Synchronism vs asynchronism in Boolean automata networks

Sylvain Sené

MOVE seminar
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Outline

1. Introduction
2. Main definitions
3. Deterministic periodic updates
4. Non-deterministic updates
1 Introduction

2 Main definitions

3 Deterministic periodic updates

4 Non-deterministic updates
Introduction

BANs, non formally

- A discrete computational model of interaction systems.

- From a theoretical standpoint:
  - Simple setting and representation.
  - Able to capture dynamically a lot of behavioural intricacies and heterogeneities.

- From a more practical/applied standpoint:
  - Originate from neural theoretical modelling (McCulloch, Pitts, 1943).
  - Developed in the context of genetics (Kauffman, 1969; Thomas, 1973).
  - The most used mathematical objects for genetic regulation qualitative modelling.
The causality of events along time depends on the relation between automata updates and "time" but...
- How to define this relation?
- How to study the causal perturbations due to changes of this relation?

Mathematical pertinence:
- Neat problematic at the frontier of dynamical systems, combinatorics, complexity and computability.

Biological pertinence:
- Genetic expression and chromatin dynamics.

A remaining question: does model synchronicity stand for modelled system simultaneity?
Main definitions

1. Introduction

2. Main definitions

3. Deterministic periodic updates

4. Non-deterministic updates
Main definitions

BANs and interaction graphs

A Boolean automata network (BAN) of size $n$ is a function

$$f : \mathbb{B}^n \rightarrow \mathbb{B}^n$$

$$x = (x_0, x_1, \ldots, x_{n-1}) \mapsto f(x) = (f_0(x), f_1(x), \ldots, f_{n-1}(x))$$

where $\forall i \in \{0, \ldots, n-1\}$, $x_i \in \mathbb{B}$ is the state of automaton $i$, and $\mathbb{B}^n$ is the set of configurations.

The interaction graph of $f$ is the signed digraph $G(f) : (V, E \subseteq V \times V)$ where:

- $V = \{0, \ldots, n-1\}$;
- $(i,j) \in E$ is positive if $\exists x \in \mathbb{B}^n$ s.t.
  $$f_j(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1}) = 0 \text{ and } f_j(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n-1}) = 1;$$
- $(i,j) \in E$ is negative if $\exists x \in \mathbb{B}^n$ s.t.
  $$f_j(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1}) = 1 \text{ and } f_j(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n-1}) = 0.$$
Main definitions
BANs and interaction graphs

A *Boolean automata network* (BAN) of size $n$ is a function

$$f : \mathbb{B}^n \rightarrow \mathbb{B}^n$$

$$x = (x_0, x_1, \ldots, x_{n-1}) \mapsto f(x) = (f_0(x), f_1(x), \ldots, f_{n-1}(x))$$

where $\forall i \in \{0, \ldots, n-1\}$, $x_i \in \mathbb{B}$ is the **state** of automaton $i$, and $\mathbb{B}^n$ is the set of configurations.

$$f : \mathbb{B}^4 \rightarrow \mathbb{B}^4$$

$$f = \begin{cases} 
  f_0(x) = \neg x_0 \lor x_1 \land x_3 \\
  f_1(x) = x_0 \land (x_1 \lor x_2) \\
  f_2(x) = \neg x_3 \\
  f_3(x) = x_0 \lor \neg x_1 
\end{cases}$$
### Automata updates

- \[ f_0(x) = \neg x_0 \lor x_1 \land x_3 \]
- \[ f_1(x) = x_0 \land (x_1 \lor x_2) \]
- \[ f_2(x) = \neg x_3 \]
- \[ f_3(x) = x_0 \lor \neg x_1 \]
Main definitions

Automata updates

\[ f_0(x) = \neg x_0 \lor x_1 \land x_3 \]

\[ f_1(x) = x_0 \land (x_1 \lor x_2) \]

\[ f_2(x) = \neg x_3 \]

\[ f_3(x) = x_0 \lor \neg x_1 \]

\[ \equiv_0 \]

