Modal Logic

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In this chapter, we present logics with modal operators. When we want to describe a modal relation, we first will fix the properties this relation should have. For example, we might want that the relation be linear or transitive. According to possible properties of the binary tense relation, different axiomatics may be defined. In this chapter, we first present in detail the "smallest" modal system, we can think of: the modal relation does not have any particular property. Subsequently, we present some results on modal systems where the transition relation has additional properties.

1 The language of modal logic

The language of modal logic, $\mathcal{L}$, is an extension of the language of classical propositional logic. $\mathcal{L}$ consists of

- a set ATM of popositional variables
- the logical symbols $\neg$, $\vee$
- the modal operators $\Box$ and $\Diamond$

Terms and formulas are defined as usual, and if ϕ is a formula then so are $\Box\phi$ $\Diamond\phi$. $\wedge$, $\rightarrow$, $\Diamond$ are defined as in classical logic.

$\Diamond$ is defined in terms of $\Box$:

$\Diamond\phi \leftrightarrow \neg\Box\neg\phi$

In order to omit parentheses, we assume that the one-place operators bind closer that the two place operators and $\wedge$, $\vee$ bind closer than $\rightarrow$, (i.e. $\phi \wedge \psi \rightarrow \rho \vee \sigma$ is $(\phi \wedge \psi) \rightarrow$

$(\rho \vee \sigma)$, and a sequence of $\rightarrow$ is parenthesized to the right (i.e. $\phi \rightarrow \psi \rightarrow \rho \rightarrow \sigma$ is $\phi \rightarrow (\psi \rightarrow (\rho \rightarrow \sigma))$. We call atomic formula or atom any positive literal containing no free variables (i.e. containing only constants or terms containing only constants).

2 Axioms and inference rules of modal logic

According to the properties one wants to give to the transition (e.g. transitiveness, symmetry, linearity, etc.) there are different axiom systems for modal logic. We will subsequently give the axioms of the system $\mathcal{K}$ of minimal modal logic. Minimal means here that the modal relation has no special property, the set of axioms for $\mathcal{K}$ will be shown to be true in any modal structure.

Axiom 1 *all valid formulas of classical propositional logic*

Axiom 2 *(K)* $\Box(\phi \rightarrow \psi) \rightarrow \Box\phi \rightarrow \Box\psi$

Rule 1 *all valid inference rules of classical predicate logic*

Rule 2 $\Box$ *- introduction :* ϕ *:* $\Box\phi$

We write $\vdash_{\mathcal{K}} \phi$ (or $\vdash \phi$, when no confusion can occur) if ϕ is a theorem of $\mathcal{K}$. When we introduce a set of non-logical axioms $\mathcal{M}$ from which ϕ can be deduced then we write $\mathcal{M} \vdash \phi$.

One might object that $\mathcal{K}$ does not include one axiom $\Box\phi \rightarrow \neg\Box\neg\phi$ which intuitively appears to be true. But $\Box\phi \rightarrow \neg\Box\neg\phi$ is not true in a modal structure which has ending (resp. beginning) points.

3 Theorems and derivation rules which can be proven from the above calculus

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D1 from $\phi \rightarrow \psi$ infer

(a) $\Box\phi \rightarrow \Box\psi$ b

(b) $\Diamond\phi \rightarrow \Diamond\psi$

Proof of 1:

(1) $\Box(\phi \to \psi)$ by R1
(2) $\Box\phi \to \Box\psi$ from (1) by axiom 1 and R0

Proof of 2.b:

(1) $\neg\psi \to \neg\phi$
(2) $\Box\neg\psi \to \Box\neg\phi$ by D1 a
(3) $\neg\Box\neg\phi \to \neg\Box\neg\psi$ by R0.

D2 from $\phi \to \psi$ infer $\heartsuit\phi \to \heartsuit\psi$, where $\heartsuit$ is any sequence of unary operators $\Box$ and $\neg$. Proof : immediately from D1.

