Classical Logic

II. First Order Logic

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1 The language of classical predicate logic CPL

The language of classical predicate logic (CPL), $\mathcal{L}$ consists of

- individual variables x, y,...
- ω-many m_i - place function symbols, F_i for $i \in \omega$
- ω-many m_j - place predicate symbols P_j for $j \in \omega$
- the logical symbols $\neg$, $\vee$,$\exists$
- the equality symbol $=$

Terms and formulas are defined as usual, and if ϕ and ψ are formulas and x is an individual variable then $\neg\phi$, $\phi \vee \psi$, $\exists x\phi$ are formulas. $\wedge$, $\rightarrow$, ... are defined as in propositional logic and $\forall x\phi \equiv \neg\exists x\neg\phi$.

If ϕ is a formula containing a term t and t' is a term, then we denote by $\phi_t[t']$ the formula obtained from ϕ by replacing every free occurrence of t by t'. If t' is the name of an element of a set $\mathcal{O}$ then $\phi_t[t']$ is called $\mathcal{O}$-instance of ϕ. In order to omit parentheses, we assume that the one-place operators bind closer that the two place operators and $\wedge$, $\vee$ bind closer than $\rightarrow$, (i.e. $\phi \wedge \psi \rightarrow \rho \vee \sigma$ is $(\phi \wedge \psi) \rightarrow (\rho \vee \sigma)$, and a sequence of $\rightarrow$ is parenthesized to the right (i.e. $\phi \rightarrow \psi \rightarrow \rho \rightarrow \sigma$ is $\phi \rightarrow (\psi \rightarrow (\rho \rightarrow \sigma))$. We call atomic formula or atom any positive literal containing no free variables (i.e. containing only constants or terms containing only constants).

2 Axioms and inference rules

Axiom 1 $\phi \vee \neg\phi$

Axiom 2 $\phi[c] \rightarrow \exists x\phi_c[x]$

Axiom 3 *identity:* $t = t$

Axiom 4 *equality:* $x_1 = y_1 \wedge \ldots x_n = y_n \rightarrow f(x_1, x_2, \ldots, x_n) = f(y_1, y_2, \ldots, y_n)$

Axiom 5 $x_1 = y_1 \wedge \ldots x_n = y_m \rightarrow p(x_1, x_2, \ldots, x_m) \rightarrow p(y_1, y_2, \ldots, y_m)$

Rule 1 *Expansion* $\phi : \phi \vee \psi$

Rule 2 *Contraction* $\phi \vee \phi : \phi$

Rule 3 *Associativity* $\phi \vee (\psi \vee \chi) : (\phi \vee \psi) \vee \chi$

Rule 4 *Cut* $\phi \vee \psi, \chi \vee \neg\psi : \phi \vee \chi$

Rule 5 $\exists$ *- introduction :* $\phi \rightarrow \psi$ *:* $(\exists x \phi) \rightarrow \psi$ *provided that* x *does not occur unbound in* ψ*.*

Every instance of an axiom is a classical theorem and if A and B are classical theorems and C is obtained from them by application of one of the rules, C is a classical theorem. We write $\vdash \phi$ if ϕ is a theorem. When we introduce a set of non-logical axioms $\mathcal{M}$ from which ϕ can be deduced then we write $\mathcal{M} \vdash \phi$.

3 Theorems and derivation rules which can be proven from the above calculus

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Furthermore, we shall often use theorems or derivation rules proven in classical predicate logic as in [2], for example.

4 Semantical characterization of classical logic

4.1 Classical structures

Definition 1 *The tuple* $\mathcal{A} = (\mathcal{O}, (f_i)_{i\in\omega}, (p_j)_{j\in\omega}, =)$*,* ω *being the set of natural numbers, is a classical structure, where*

- $\mathcal{O} \neq \emptyset$ *is the set of objects*
- $f_i : \mathcal{O}^{n_i} \rightarrow \mathcal{O}$ *is an* n_i*-place function on* $\mathcal{O}$*, for* $n_i \in \omega$
- $p_j \subseteq \mathcal{O}^{m_j}$ *is an* m_j*-place predicate on* $\mathcal{O}$ *for* $m_j \in \omega$*.*

4.2 Definition of a truth function T

We define by recursion a function T which assigns elements of $\mathcal{O}$ to terms and truth values $\{\top, \bot\}$ to formulas of $\mathcal{L}$. Let t be a term without variables.