0101
Main definitions

Automata updates

Asynchronous transitions

\[ f_0(x) = \neg x_0 \lor x_1 \land x_3 \]

\[ f_1(x) = x_0 \land (x_1 \lor x_2) \]

\[ f_2(x) = \neg x_3 \]

\[ f_3(x) = x_0 \lor \neg x_1 \]
Main definitions

Automata updates

\[
\begin{align*}
0101 \\
\{2\} \\
\{1\} \\
\{3\} \\
\{0, 1\} \\
\{0\}
\end{align*}
\]

\[
\begin{align*}
0001, 0100, 1001, 1101
\end{align*}
\]

\[
\begin{align*}
I & \subseteq \{ x = x_0 \lor x_1 \land x_3 \} \\
f_0(x) & = \neg x_0 \lor x_1 \land x_3 \\
f_1(x) & = x_0 \land (x_1 \lor x_2) \\
f_2(x) & = \neg x_3 \\
f_3(x) & = x_0 \lor \neg x_1
\end{align*}
\]
Main definitions

Automata updates

\[ f_0(x) = \lnot x_0 \lor x_1 \land x_3 \]

\[ f_1(x) = x_0 \land (x_1 \lor x_2) \]

\[ f_2(x) = \lnot x_3 \]

\[ f_3(x) = x_0 \lor \lnot x_1 \]

Synchronism vs asynchronism in BANs
Main definitions

Update modes and BAN behaviours

- An **update mode** is a way of organising the automata updates along time.
- It can be deterministic (**periodic** or not) or non-deterministic (**stochastic** or **not**).
- There exists an infinite number of update modes.
Main definitions
Update modes and BAN behaviours

- An update mode is a way of organising the automata updates along time.
- It can be deterministic (periodic or not) or non-deterministic (stochastic or not).
- There exists an infinite number of update modes.

- The update mode defines the network behaviour.
- The behaviour of a BAN $f$ is described by a transition graph

$$\mathcal{G}(f) = (\mathbb{B}^n, T \subseteq \mathbb{B}^n \times (\mathcal{P}(V)\setminus\emptyset) \times \{0, 1\}^n),$$

where $\bullet$ represents a given “fair” update mode.
Main definitions

Some examples

\[ f : \mathbb{B}^3 \to \mathbb{B}^3 \]

\[ f = \begin{cases} 
  f_0(x) = x_1 \lor x_2 \\
  f_1(x) = \neg x_0 \land x_2 \\
  f_2(x) = \neg x_2 \land (x_0 \lor x_1) 
\end{cases} \]

Parallel evolution

Sylvain Sené
Main definitions

Some examples

\[ f : \mathbb{B}^3 \rightarrow \mathbb{B}^3 \]

\[ f = \begin{cases} 
  f_0(x) = x_1 \lor x_2 \\
  f_1(x) = \neg x_0 \land x_2 \\
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\end{cases} \]

Parallel evolution

- An attractor of \((f, \bullet)\) is a terminal SCC of \(G_\bullet(f)\).
- A fixed point (stable configuration) is a trivial attractor.
- A limit cycle (stable oscillation) is a non-trivial attractor.
Main definitions

Some examples

\[ f : \mathbb{B}^3 \rightarrow \mathbb{B}^3 \]

\[ f = \begin{cases} 
  f_0(x) = x_1 \lor x_2 \\
  f_1(x) = \neg x_0 \land x_2 \\
  f_2(x) = \neg x_2 \land (x_0 \lor x_1) 
\end{cases} \]

\((\{0\}, \{1\}, \{2\})\)-sequential evolution
Main definitions

Some examples

\[ f : \mathbb{B}^3 \rightarrow \mathbb{B}^3 \]

\[
\begin{align*}
    f_0(x) &= x_1 \lor x_2 \\
    f_1(x) &= \neg x_0 \land x_2 \\
    f_2(x) &= \neg x_2 \land (x_0 \lor x_1)
\end{align*}
\]

\(^{(\{0\}, \{1\}, \{2\})\text{-sequential evolution}}\)
Main definitions

Some examples

\[ f : \mathbb{B}^3 \rightarrow \mathbb{B}^3 \]

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\end{cases} \]

\{(0, 2), \{1\}\)-block-sequential evolution
Main definitions

Some examples

\[ f : \mathbb{B}^3 \rightarrow \mathbb{B}^3 \]

\[ f = \begin{cases} 
    f_0(x) = x_1 \lor x_2 \\
    f_1(x) = \neg x_0 \land x_2 \\
    f_2(x) = \neg x_2 \land (x_0 \lor x_1) 
\end{cases} \]

\[(\{0,2\},\{1\})\text{-block-sequential evolution}\]
Main definitions

Some examples

\[ f : \mathbb{B}^3 \to \mathbb{B}^3 \]

\[ f = \begin{cases} 
    f_0(x) = x_1 \lor x_2 \\
    f_1(x) = \neg x_0 \land x_2 \\
    f_2(x) = \neg x_2 \land (x_0 \lor x_1) 
\end{cases} \]