T1 $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$

Proof :

(1) $\phi \wedge \psi \to \phi$ by AO
(2) $\Box(\phi \wedge \psi) \to \Box\phi$ by D1
(3) $\Box(\phi \wedge \psi) \to \Box\psi$ analogous (2)
(4) $\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$ (2) and (3)
(5) $\phi \to (\psi \to \phi \wedge \psi)$ AO
(6) $\Box\phi \to \Box(\psi \to \phi \wedge \psi)$ by D1
(7) $\Box(\psi \to \phi \wedge \psi) \to \Box\psi \to \Box(\phi \wedge \psi)$
(8) $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$ (6) and (7)

T2 $\Diamond(\phi \wedge \psi) \to \Diamond\phi$

Proof : by AO, D1. 2.

T4 $\Box(\phi \to \psi) \wedge \Diamond\phi \to \Diamond\psi$

Proof :

(1) $\neg\psi \wedge (\neg\psi \to \neg\phi) \to \neg\phi$
(2) $\Box(\neg\psi \wedge (\neg\psi \to \neg\phi)) \to \Box\neg\phi$ from (1) by D1.a.
(3) $\Box\neg\psi \wedge \Box(\neg\psi \to \neg\phi) \to \Box\neg\phi$ from (2) by T1.a.
(4) $\Box(\neg\psi \to \neg\phi) \wedge \neg\Box\neg\phi \to \neg\Box\neg\psi$ from (3) by R0.
(5) $\Box(\phi \to \psi) \wedge \Diamond\phi \to \Diamond\psi$ from (4) by contraposition.

T5 $\Diamond(\phi \to \psi) \wedge \Box\phi \to \Diamond\psi$

Proof :

(1) $\phi \wedge \neg\psi \to \neg(\phi \to \psi)$
(2) $\Box(\phi \wedge \neg\psi) \to \Box\neg(\phi \to \psi)$ from (1) by D1.a.
(3) $\Box\phi \wedge \Box\neg\psi \to \Box\neg(\phi \to \psi)$ from (2) by T1.a.
(4) $\neg\Box\neg(\phi \to \psi) \wedge \Box\phi \to \neg\Box\neg\psi$ from (3) by R0.

Furthermore, we shall often use theorems or derivation rules proven in classical predicate logic as in [4], for example.

4 Semantical characterization of the modal logic $\mathcal{K}$

Modal structures for $\mathcal{K}$ ($\mathcal{K}$- structures) are Kripke structures [1], i.e. sets of classical structures.

Definition 1 *The pair (S, R) is called modal frame, if*

- *$S \neq \emptyset$ is a set of states*
- *$R \subseteq \mathcal{S} \times \mathcal{S}$ is a binary relation on $\mathcal{S}$, called state transition relation*

We set $R(s) = \{s' : (s, s') \in R\}$.

Definition 2 *The tuple $(\mathcal{S}, R, v)$ is called modal interpretation (or modal structure), if*

- *$(\mathcal{S}, R)$ is a modal frame*
- *$v : \mathcal{S} \times ATM \longrightarrow \{\top, \bot\}$ is a valuation function*

v is extended to formulas by

- $v(s, \neg\phi) = \top$ iff $v(\phi, s) = \bot$
- $v(s, \phi \wedge \psi) = \top$ iff $v(\phi, s) = \top$ and $v(\psi, s) = \top$
- $v(s, \Box\phi) = \top$ iff for all $s' \in R(s), v(s', \phi) = \top$

Given a modal interpretation $\mathcal{M} = (\mathcal{S}, R, v)$.

A formula ϕ is called valid in state $s \in \mathcal{S}$ of a modal interpretation $\mathcal{M} = (\mathcal{S}, R, v)$ iff $v(s, \phi) = \top$. This is denoted by $\mathcal{M}, s \models \phi$. A formula ϕ is called valid in $\mathcal{M}$ iff ϕ is valid in every $s \in \mathcal{S}$. We denote that by $\mathcal{M} \models_{\mathcal{K}} \phi$. A formula phi is called valid wrt a modal frame (S, R) iff for every valuation v, ϕ is valid in (S, R, v). A formula ϕ is called valid iff ϕ is valid in every modal interpretation. This is denoted by $\models_{\mathcal{K}} \phi$.