- If t is the name of an $a \in \mathcal{O}$, then $T(t) = a$.
- If t is not a name then it has the form $F_i(t_1 \ldots t_{n_i})$ since it is devoid of variables, where $\cdot_1, \cdot_2, \ldots, \cdot_T$hen $T(F_i(t_1 \ldots t_{n_i})) = f_i(T(t_1), \ldots T(t_{n_i}))$

Let A be a closed formula of $\mathcal{L}$, and t_1, t_2 be terms. Then

- $T(t_1 = t_2) = \top$ iff $T(t_1) = T(t_2)$
- $T(P_j(t_1, \ldots t_{m_j})) = \top$ iff $(T(t_1), \ldots T(t_{m_j})) \in p_j$
- $T(\neg\phi) = \top$ iff $T(\phi) = \bot$
- $T(\phi \vee \psi) = \top$ iff $T(\phi) = \top$ or $T(\psi) = \top$
- $T(\exists x \phi) = \top$ iff there is $c \in O$ such that $T(\phi_x[c]) = \top$

A formula ϕ is called valid in a structure $\mathcal{M}$ iff $T(\phi') = \top$ for every $\mathcal{O}$-instance ϕ' of ϕ. This is denoted by $\mathcal{M} \models \phi$. A formula ϕ is called valid iff ϕ is valid in every classical structure. This is denoted by $\models \phi$.

5 Soundness

Theorem 1 *If $\vdash \phi$ then ϕ is valid.*
We show that all axioms are valid and that Proof Axiom 1: $T(A \vee \neg A) = \top$ iff $T(A) = \top$ or $T(\neg A) = \top$. In the first case, we are finished, in the second case we have $T(\phi) = \bot$ according to the
Axiom 2:

Q.E.D.

6 Completeness

In this part we will show that every valid formula is a theorem of. Our proof is along the same lines as Henkin's completeness proof [1].

Definition 2 *A set of formulas, s, is called inconsistent if it has a finite subset $\{\phi_1, \phi_2, \ldots, \phi_k\}$ with $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k$. Otherwise s is called consistent.*

Let s be a set of formulas. Then we denote by $V(s)$ the set of all terms occurring in formulas of s. We denote by $P(s)$ the set of all formulas of $\mathcal{L}(\mathcal{K})$ containing only nonlogical symbols (terms, function and predicate symbols) of formulas of s.

Definition 3 *A set of formulas s is called complete (or complete) if*

- *s is consistent.*
- *s is maximal, i.e. for all ϕ in $P(s)$ holds: if $\phi \notin s$ then $s \cup \{\phi\}$ is inconsistent.*
- *s is saturated, i.e. for every existential formula $\exists x\phi \in s$ there is a formula $\phi_x[c] \in s$ for some constant $c \in V(s)$.*

Lemma 1 *Every consistent set of formulas can be extended to a complete set of formulas. Proof analogous to that by Henkin-Hasenjäger.*

Let s be a consistent set of formulas and let a_1, a_2, ... be a sequence of 'new' object variables not in $V(s)$. We define $P^\star(s) = P(s) \cup \{j : j$ is a formula with variables from $a_1, a_2 \ldots\}$. Let $\{\exists x_i \phi_i(x_i)\}_{i=1,\ldots}$ be an enumeration of the existential formulas of P(s). Then we form a set of formulas $s^\sim$, by adding to s a set of formulas $\exists x_i \phi_i(x_i) \rightarrow \phi(a_i)$ for every existential formula of $P(s)$. $s^\sim$ is consistent : assume for the contrary that there are formulas $\phi_1, \phi_2, \ldots, \phi_k$ in s, such that $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg(\neg\exists x_{i_0}\phi_{i_0}(x_{i_0}) \vee \phi_{i_0}(a_{i_0}))$*