\((\{0, 2\}, \{1\})\)-block-sequential evolution

Number of ordered partitions:

\[ \mathcal{B}_n = \sum_{k=0}^{n-1} \binom{n}{k} \mathcal{B}_k \]

with \( \mathcal{B}_0 = 1 \).
Main definitions

Some examples

\[ f : \mathbb{B}^3 \rightarrow \mathbb{B}^3 \]

\[ f = \begin{cases} 
  f_0(x) = x_1 \lor x_2 \\
  f_1(x) = \neg x_0 \land x_2 \\
  f_2(x) = \neg x_2 \land (x_0 \lor x_1) 
\end{cases} \]

Asynchronous evolution
Main definitions

Some examples

\[ f : \mathbb{B}^3 \to \mathbb{B}^3 \]

\[ f = \begin{cases} 
  f_0(x) = x_1 \lor x_2 \\
  f_1(x) = \neg x_0 \land x_2 \\
  f_2(x) = \neg x_2 \land (x_0 \lor x_1) 
\end{cases} \]

Asynchronous evolution + \{0,2\}-synchronous transitions
Deterministic periodic updates

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Deterministic periodic updates

Update graphs

Given an interaction graph $G = (V, E)$, a labelled graph is a graph $(G, \text{lab})$, with $\text{lab} : E \rightarrow \{\oplus, \square\}$.

A labelled graph $(G, \text{lab})$ is an update graph if there exist $s : V \rightarrow \{1, \ldots, n\}$ s.t.

$$\forall (i, j) \in E, \text{lab}(i, j) = \begin{cases} \oplus & \text{if } s(i) \geq s(j) \\ \square & \text{if } s(i) < s(j) \end{cases}.$$
Let $f$ be a BAN and $G(f) = (V, E)$ its interaction graph, let $\pi$ be the parallel update mode, and let $s \neq s'$ be two distinct block-sequential modes different from $\pi$.

Theorem 1 (Aracena et al., 2009)

If $G(f, \text{lab}_s) = G(f, \text{lab}_{s'})$ then $\mathcal{G}_s(f) = \mathcal{G}_{s'}(f)$.

Theorem 2 (Tchuente, 1988; Aracena et al., 2009)

If $s$ is defined as $\forall j \in \{0, \ldots, n-1\}, \forall i \text{ s.t. } (i, j) \in E, s(i) \geq s(j)$ then $\mathcal{G}_s(f) = \mathcal{G}_\pi(f)$.

Theorem 3 (Aracena et al., 2009)

Consider $s$ and $f$ s.t. all the loops in $G(f)$ are positive. Then there exists $s'$ such that $\mathcal{G}_s(f)$ and $\mathcal{G}_{s'}(f)$ do not have any common limit cycle.
Deterministic periodic updates

Update graphs and dynamics

Theorem 1 (Aracena et al., 2009)
If $G(f, \text{lab}_s) = G(f, \text{lab}_{s'})$ then $G_s(f) = G_{s'}(f)$.

$$f = \begin{cases} f_0(x) = x_1 \land x_3 \\ f_1(x) = x_0 \\ f_2(x) = x_1 \lor x_2 \\ f_3(x) = x_2 \land x_3 \end{cases}$$

$$s_1 \equiv (\{1\}, \{0\}, \{2\}, \{3\})$$
$$s_2 \equiv (\{1\}, \{2\}, \{0\}, \{3\})$$
$$s_3 \equiv (\{1\}, \{2\}, \{0,3\})$$
Theorem 1 (Aracena et al., 2009)

If \( G(f, \text{lab}_s) = G(f, \text{lab}_{s'}) \) then \( G_s(f) = G_{s'}(f) \).

\[
f = \begin{cases} 
  f_0(x) &= x_1 \land x_3 \\
  f_1(x) &= x_0 \\
  f_2(x) &= x_1 \lor x_2 \\
  f_3(x) &= x_2 \land x_3 
\end{cases}
\]

\( s_1 = (\{1\}, \{0\}, \{2\}, \{3\}) \)

\( s_2 = (\{1\}, \{2\}, \{0\}, \{3\}) \)

\( s_3 = (\{1\}, \{2\}, \{0,3\}) \)
2 types of interaction cycles, the **positive** and the **negative** ones:

- **Positive cycle** ($C^+_6$) with an even number of negative arcs
- **Negative cycle** ($C^-_6$) with an odd number of negative arcs

**Seminal results:**

**Theorem 4 (Robert, 1986)**

If $G(f)$ is acyclic, then $f$ admits a unique attractor which is a fixed point.

