \geqslant f_i(x_0,\dots,x_{j-1},1,x_{j+1},\dots,x_{n-1}) \quad .$$

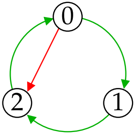
An automaton $i \in V$ is unstable (resp. stable) in $x \in \mathbb{B}^n$ if it belongs to the set

$$\mathrm{U}(x) = \{i \in V \mid f_i(x) \neq x_i\} \quad \text{(resp. } \overline{\mathrm{U}}(x) = V \setminus \mathrm{U}(x)\text{)}.$$

An influence $(i,j) \in E$ is frustrated in x iff it belongs to

$$\mathrm{FRUS}(x) = \{(i,j) \in E \mid \mathsf{s}(x_i)\cdot\mathsf{s}(x_j) = -\mathrm{sign}(i,j)\}.$$

$$f = \begin{cases} f_0(x) = x_2 \\ f_1(x) = x_0 \vee \neg x_1 \\ f_2(x) = \neg x_0 \wedge x_1 \end{cases}$$



$\mathrm{FRUS}(000) = \{(0,2)\}$

$\mathrm{FRUS}(001) = \{(1,2),(2,0)\}$

$\mathrm{FRUS}(010) = \{(0,1),(0,2),(1,2)\}$

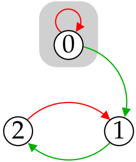
$\mathrm{FRUS}(011) = \{(0,1),(2,0)\}$

Relations between unstabilities and frustrations

Remark (Noual, S., 2017)

If $j \in \mathrm{U}(x)$ then $\exists i \in V^-(j),\ (i,j) \in \mathrm{FRUS}(x)$.

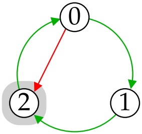
$$f = \begin{cases} f_0(x) = \neg x_0 \\ f_1(x) = x_0 \vee \neg x_2 \\ f_2(x) = x_1 \end{cases}$$



$\mathrm{FRUS}(000) = \{(0,0),(2,1)\}$
$\mathrm{FRUS}(001) = \{(0,0),(1,2)\}$
$\mathrm{FRUS}(110) = \{(0,0),(1,2)\}$
$\mathrm{FRUS}(111) = \{(0,0),(2,1)\}$

N.B: The reciprocal does not hold.

$$f = \begin{cases} f_0(x) = x_2 \\ f_1(x) = x_0 \vee \neg x_1 \\ f_2(x) = \neg x_0 \wedge x_1 \end{cases}$$



$\mathrm{FRUS}(000) = \{(0,2)\}$
$\mathrm{FRUS}(001) = \{(1,2),(2,0)\}$
$\mathrm{FRUS}(010) = \{(0,1),(0,2),(1,2)\}$
$\mathrm{FRUS}(011) = \{(0,1),(2,0)\}$

Relations between unstabilities and frustrations

Lemma 1 (Noual, S., 2017)

Adding frustrated influences incoming an unstable automaton cannot stabilise it. Formally, noting $V^-_{\mathrm{FRUS}(x)}(j) = V^-(j) \cap \{i \in V \mid (i,j) \in \mathrm{FRUS}(x)\}$, we have:

$$\forall x, y \in \mathbb{B}^n,\ j \in \mathsf{U}(x) \wedge \left(V^-_{\mathrm{FRUS}(x)}(j) \subseteq V^-_{\mathrm{FRUS}(y)}(j)\right) \implies j \in \mathsf{U}(y).$$

Proof

Input provided by i to j: $\mathsf{b}^j_i(x) = \mathsf{b}(\mathrm{sign}(i,j) \cdot \mathsf{s}(x_i)) = \begin{cases} x_j & \text{if } (i,j) \notin \mathrm{FRUS}(x) \\ \neg x_j & \text{otherwise} \end{cases}$. By local monotonicity,

$$f_j(x) = \bigwedge_{k \leqslant m} c_k(x) = \bigwedge_{k \leqslant m} \Big(\bigvee_{i \in V^j_k} \mathsf{b}^j_i(x) = \bigvee_{\substack{i \in V^j_k \\ (i,j) \in \mathrm{FRUS}(x)}} \neg x_j \vee \bigvee_{\substack{i \in V^j_k \\ (i,j) \notin \mathrm{FRUS}(x)}} x_j \Big),$$

where V^j_k is the set of in-neighbours of j involved in the kth clause.