5 Soundness

Theorem 1 *If $\vdash_{\mathcal{K}} \phi$ then ϕ is $\mathcal{K}$- valid.*
Proof: Axiom 1: All classically valid formulas are $\mathcal{K}$- valid, the definition of the truth function for the classical connectives being the same as in classical logic.
Axiom 2: Assume there is a $\mathcal{K}$- structure A where $v(s, \Box(\phi \to \psi) \to \Box\phi \to \Box\psi) = f$

for some state s. Then $v(s, \Box(\phi \to \psi)) = v(s, \Box\phi) = f$ and $v(s, \Box\psi) = f$, i.e. $v(s', \phi \to \psi) = v(s', \phi) = t$ for all s' with sRs' and there is s'' with sRs'' such that $v(s", \psi) = f$. This is a contradiction, since from $v(s', \phi \to \psi) = v(s', \phi) = t$ follows that $v(s', \psi) = t$.
Rule 1 is as in predicate logic and hence produces only valid formulae.
Rule 2: If ϕ is a theorem of $\mathcal{K}$ then by induction assumption ϕ is valid, i.e. ϕ is true in every state of every modal structure, hence $\Box\phi$ is true in every state of every $\mathcal{K}$-structure, i.e. $\Box\phi$ is a theorem of $\mathcal{K}$. Q.E.D.

6 Completeness

In this part we will show that every $\mathcal{K}$ - valid formula is a theorem of $\mathcal{K}$. Our proof is based on Henkin's completeness proof for classical logic [2] and is along the same lines as [3].

Given a set of formulas s, we denote by $P(s)$ the set of all formulas with atoms from formulas in s.

Definition 3 *A set of formulas, s, is called $\mathcal{K}$ -inconsistent if it has a finite subset $\{\phi_1, \phi_2, \ldots, \phi_k\}$ with $\vdash_{\mathcal{K}} \neg\phi_1 \vee \ldots \vee \neg\phi_k$. Otherwise s is called $\mathcal{K}$ - consistent (or consistent).*

Definition 4 *A set of formulas s is called complete (or $\mathcal{K}$ -complete) if*

- *s is $\mathcal{K}$ - consistent.*
- *s is maximal, i.e. for all ϕ in $P(s)$ holds: if $\phi \notin s$ then $s \cup \{\phi\}$ is inconsistent.*

Lemma 1 *Every consistent set of formulas can be extended to a complete set of formulas. Proof: analogous to that by Henkin-Hasenjäger.*

Let s be a consistent set of formulas s is extended to a complete set $s^\star$ in the following way: let Θ be the set of all consistent formula sets, containing s: $\Theta = \{s' : s \subset s'$ and s' consistent$\}$ Let H be a chain in Θ, i.e. $H \subset \Theta$ and s_1, $s_2 \in H$ implies $s_1 \subset s_2$ or $s_2 \subset s_1$. Then $\bigcup H$ is an upper bound of H in Θ, since $s \subset \bigcup H$ for all $s \in H$. $\bigcup H$ is consistent: if not, assume that there are formulas $\phi_1, \phi_2, \ldots, \phi_k \in \bigcup H$ such that $\vdash_{\mathcal{K}} \neg\phi_1 \vee \ldots \vee \neg\phi_k$. Then there are $s_1, s_2, \ldots, s_k \in H$ and $\phi_i \in s_i$ for $1 \leq i \leq k$. There is $s_{k_0} \in \{s_1, s_2, \ldots, s_k\}$, such that $s_i \subset s_{k_0}$ for $1 \leq i \leq k$, hence $\phi_i \in s_{k_0}$ for $1 \leq i \leq k$, which contradicts the consistency of s_{k_0}. Therefore $\bigcup H \in \Theta$ (since $\bigcup H$ is consistent). $\bigcup H$ is an upper bound of H and $\bigcup H \in \Theta$. By Zorn's Lemma Θ has a maximal element $s^\star$, i.e. for all $s' \in \Theta$: $s^\star \subset s'$ entails $s' = s^\star$. $s^\star$ is complete:

- *$s^\star$ is consistent, because $s^\star \in \Theta$*

- *$s^\star$ is maximal: let $\psi \notin s^\star$. Then $s^\star \cup \{\psi\} \notin \Theta$ because $s^\star$ is a maximal element in Θ. Therefore, from the definition of Θ, $s^\star \cup \{\psi\}$ is not consistent.*

Q.E.D.