**Theorem 5 (Thomas, 1981; Richard, Comet, 2007)**

If there are no positive cycles in $G(f)$, $f$ admits no more than one fixed point.
Deterministic periodic updates
Impact of update modes on cycles

Block-sequential mode
$s \equiv \{0, 1\}, \{2\}$

$s(0) = 1$

$s(2) = 2$

$s(1) = 1$
Deterministic periodic updates

Impact of update modes on cycles

Block-sequential mode
$s \equiv (\{0, 1\}, \{2\})$

$s(0) = 1$
$s(1) = 1$
$s(2) = 2$

$x_0(t + 1) = f_0(x_2(t))$
$x_1(t + 1) = f_1(\neg x_0(t))$
Deterministic periodic updates

Impact of update modes on cycles

Block-sequential mode

\[ s \equiv (\{0, 1\}, \{2\}) \]

\[ s(0) = 1 \]

\[ x_0(t + 1) = f_0(x_2(t)) \]

\[ x_1(t + 1) = f_1(\neg x_0(t)) \]

\[ x_2(t + 1) = f_2(x_1(t + 1)) = f_2(f_1(\neg x_0(t))) \]

\[ s(2) = 2 \]

\[ s(1) = 1 \]
Deterministic periodic updates

Impact of update modes on cycles

Block-sequential mode

\[ s \equiv (\{0, 1\}, \{2\}) \]

Interaction graph \( G(f, s) = (V, E(s)) \)

Each arc \((i, j) \in E(s)\) represents the dependence of \(x_j(t+1)\) on \(x_i(t)\).
Deterministic periodic updates

Impact of update modes on cycles

\[ \text{inv}(s) = \{(i, i+1) \mid s(i) < s(i+1)\} \]
Deterministic periodic updates

Impact of update modes on cycles

\[ \text{inv}(s) = \{(i, i+1) | s(i) < s(i+1)\} \]
**Deterministic periodic updates**

**Impact of update modes on cycles**

\[ \text{inv}(s) = \{(i,i+1) \mid s(i) < s(i+1)\} \]

**Theorems (Goles, Noual, 2010)**

- The dynamics induced by two update modes \( s \) and \( s' \) are equal iff \( \text{inv}(s) = \text{inv}(s') \).

\[ \Rightarrow \text{Given a cycle of size } n, \text{ the total number of distinct dynamics induced by block-sequential update modes is:} \]

\[ \sum_{k=0}^{n-1} \binom{n}{k} = 2^n - 1. \]

- \( \text{inv}(s) \neq \text{inv}(s') \implies \) no common limit cycles.

- Iterating a cycle of size \( n \) with an update mode \( s \) with \( |\text{inv}(s)| = k \) corresponds to iterating a cycle of same sign and of size \( n - k \) in parallel.
Deterministic periodic updates

Impact of update modes on cycles

Theorem 6 (Goles, Noual, 2010)

\( \text{inv}(s) \neq \text{inv}(s') \implies \text{no common limit cycles.} \)

Proof

First, let us note that \( \forall i, j \in V, f[j, i] : \begin{cases} f_j \circ f_{j-1} \circ \cdots \circ f_i & \text{if } i \leq j \\ f_j \circ f_{j-1} \circ \cdots \circ f_0 \circ f_{n-1} \circ \cdots \circ f_i & \text{if } i > j \end{cases} \).

Suppose that \( (i, i + 1) \in \text{inv}(s) \setminus \text{inv}(s') \) and that \( \exists x = x^s(t) = x^{s'}(t) \) s.t. \( x^s(t + 1) = x^{s'}(t + 1) \). Then:

\[ x_{i+1}^s(t + 2) = f_{i+1}(x_i^s(t + 2)) = f[i + 1, i^* + 1](x_{i^*}^s(t + 1)), \]

and

\[ x_{i+1}^{s'}(t + 2) = f_{i+1}(x_i^{s'}(t + 2)) = f_{i+1}(x_i^s(t + 1)) = f[i + 1, i^* + 1](x_{i^*}^{s'}(t)), \]

where \( i^* = \max(\{k < i \mid s(k) \geq s(k + 1)\}) \).