Let x be unstable, admitting thus at least one frustrated incoming influence. Let y be such that it admits at least one more frustrated incoming influence than x. Since f_j can be written as a conjunction of disjunctive clauses, the values of these clauses for y are necessarily the same as for x. □

Critical cycles

Let f be a BAN, $G = (V, E)$ its interaction graph, and x a configuration in $\mathbb{B}^n$. A cycle $C = (V_C, E_C)$ of G is x-critical if $E_C \subseteq \text{FRUS}(x)$.
A cycle C is critical if it is x-critical for some x.

Proposition 1 (Noual, S., 2017)

A critical cycle is a NOPE-cycle, *i.e.* negative of odd length or positive of even length.

Proof

Let $x \in \mathbb{B}^n$. By definition of frustrated influences, if $C = (V_C, E_C)$ is x-critical, has length ℓ and sign s then: $\prod_{(i,j)\in E_C} -\text{sign}(i,j) = (-1)^\ell \times \mathsf{s} = \prod_{(i,j)\in E_C} \mathsf{s}(x_i) \cdot \mathsf{s}(x_j) = 1.$ □

$$f = \begin{cases} f_0(x) = x_0 \wedge \neg x_1 \\ f_1(x) = \neg x_0 \wedge x_1 \end{cases}$$

x	$f_0(x)$	$f_1(x)$	$\text{FRUS}(x)$
$(0,0)$	0	0	$\{(0,1),(1,0)\}$
$(0,1)$	0	1	$\varnothing$
$(1,0)$	1	0	$\varnothing$
$(1,1)$	0	0	$\{(0,1),(1,0)\}$

Transitions and trajectories

Name	Notation	Definition
Asynchronous	$x \longrightarrow y$	$d(x,y) \leqslant 1$
Synchronous	$x \color{teal}{\longrightarrow} y$	$d(x,y) > 1$
Elementary	$x \rightarrowtriangle y$	$x \rightarrowtriangle y \in \{x \longrightarrow y\} \cup \{x \color{teal}{\longrightarrow} y\}$
Non-sequentialisable	$x \color{purple}{\longrightarrow} y$	$x \color{teal}{\longrightarrow} y$ not decomposable into smaller elementary transitions

For all $x, y \in \mathbb{B}^n$ s.t. $x \neq y$, x is willing (resp. unwilling) towards y if $\mathrm{D}(x,y) \subseteq \mathrm{U}(x)$ (resp. $\mathrm{D}(x,y) \cap \mathrm{U}(x) = \emptyset$).

A trajectory from x to y is a path $x \rightarrowtriangle \ldots \rightarrowtriangle y$ in the transition graph.

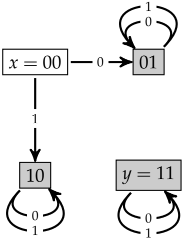
Let $x = x(0) \rightarrowtriangle x(1) \rightarrowtriangle \ldots \rightarrowtriangle x(m-1) \rightarrowtriangle y = x(m)$ be a trajectory from x to y. If $\forall t < m,\ \mathrm{D}(x(t+1), y) \subsetneq \mathrm{D}(x(t), y)$, this trajectory is direct. It performs no reversed changes, *i.e.* $\forall t < m,\ x(t)_i = y_i \implies \forall t < t' \leqslant m,\ x(t')_i = y_i$.

Results relating trajectories and critical cycles

Proposition 2 (Noual, S., 2017)

Let x a willing configuration towards y.

1. If there are no asynchronous trajectories from x to y, then $\mathrm{D}(x,y)$ induces a NOPE-cycle that is x-critical.
2. If $\mathrm{D}(x,y)$ does not induce an x-critical cycle, then there is a *direct* asynchronous trajectory from x to y.



$$f = \begin{cases} f_0(x) = x_0 \vee \neg x_1 \\ f_1(x) = \neg x_0 \vee x_1 \end{cases}$$

x	$f_0(x)$	$f_1(x)$	$\mathrm{U}(x)$
$(0,0)$	1	1	$\mathrm{D}(x,y)$
$(0,1)$	0	1	$\varnothing$
$(1,0)$	1	0	$\varnothing$
$(1,1)$	1	1	$\varnothing$

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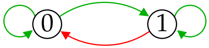

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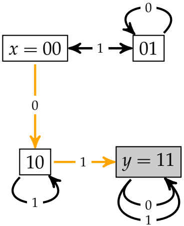
(0,1)	0	0	{1}
(1,0)	1	1	{1}
(1,1)	1	1	∅



Results relating trajectories and critical cycles

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Implication

When m local changes are possible in x, then, unless there is a NOPE-cycle of size m, these m changes can be made asynchronously without risking a deadlock, *i.e.* a situation in which some transitions would have transformed x into a configuration $x(t)$ from which y is not reachable anymore.