Lemma 2 *Let s be a complete set of formulas. Then*

1. *If $\phi \in s$ then $\neg\phi \notin s$*
2. *If $\phi \in P(s)$ and $\phi \notin s$ then $\neg\phi \in s$*
3. *If $\phi \in s$ and $\psi \in P(s)$ and $\vdash_{\mathcal{K}} \phi \to \psi$ then $\psi \in s$*
4. *If $\phi \vee \psi \in s$ then $\phi \in s$ or $\psi \in s$*
5. *If $\phi \vee \psi \in P(s)$ and $\phi \in s$ or $\psi \in s$ then $\phi \vee \psi \in s$*

Proof:

1. *Since s is consistent and $\vdash_{\mathcal{K}} \neg\phi \vee \neg\neg\phi$.*
2. *$s \cup \{\phi\}$ is inconsistent. Hence there are formulas $\phi_1, \phi_2, \ldots, \phi_k \in s$ and $\vdash_{\mathcal{K}} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\phi$. Assume $\neg\phi \notin s$. Then $s \cup \{\neg\phi\}$ is inconsistent and there are formulas $\psi_1, \psi_2, \ldots, \psi_k \in s$ such that $\vdash_{\mathcal{K}} \neg\psi_1 \vee \ldots \vee \neg\psi_m \vee \neg\neg\phi$. It follows that $\vdash_{\mathcal{K}} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\psi_1 \vee \ldots \vee \psi_m$, which contradicts the consistency of s.*
3. *is proven along the same lines as 2 and by 2.*
4. *is proven along the same lines as 2 and by 2.*
5. *is proven along the same lines as 2 and by using 2.*

Q.E.D.

We note S the set of all complete formula sets.

Definition 5 *Let s be a set of formulas. Then we define $s^\Box := \{\phi : \Box\phi \in s\}$*

Lemma 3 *Let $s \in S$. Then $\Box\phi \in s$ iff for every $s' \in S$ such that $s^\Box \subseteq s'$, $\phi \in s'$*
Proof: :
$(\Rightarrow)$: $\Box\phi \in s$ iff $\phi \in s^\Box$ and therefore $\phi \in s'$.
$(\Leftarrow)$: We show first that $s^\Box \cup \{\neg\phi\}$ is inconsistent. Assume that $s^\Box \cup \{\neg\phi\}$ is consistent. Then there exists $s' \in S$ such that $s^\Box \cup \{\neg\phi\} \subseteq s'$. by lemma 1. Hence $s^\Box \subseteq s'$ and $\phi \in s'$ from the precondition and therefore $\neg\phi \notin s'$ by Lemma 2,1 (since s' is

complete). This contradicts $s^\Box \cup \{\neg\phi\} \subseteq s'$! Hence $s^\Box \cup \{\neg\phi\}$ is inconsistent. So there exist formulas $\phi_1, \phi_2, \ldots, \phi_k \in s^\Box$ such that
(1) $\vdash_\mathcal{K} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \phi$
(2) $\vdash_\mathcal{K} \phi_1 \wedge \ldots \wedge \phi_k \rightarrow \phi$
(3) $\vdash_\mathcal{K} \Box(\phi_1 \wedge \ldots \wedge \phi_k) \rightarrow \Box\phi$ *by D2*
(4) $\vdash_\mathcal{K} \Box\phi_1 \wedge \ldots \wedge \Box\phi_k \rightarrow \Box\phi$ *by T3*
(5) $\vdash_\mathcal{K} \neg\Box\phi_1 \vee \ldots \vee \neg\Box\phi_k \vee \Box\phi$

$\Box\phi_1, \Box\phi_2, \ldots, \Box\phi_k \in s$*, therefore* $\neg\Box\phi \notin s$*, since* s *is consistent and hence* $\Box\phi \in s$ *by Lemma 2,2 and the completeness of* s*.* Q.E.D.

We are now in a position to define the canonical model.