By the injectivity of \( f[i + 1, i^* + 1] \), if \( x^s(t + 2) = x^{s'}(t + 2) \) then \( x_{i^*}(t + 1) = x_{i^*}(t) \). Now, if \( x \) belongs to an attractor that is induced identically by both \( s \) and \( s' \), then \( x^s(t) = x^{s'}(t) \) \( \forall t \). As result, in this case, \( \forall t, x_{i^*}^s(t + 1) = x_{i^*}^{s'}(t) = x_{i^*}^s(t) \). In other terms, the state of node \( i^* \) is fixed in the attractor. Hence the states of all nodes are fixed in the attractor which therefore is a fixed point.

\( \square \)
Deterministic periodic updates
Update graphs other related results

Q: Is a labelled graph an update graph?

Labelled graph \((G, \text{lab})\)

Reduced labelled graph \((G, \text{lab})^\oplus\)

Reversed labelled graph \((G, \text{lab})^\ominus_R\)
**Deterministic periodic updates**

**Update graphs other related results**

**Q:** Is a labelled graph an update graph?

<table>
<thead>
<tr>
<th>Labelled graph</th>
<th>Reduced labelled graph</th>
<th>Reversed labelled graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>((G, \text{lab}))</td>
<td>((G, \text{lab})^\oplus)</td>
<td>((G, \text{lab})^\oplus_R)</td>
</tr>
</tbody>
</table>

**Theorem 7 (Aracena et al., 2011)**

A labelled digraph \((G, \text{lab})\) is an update graph iff \((G, \text{lab})^\oplus_R\) does not contain any forbidden cycle.

**Idea**

\[
\begin{align*}
s(1) < s(2) \\
s(1) \geq (s(0) = s(4)) \\
s(3) < (s(0) = s(4)) \\
s(3) \geq s(2)
\end{align*}
\]

\[
\implies s(1) < s(3) \\
s(3) < s(1)
\]
Question: How to find the most compact update mode on $(G, \text{lab})$?

$(G', \text{lab})$

$(G', \text{lab})^\oplus$

$(G', \text{lab})^R_{\oplus}$
Deterministic periodic updates

Update graphs other related results

¿: How to find the most compact update mode on \((G, \text{lab})\)?

\[ (G', \text{lab}) \]

\[ (G', \text{lab})^{\oplus} \]

\[ (G', \text{lab})^{\otimes}_{R} \]

\[ s \equiv (\{0,4\}, \{1,3\}, \{2\}) \]

Algorithm

Init. Take \(G' := (G, \text{lab})^{\otimes}_{R}\) and \(t := 1\).

1. Compute the paths \(P_{\square} = \{P \mid \#(\square \in P) \text{ is max.}\}\) on \(G'\). If \(P_{\square} = \emptyset\), goto (4).

2. The targets \(T\) of the last negative arc of each \(P\) of \(P_{\square}\), and their successors \(S(T)\) are scheduled at time step \(t\). \(t := t + 1\).

3. Remove \(T, S(T)\) and all their incoming arcs from \(G'\), and go back to (1).

4. All the remaining nodes are scheduled all at once, at time step \(t\).
Outline

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Non-deterministic updates

Basic definitions and notations

\( \forall x = (x_0, \ldots, x_{n-1}) \in \mathbb{B}^n, \forall i \in V, \overline{x}^i = (x_0, \ldots, x_{i-1}, \neg x_i, x_{i+1}, \ldots, x_{n-1}) \)

\( \forall x \in \mathbb{B}^n, \forall W = W' \cup \{i\} \subseteq V, \quad \overline{x}^W = (\overline{x}^i)^{W'} = (\overline{x}^{W'})^i \)

The sign of an influence of \( i \) on \( j \) in \( x \) is

\[
\text{sign}_x(i,j) = \frac{f_j(x) - f_j(\overline{x}^i)}{x_i - \overline{x}^i} = s(x_i) \cdot (f_j(x) - f_j(\overline{x}^i)),
\]

where \( s : b \in \mathbb{B} \mapsto b - b \in \{-1, 1\} \).

Given \( x, y \in \mathbb{B}^n, \quad D(x,y) = \{i \in V \mid x_i \neq y_i\} \) and \( d(x,y) = |D(x,y)| \).