Results relating trajectories and critical cycles

Proposition 2 (Noual, S., 2017)

Let x a willing configuration towards y.

1. If there are no asynchronous trajectories from x to y, then $\mathrm{D}(x,y)$ induces a NOPE-cycle that is x-critical.
2. If $\mathrm{D}(x,y)$ does not induce an x-critical cycle, then there is a *direct* asynchronous trajectory from x to y.

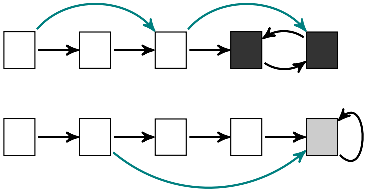
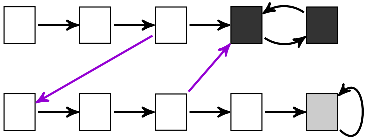
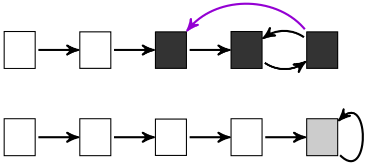
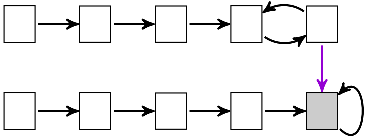
Corollary 1 (Noual, S., 2017)

If $x \longrightarrow y$ *exists, then* $\mathrm{D}(x,y)$ *induces a* NOPE*-cycle which is* x*-critical.*

Implication

In a BAN with no NOPE-cycles of size smaller or equal than $m \in \mathbb{N}$, any synchronous change affecting no more than m automata states can be totally sequentialised.

Structural sensitivity: impact of synchronism

Class **N** “null” sensitivity	Class **F** “weak” sensitivity
	
Class **G** “medium” sensitivity	Class **D** “strong” sensitivity
	

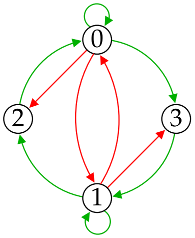
Structural sensitivity: main result

Theorem 8 (Noual, S., 2017)

1) Synchronism-sensitivity requires the existence of a NOPE-cycle.
2) Significant sensitivity requires the existence of a NOPE-cycle of length strictly smaller than the BAN size as well as of a negative cycle.
3) In the absence of a Hamiltonian NOPE-cycle and positive loops on all automata, little sensitivity also requires a NOPE-cycle of length strictly smaller than the BAN size.

A monotonic BAN belonging to sensitivity class **D**:

$$f = \begin{cases} f_0(x) = x_2 \vee (x_0 \wedge \neg x_1) \\ f_1(x) = x_3 \vee (\neg x_0 \wedge x_1) \\ f_2(x) = \neg x_0 \wedge x_1 \\ f_3(x) = x_0 \wedge \neg x_1 \end{cases}$$



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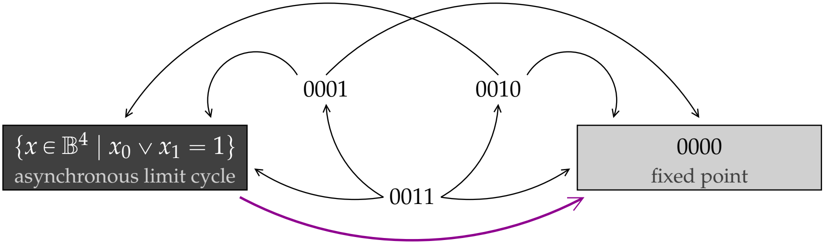
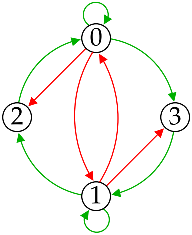


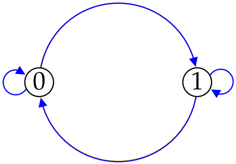
Class **D** and local (non-)monotonicity

$\mathcal{Q}$: How are these two BANs related?

$$f = \begin{cases} f_0(x) = x_2 \vee (x_0 \wedge \neg x_1) \\ f_1(x) = x_3 \vee (\neg x_0 \wedge x_1) \\ f_2(x) = \neg x_0 \wedge x_1 \\ f_3(x) = x_0 \wedge \neg x_1 \end{cases}$$



$$g = \begin{cases} g_0(x) = x_0 \oplus x_1 \\ g_1(x) = x_0 \oplus x_1 \end{cases}$$



(S., 2012)

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