Definition 6 *Canonical model* $MC = (S, R, v)$

- S *is the set of all complete formula sets*
- $(s, s') \in R$ *iff* $s^\Box \subseteq s'$ *Sometimes we note* $R(s) = \{s' : s\ R\ s'\}$*, the set of s' such that* $s\ R\ s'$*.*
- *For any* $p \in ATM$*,* $v(s, p) = \top$ *iff* $p \in s$

Lemma 4 *Let* MC *be the canonical model defined in /refdecong and let be* $s \in S$ *a complete set of formulas. For every formula* $\phi \in P(s)$*, we have* $v(s, \phi) = \top$ *iff* $\phi \in s$*.*
Proof: by induction over formulas:

1. $v(s, \neg\psi) = \top$ *iff* $v(s, \psi) = \bot$ *iff* $\psi \notin s$ *by induction hypothesis iff* $\neg\psi \notin s$ *from Lemma 2,1. and 2.*
2. $v(s, \psi \vee \chi) = \top$ *iff* $v(s, \psi) = \top$ *or* $v(s, \chi) = \top$ *iff* $\psi \in s$ *or* $\chi \in s$ *by induction hypothesis iff* $\psi \vee \chi \in s$ *by Lemma 2, 4. and 5.*
3. $v(s, \Box\psi) = \top$ *iff for all* $s' \in R(s')$*,* $v(s', \psi) = \top$ *iff for every* $s' \in R(s')$*,* $\psi \in s'$ *by induction hypothesis iff* $\Box\psi \in s$ *by Lemma 3.*

Q.E.D.

Theorem 2 (Completeness theorem) . *Every* $\mathcal{K}$*- valid formula is deducible in* $\mathcal{K}$*.*
Proof: Assume for the contrary that there is a valid formula ϕ *which is not deducible in* $\mathcal{K}$*. Then, by definition 3,* $\{\neg\phi\}$ *is a* $\mathcal{K}$*- consistent set of formulae.* $\{\neg\phi\}$ *can be extended to a complete set of formulae* s*, by Lemma 1, where* $\neg\phi \in s$*.* s *is complete, hence there is a model* $MC = (S, R, v)$ *from definition 6 and* $s \in S$*. By Lemma 4,* $v(s, \neg\phi) = \top$ *and hence* $v(s, \phi) = \bot$*, which contradicts the validity of* ϕ*.* Q.E.D.

7 Other modal system

Other modal systems are defined by additional axioms. Semantically, every system corresponds to an additional property of the transition relation.

Definition 7 *A modal frame (S, R) has the property **P** iff the transition relation R has the property **P**.*

Definition 8 *An axiom A characterizes a modal frame (S, R) that has property **P** iff $\forall v$, valuation, A is valid in the interpretation (S, R, v). Subsequently, we present the most important additional axioms characterizing particular modal logics.*

Not any property of a binary relation can be characterized by an axiom of a modal logic. There are properties that can be characterized and there are properties that cannot be characterized. We will give examples for both.

Definition 9 *We say that a property **P** is expressible in modal logic iff there is a formula ϕ that characterizes **P**.*

7.1 System T

Axiom 3 *(**T**)* $\Box\phi \rightarrow \phi$

This axiom characterizes reflexivity of the tranisition relation; it is true for modal structures where the transition relation is reflexive.

Theorem 3 *Axiom (T) characterizes the class of reflexive frames.*

*Proof: We show first that axiom (**T**) is valid in every interprestation (S,R,v) where R is reflexive. Suppose for the contrary that ther(T) is not valid. Then there is $s \in S$ such that $v(s, \Box\phi) = \top$ and $v(s, \phi) = \bot$. This is the case when for all $s' \in R(s)$, $v(s, \phi) = \top$. But by reflexivity, we have $s \in R(s)$, and $v(s, \phi) = \bot$, contradiction!*
On the other hand, axiom (T) is not valid in nonreflexive structures: Let be $M_0 = (S, R)$ a frame where R is not reflexive. Then there is $s_0 \in S$ and $s_0 \notin R(s)$. We define a valuation v by $v(s_0, p) = \bot$ and $v(s', p) = \top$ for all $s' \in R(s_0)$, hence $v(s_0, \Box p) = \top$. But this means that $v(s_0, \Box p \rightarrow p) = \bot$. Q.E.D.