\( E(x) = \{(i,j) \in V \times V \mid \text{sign}_x(i,j) \neq 0\} \) represents the set of effective influences of \( G(f) \) in \( x \), which formally means that

\[
\forall i,j \in V, \exists x \in \mathbb{B}^n, f_j(x) \neq f_j(\overline{x}^i) \iff (i,j) \in E.
\]
Non-deterministic updates

Monotonicity, unstabilities and frustrations

A local function $f_i$ is **locally monotonic** in $j$ if either:

\[
\forall x, f_i(x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n-1}) \leq f_i(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n-1})
\]

or:

\[
\forall x, f_i(x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n-1}) \geq f_i(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n-1})
\]

**Example**:

\[
f = \begin{cases} 
    f_0(x) = x_1 \land x_3 \\
    f_1(x) = \neg x_0 \\
    f_2(x) = x_1 \lor x_2 \\
    f_3(x) = \neg x_2 \lor x_3 
\end{cases}
\]

is monotonic.

\[
g = \begin{cases} 
    g_0(x) = x_1 \land x_3 \\
    g_1(x) = \neg x_0 \\
    g_2(x) = x_1 \oplus x_2 \\
    g_3(x) = \neg x_2 \lor x_3 
\end{cases}
\]

is not.
Non-deterministic updates

Monotonicity, unstabilities and frustrations

A local function $f_i$ is **locally monotonic** in $j$ if either:

$$\forall x, f_i(x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n-1}) \leq f_i(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n-1})$$

or:

$$\forall x, f_i(x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n-1}) \geq f_i(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n-1})$$

An automaton $i \in V$ is **unstable** (resp. **stable**) in $x \in \mathbb{B}^n$ if it belongs to the set

$$U(x) = \{i \in V \mid f_i(x) \neq x_i\} \quad (\text{resp. } \overline{U}(x) = V \setminus U(x)).$$

**Example:**

$$f = \begin{cases} f_0(x) = \neg x_1 \\ f_1(x) = x_0 \end{cases}$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f_0(x)$</th>
<th>$f_1(x)$</th>
<th>$U(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>1</td>
<td>0</td>
<td>{0}</td>
</tr>
<tr>
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Non-deterministic updates

Monotonicity, unstabilities and frustrations

A local function $f_i$ is locally monotonic in $j$ if either:

$$\forall x, f_i(x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n-1}) \leq f_i(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n-1})$$

or:

$$\forall x, f_i(x_0, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{n-1}) \geq f_i(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n-1})$$

An automaton $i \in V$ is unstable (resp. stable) in $x \in \mathbb{B}^n$ if it belongs to the set

$$U(x) = \{i \in V \mid f_i(x) \neq x_i\} \quad (\text{resp. } \overline{U}(x) = V \setminus U(x))$$

An influence $(i,j) \in E$ is frustrated in $x$ iff it belongs to

$$\text{FRUS}(x) = \{(i,j) \in E \mid s(x_i) \cdot s(x_j) = -\text{sign}(i,j)\}$$

$$f = \begin{cases} 
  f_0(x) = x_2 \\
  f_1(x) = x_0 \lor \neg x_1 \\
  f_2(x) = \neg x_0 \land x_1 
\end{cases}$$

FRUS(000) = \{(0,2)\}
FRUS(001) = \{(1,2), (2,0)\}
FRUS(010) = \{(0,1), (0,2), (1,2)\}
FRUS(011) = \{(0,1), (2,0)\}
Non-deterministic updates

Relations between unstabilities and frustrations

Remark (Noual, S., 2017)

If \( j \in U(x) \) then \( \exists i \in V^-(j), (i,j) \in \text{FRUS}(x) \).

\[
\begin{align*}
f &= \begin{cases} 
f_0(x) = \neg x_0 \\
f_1(x) = x_0 \lor \neg x_2 \\
f_2(x) = x_1 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{FRUS}(000) &= \{ (0,0), (2,1) \} \\
\text{FRUS}(001) &= \{ (0,0), (1,2) \} \\
\text{FRUS}(110) &= \{ (0,0), (1,2) \} \\
\text{FRUS}(111) &= \{ (0,0), (2,1) \}
\end{align*}
\]

N.B: The reciprocal does not hold.

\[
\begin{align*}
f &= \begin{cases} 
f_0(x) = x_2 \\
f_1(x) = x_0 \lor \neg x_1 \\
f_2(x) = \neg x_0 \land x_1 
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\end{align*}
\]
Non-deterministic updates

Relations between unstabilities and frustrations

Lemma 1 (Noual, S., 2017)
Adding frustrated influences incoming an unstable automaton cannot stabilise it. Formally, noting $V_{\text{FRUS}}(x)(j) = V^-(j) \cap \{i \in V \mid (i, j) \in \text{FRUS}(x)\}$, we have:

$$\forall x, y \in \mathbb{B}^n, j \in U(x) \land \left( V_{\text{FRUS}}(x)(j) \subseteq V_{\text{FRUS}}(y)(j) \right) \implies j \in U(y).$$