This gives
(1) $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \exists x_{i_0}\phi_{i_0}(x_{i_0})$ and
(2) $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\phi_{i_0}(a_{i_0})$
From (2), we obtain
(3) $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \forall x\neg\phi_{i_0}(x)$and hence
(4) $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\exists x\phi_{i_0}(x)$
(1) and (4) imply $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k$, which contradicts the consistency of s.
$s^\sim$ is extended to $s^\star$ in the following way: let Θ be the set of all consistent formula sets, containing $s^\star$: $\Theta = \{s' : s \subset s'$ and s' consistent$\}$ Let H be a chain in Θ, i.e. $H \subset \Theta$ and s_1, $s_2 \in H$ implies $s_1 \subset s_2$ or $s_2 \subset s_1$. Then $\bigcup H$ is an upper bound of H in Θ, since $s \subset \bigcup H$ for all $s \in H$. $\bigcup H$ is consistent: if not, assume that there are formulas $\phi_1, \phi_2, \ldots, \phi_k \in \bigcup H$ such that $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k$. Then there are $s_1, s_2, \ldots, s_k \in H$ and $\phi_i \in s_i$ for $1 \leq i \leq k$. There is $s_{k_0} \in \{s_1, s_2, \ldots, s_k\}$, such that $s_i \subset s_{k_0}$ for $1 \leq i \leq k$, hence $\phi_i \in s_{k_0}$ for $1 \leq i \leq k$, which contradicts the consistency of s_{k_0}. Therefore $\bigcup H \in \Theta$ (since $\bigcup H$ is consistent). $\bigcup H$ is an upper bound of H and $\bigcup H \in \Theta$. By Zorn's Lemma Θ has a maximal element $s^\star$, i.e. for all $s' \in \Theta$: $s^\star \subset s'$ entails $s' = s^\star$. $s^\star$ is complete:

- *$s^\star$ is consistent, because $s^\star \in \Theta$*
- *$s^\star$ is maximal: let $\psi \notin s^\star$. Then $s^\star \cup \{\psi\} \notin \Theta$ because $s^\star$ is a maximal element in Θ. Therefore, from the definition of Θ, $s^\star \cup \{\psi\}$ is not consistent.*
- *$s^\star$ is saturated: let $\exists x\phi \in s^\star$. From the definition of $s^\sim$ and because of $s^\sim \subset s^\star$, we have $(\exists x\phi(x) \rightarrow \phi(a)) \in s^\star$. Hence $(\neg\exists x\phi(x)) \notin s^\star$ and $\neg(\exists x\phi(x) \rightarrow \phi(a)) \notin$*

$s^\star$ because $s^\star$ is consistent. Assume now that $\phi(a) \notin s^\star$. Then $s^\star \cup \{\phi(a)\}$ is inconsistent, i.e. there are formulas $\phi_1, \phi_2, \ldots, \phi_k \in s^\star$, such that
(1) $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\phi(a)$.
By the same reasoning, there are formulas $\psi_1, \psi_2, \ldots, \psi_m \in s^\star$ and $\chi_1, \chi_2, \ldots, \chi_n \in s^\star$, such that
(2) $\vdash \neg\psi_1 \vee \ldots \vee \neg\psi_m \vee \neg\neg\exists x\phi$ and
(3) $\vdash \neg\chi_1 \vee \ldots \vee \neg\chi_n \vee \neg\neg(\exists x\phi(x) \rightarrow \phi(a))$.
From (1) and (2) and (3) it follows easily
(4) $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\psi_1 \vee \ldots \vee \neg\psi_m \vee \neg\chi\chi_1 \vee \ldots \vee \neg\chi\chi_n$
which contradicts the consistency of $s^\star$.

Q.E.D.