Proof
Input provided by $i$ to $j$: $b^i_j(x) = b(\text{sign}(i, j) \cdot s(x_i)) = \begin{cases} x_j & \text{if } (i, j) \notin \text{FRUS}(x) \\ \neg x_j & \text{otherwise} \end{cases}$. By local monotonicity,

$$f_j(x) = \bigwedge_{k \leq m} c_k(x) = \bigwedge_{k \leq m} \left( \bigvee_{i \in V_k^j} b^i_j(x) = \bigvee_{i \in V_k^j} \neg x_j \lor \bigvee_{(i, j) \notin \text{FRUS}(x)} x_j \right),$$

where $V_k^j$ is the set of in-neighbours of $j$ involved in the $k$th clause.

Let $x$ be unstable, admitting thus at least one frustrated incoming influence. Let $y$ be such that it admits at least one more frustrated incoming influence than $x$. Since $f_j$ can be written as a conjunction of disjunctive clauses, the values of these clauses for $y$ are necessarily the same as for $x$. □
Non-deterministic updates

Critical cycles

Let $f$ be a BAN, $G = (V, E)$ its interaction graph, and $x$ a configuration in $\mathbb{B}^n$. A cycle $C = (V_C, E_C)$ of $G$ is $x$-critical if $E_C \subseteq \text{FRUS}(x)$. A cycle $C$ is critical if it is $x$-critical for some $x$.

Proposition 1 (Noual, S., 2017)

A critical cycle is a NOPE-cycle, i.e. negative of odd length or positive of even length.

Proof

Let $x \in \mathbb{B}^n$. By definition of frustrated influences, if $C = (V_C, E_C)$ is $x$-critical, has length $\ell$ and sign $s$ then:

$$\prod_{(i,j) \in E_C} -\text{sign}(i,j) = (-1)^\ell \times s = \prod_{(i,j) \in E_C} s(x_i) \cdot s(x_j) = 1.$$
Non-deterministic updates

Transitions and trajectories

<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asynchronous</td>
<td>$x \rightarrow y$</td>
<td>$d(x, y) \leq 1$</td>
</tr>
<tr>
<td>Synchronous</td>
<td>$x \rightarrow y$</td>
<td>$d(x, y) &gt; 1$</td>
</tr>
<tr>
<td>Elementary</td>
<td>$x \rightarrow y$</td>
<td>$x \rightarrow y \in {x \rightarrow y} \cup {x \rightarrow y}$</td>
</tr>
<tr>
<td>Non-sequentialisable</td>
<td>$x \rightarrow y$</td>
<td>$x \rightarrow y$ not decomposable into smaller elementary transitions</td>
</tr>
</tbody>
</table>

For all $x, y \in \mathbb{B}^n$ s.t. $x \neq y$, $x$ is willing (resp. unwilling) towards $y$ if $D(x, y) \subseteq U(x)$ (resp. $D(x, y) \cap U(x) = \emptyset$).

A trajectory from $x$ to $y$ is a path $x \rightarrow \ldots \rightarrow y$ in the transition graph.

Let $x = x(0) \rightarrow x(1) \rightarrow \ldots \rightarrow x(m-1) \rightarrow y = x(m)$ be a trajectory from $x$ to $y$. If $\forall t < m$, $D(x(t+1), y) \subsetneq D(x(t), y)$, this trajectory is direct. It performs no reversed changes, i.e. $\forall t < m$, $x(t)_i = y_i \implies \forall t < t' \leq m$, $x(t')_i = y_i$. 
Proposition 2 (Noual, S., 2017)

Let \( x \) a willing configuration towards \( y \).

1. If there are no asynchronous trajectories from \( x \) to \( y \), then \( D(x,y) \) induces a NOPE-cycle that is \( x \)-critical.

2. If \( D(x,y) \) does not induce an \( x \)-critical cycle, then there is a direct asynchronous trajectory from \( x \) to \( y \).
Proposition 2 (Noual, S., 2017)

Let $x$ a willing configuration towards $y$.

1. If there are no asynchronous trajectories from $x$ to $y$, then $D(x,y)$ induces a NOPE-cycle that is $x$-critical.

2. If $D(x,y)$ does not induce an $x$-critical cycle, then there is a direct asynchronous trajectory from $x$ to $y$. 

$$
\begin{array}{cccc}
\text{x} & f_0(x) & f_1(x) & \text{U}(x) \\
(0,0) & 1 & 1 & \{0,1\} \\
(0,1) & 0 & 0 & \{1\} \\
(1,0) & 1 & 1 & \{1\} \\
(1,1) & 1 & 1 & \emptyset \\
\end{array}
$$

$$
f = \begin{cases} 
  f_0(x) = x_0 \lor \neg x_1 \\
  f_1(x) = x_0 \lor \neg x_1 
\end{cases}
$$
Proposition 2 (Noual, S., 2017)

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2. If $D(x, y)$ does not induce an $x$-critical cycle, then there is a direct asynchronous trajectory from $x$ to $y$.

Implication

When $m$ local changes are possible in $x$, then, unless there is a NOPE-cycle of size $m$, these $m$ changes can be made asynchronously without risking a deadlock, i.e. a situation in which some transitions would have transformed $x$ into a configuration $x(t)$ from which $y$ is not reachable anymore.
Non-deterministic updates

Results relating trajectories and critical cycles

Proposition 2 (Noual, S., 2017)

Let $x$ a willing configuration towards $y$.

1. If there are no asynchronous trajectories from $x$ to $y$, then $D(x, y)$ induces a NOPE-cycle that is $x$-critical.

2. If $D(x, y)$ does not induce an $x$-critical cycle, then there is a direct asynchronous trajectory from $x$ to $y$.

Corollary 1 (Noual, S., 2017)

If $x \rightarrow y$ exists, then $D(x, y)$ induces a NOPE-cycle which is $x$-critical.

Implication

In a BAN with no NOPE-cycles of size smaller or equal than $m \in \mathbb{N}$, any synchronous change affecting no more than $m$ automata states can be totally sequentialised.
Non-deterministic updates

Structural sensitivity: impact of synchronism

<table>
<thead>
<tr>
<th>Class N</th>
<th>Class F</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;null&quot; sensitivity</td>
<td>&quot;weak&quot; sensitivity</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Class G</th>
<th>Class D</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;medium&quot; sensitivity</td>
<td>&quot;strong&quot; sensitivity</td>
</tr>
</tbody>
</table>

Sylvain Sené

Synchronism vs asynchronism in BANs
Theorem 8 (Noual, S., 2017)

1) Synchronism-sensitivity requires the existence of a NOPE-cycle.

2) Significant sensitivity requires the existence of a NOPE-cycle of length strictly smaller than the BAN size as well as of a negative cycle.

3) In the absence of a Hamiltonian NOPE-cycle and positive loops on all automata, little sensitivity also requires a NOPE-cycle of length strictly smaller than the BAN size.

A monotonic BAN belonging to sensitivity class $D$: 

\[
\begin{align*}
  f_0(x) &= x_2 \vee (x_0 \land \neg x_1) \\
  f_1(x) &= x_3 \vee (\neg x_0 \land x_1) \\
  f_2(x) &= \neg x_0 \land x_1 \\
  f_3(x) &= x_0 \land \neg x_1
\end{align*}
\]
Non-deterministic updates

Structural sensitivity: main result

Theorem 8 (Noual, S., 2017)

1) Synchronism-sensitivity requires the existence of a NOPE-cycle.
2) Significant sensitivity requires the existence of a NOPE-cycle of length strictly smaller than the BAN size as well as of a negative cycle.
3) In the absence of a Hamiltonian NOPE-cycle and positive loops on all automata, little sensitivity also requires a NOPE-cycle of length strictly smaller than the BAN size.

A monotonic BAN belonging to sensitivity class $D$:

\[ \{ x \in \mathbb{B}^4 \mid x_0 \lor x_1 = 1 \} \]

asynchronous limit cycle

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fixed point
Non-deterministic updates

Class $\mathcal{D}$ and local (non-)monotonicity

\( \mathcal{Q} \): How are these two BANs related?

\[
\begin{align*}
  f &= \begin{cases} 
  f_0(x) &= x_2 \lor (x_0 \land \neg x_1) \\
  f_1(x) &= x_3 \lor (\neg x_0 \land x_1) \\
  f_2(x) &= \neg x_0 \land x_1 \\
  f_3(x) &= x_0 \land \neg x_1 
  \end{cases} \\
  g &= \begin{cases} 
  g_0(x) &= x_0 \oplus x_1 \\
  g_1(x) &= x_0 \oplus x_1 
  \end{cases}
\end{align*}
\]

(S., 2012)
References