Lemma 2 *Let s be a complete set of formulas. Then*

1. *If $\phi \in s$ then $\neg\phi \notin s$*
2. *If $\phi \in P(s)$ and $\phi \notin s then \neg\phi \in s$*
3. *If $\phi \in s$ and $\psi \in P(s)$ and $\vdash \phi \rightarrow \psi$ then $\psi \in s$*
4. *If $\phi \vee \psi \in s$ then $\phi \in s$ or $\psi \in s$*
5. *If $\phi \vee \psi \in P(s)$ and $\phi \in s$ or $\psi \in s$ then $\phi \vee \psi \in s$*
6. *If $\phi[a] \in s$ then $\exists x\phi \in s$*
7. *If $\exists x\phi \in s$ then there is $a \in V(s)$ such that $\phi[a] \in s$*
8. *$t = t \in s$ for every term $t \in V(s)$*
9. *If $t_1 = t_1' \in s, \ldots t_n = t_n' \in s$ and $F(t_1, ..., t_n) \in V(s)$ and $F(t_1', t_2', \ldots, t_n') \in V(s)$ then $F(t_1, t_2, \ldots, t_n) = F(t_1', t_2', \ldots, t_n') \in s$*
10. *If $t_1 = t_1' \in s, \ldots t_n = t_n' \in s$ and $P(t_1, t_2, \ldots, t_n) \in s$ and $P(t_1', t_2', \ldots, t_n') \in P(s)$ then $P(t_1', t_2', \ldots, t_n') \in s$*

Proof

1. *Since s is consistent and $\vdash \neg\phi \vee \neg\neg\phi$.*
2. *$s \cup \{\phi\}$ is inconsistent. Hence there are formulas $\phi_1, \phi_2, \ldots, \phi_k \in s$ and $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\phi$. Assume $\neg\phi \notin s$. Then $s \cup \{\neg\phi\}$ is inconsistent and there are formulas $\psi_1, \psi_2, \ldots, \psi_k \in s$ such that $\vdash \neg\psi_1 \vee \ldots \vee \neg\psi_m \vee \neg\neg\phi$. It follows that $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\psi_1 \vee \ldots \vee \psi_m$, which contradicts the consistency of s.*
3. *is proven along the same lines as 2.*
4. *is proven along the same lines as 2*
5. *is proven along the same lines as 2*
6. *holds because s is saturated and from $\vdash \phi[a] \rightarrow \exists x\phi[x]$ and from 3.*
7. *holds by the definition of complete set (saturated)*
8. *Assume for the contrary that $t = t \notin s$. Then there are formulas $\phi_1, \phi_2, \ldots, \phi_k \in s$ such that $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg(t = t)$. But $\vdash t = t$ by (identity axiom). Hence $\vdash \neg\phi\phi_1 \vee \ldots \vee \neg\phi\phi_k$ from the cut rule contradicting the consistency of s.*

9. *Proof as in 7. by the equality axiom* $t_1 = t'_1 \ldots t_n = t'_n \to F(t_1, t_2, \ldots, t_n) = F(t'_1, t'_2, \ldots, t'_n)$ *of classical logic.*
10. *Proof as in 7. by the equality axiom* $t_1 = t'_1 \ldots t_n = t'_n \to P(t_1, t_2, \ldots, t_n) \to P(t'_1, t'_2, \ldots, t'_n)$.

Q.E.D.

We are now in a position to define the canonical model. The objects will be the equivalence classes according to the identity of terms, $=$.

Definition 4 *Let* S *be a complete formula set. Let* t_0, $t_1 \in V(s)$. *Then* $t_0 \cong t_1$ *iff*$_{def}$ $(t_0 = t_1) \in s$.

Lemma 3 $\cong$ *is an equivalence relation. Proof:*

a) $\cong$ *is reflexive by 2,9.*
b) $\cong$ *is symmetric by* $\vdash t = t' \wedge t = t" \to t' = t"$ *equality axiom.* $\vdash t = t' \wedge t = t \to t' = t$ *replace* $t"$ *by* t. *Since* $t = t \in s$ *by Lemma 2.9., we get* $t = t' \in s$ *entails* $t' = t \in s$ *by Lemma 2.11.*
c) $\cong$ *is transitive* $\vdash t' = t" \wedge t = t' \to t = t"$ $t' = t" \in s$ *and* $t = t' \in s$ *entails* $t = t" \in s$ *by Lemma 2,3.*

Q.E.D.

We denote by $[t]$ the $\cong$-equivalence class of t.

Given a complete set of formulas s, we define a the canonical structure $\mathcal{A} =$ as the classical structure $(O, (f_i), (p_j), =)$, where

(1) O is the set of $\cong$ - equivalence classes.
(2) $f_i([t_1], [t_2], \ldots, [t_{n_i}]) = [F_i(t_1, t_2, \ldots, t_{n_i})]$
(3) $([t_1], [t_2], \ldots, [t_{m_j}]) \in p_j$ iff $P_j(t_1, t_2, \ldots, t_{m_j}) \in s$
(4) $=$ is the equality of the equivalence classes.

The definition of the functions f_i and the predicates p_j depends only on the $\cong$ - classes and not on the terms, i.e.

> $[t_0] = [t'_0] \ldots$ and $[tn_i] = [t'n_i]$ entails
> $f_i([t_1], [t_2], \ldots, [t_{n_i}]) = f_i([t'_1], [t'_2], \ldots, [t'_{n_i}])$ by Lemma 2, 10.
> $[t_1] = [t'_1]$,... and $[t_{m_j}] = [t'_{m_j}]$ entails
> $([t_1], [t_2], \ldots, [t_{m_j}]) \in p_j)$ iff $([t'_1], [t'_2], \ldots, [t'_{m_j}]) \in p_j$ by Lemma 2,11.

Definition of the truth value function T:

For every term t we set $T(t) = [t]$. The truth value for a formula is given inductively by the definition of functions and predicates.

Lemma 4 *For every closed formula $\phi \in P(s)$, we have $T(\phi) = \top$ iff $\phi \in s$. Proof by induction over formulas:*

1. *$T(t = t') = \top$ iff $T(t) = T(t')$ iff $[t] = [t']$ iff $t = t' \in s$ by definition of $\cong$.*
2. *$T(P_j(t_1, t_2, \ldots, t_{m_j})) = \top$ if $(T(t_1), \ldots T(t_{m_j})) \in p_j$ iff $([t_1], [t_2], \ldots, [t_{m_j}]) \in p_j$ if $p_j(t_1, t_2, \ldots, t_{m_j}) \in s$.*
3. *$T(\neg\psi) = \top$ iff $T(\psi) = \bot$ if $\psi \notin s$ by induction hypothesis iff $\neg\psi \notin s$ from Lemma 2,1. and 2.*
4. *$T(\psi \vee \chi) = t$ if $T(\psi) = \top$ or $T(\chi) = \top$ iff $\psi \in s$ or $\chi \in s$ by induction hypothesis if $\psi \vee \chi \in s$ by Lemma 2, 4. and 5.*
5. *$T(\exists x\psi) = \top$ if there is $c \in O$ such that $T(\psi_x[c]) = t$ if $\psi_x[c] \in s$ by induction hypothesis iff $\exists x\psi \in s$ (from Lemma 2,6 and 7.*

Q.E.D.

Theorem 2 *Theorem 2. Completeness theorem. Every valid formula is deducible. Proof Assume for the contrary that there is a valid formula ϕ which is not deducible. Then, by definition 2, $\{\neg\phi\}$ is a consistent set of formulae. $\{\neg\phi\}$ can be extended to a complete set of formulae s, by Lemma 1, where $\neg\phi \in s$. By Lemma 4, $T(\neg\phi) = \top$ and hence $T(\phi) = \bot$, which contradicts the validity of ϕ.* Q.E.D.

References

1. L.Henkin. The completeness of the first-order functional calculus. *Journal of Symbolic Logic*, (14):159–166, 1949.
2. J. R. Shoenfield. *Mathematical Logic*. Addison - Wesley Publishing Company, Mass., 1967.