Corollary 1 *The class of reflexive structures can be expressed in modal logic.*

7.2 System S4

Axiom 4 *(**S4**)* $\Box\phi \to \Box\Box\phi$

If a formula ϕ is necessarily true then necessarily ϕ is necessarily true. This axiom is only true for modal structures where the modal relation R is transitive.

Theorem 4 *Axiom (**S4**) characterizes the class of transitive frames.*

*Proof: We show first that axiom (**S4**) is valid in every transitive interpretation. Suppose for the contrary that there is a transitive interpretation $M_0 = (S, R, v)$ in which axiom S4 is not valid. Then there is a state $s_0 \in S$ such that $v(s_0, \Box\phi \to \Box\Box\phi) = f$. Hence, $v(s_0, \Box\phi) = t$ and $v(s_0, \Box\Box\phi) = f$, i.e. for all $s' \in R(s_0)$, $v(s_0, \phi) = t$ and there is $s'' \in R(s')$ and there is $s''' \in R(s'')$ and $v(s_0, \phi) = f$. But by transitivity of R, we have also $s''' \in R(s_0)$. Contradicton!*
On the other hand, axiom 4 is not valid in non-transitive interpretations: Let $M_0 = (S, R)$ be a frame where R is not transitive. Then there is s, s', $s'' \in S$ and $s' \in R(s)$, $s'' \in R(s')$ but $s'' \notin R(s)$. We define a valuation v_0 by $v_0(s_1, p) = \top$ for every $s_1 \in R(s)$ and $v_0(s'', p) = \bot$. Then we have $v_0(s, \Box p) = \top$ and $v_0(s, \Box\Box p) = \bot$. Therefore $v_0(s, \Box p \to \Box\Box p) = \bot$. Hence S4 is not valid in the interpretation (S, R, v_0). Q.E.D.

Corollary 2 *The class of transitive structures can be expressed in modal logic.*

7.3 System S5

Axiom 5 *(**S5**)* $\Diamond\Box\phi \to \phi$

This axiom is only true for Kripke structures where the transition relation is symmetric.

Theorem 5 *Axiom (**S5**) characterizes the class of symmetric frames.*

*Proof: We show first that axiom (**S5**) is valid in every symmetric structure. Suppose for the contrary that there is a symmetric interpretation $M_0 = (S, R, v)$ in which axiom 5 is not valid. Then there is $s \in S$ such that*
(1) $v(s, \Diamond\Box\phi) = \top$ and
(2) $v(s, \phi) = \bot$.
Then $v(s, \Box\neg\Box\phi) = \bot$, i.e. there is $s' \in R(s)$ and $v(s', \neg\Box\phi) = \bot$, i.e. $v(s', \Box\phi) = t$, from which we get for all $s'' \in R(s')$, $v(s'', \phi) = \top$. But by symmetry, we have also $s \in R(s')$ and therefore $v(s, \phi) = \top$, which contradicts (2)!
On the other hand, axiom 5 is not valid in nonsymmetric structures: Let $M_0 = (S, R)$ a frame where R is not symmetric. Then there are s, $s' \in S$ and $s' \in R(s)$ but not

$s \in R(s')$. Let us define an interpretation (S, R, v) where $v(s, p) = \bot$ and $v(s_1, p) = \top$ for all s_1 such that $s_1 \in R(s')$. Then $v(s, \Diamond\Box p) = \top$ and since $v(s, p) = \bot$, $v(s, \Diamond\Box p \to p) = \bot$. Q.E.D.

Corollary 3 *The class of symmetric structures can be expressed in modal logic.*

7.4 System eucleadian

Axiom 6 ***(Eu)*** $\Diamond\phi \to \Box\Diamond\phi$

A binary relation R on a set M is eucleadian iff for any $a, b, c \in M$, if $(a, b) \in R$ and $(a, c) \in R$ then $(c, b) \in R$

This axiom is true for Kripke structures where the transition relation is eucleadian.

Theorem 6 *Axiom (**Eu**) characterizes the class of eucleadian frames.*

*Proof: We show first that axiom (**Eu**) is valid in every eucleadian structure. Suppose for the contrary that there is a eucleadian structure $M_0 = (S, R, v)$ in which axiom (**Eu**) is not valid. Then there is $s \in S$ such that*
(1) $v(s, \Diamond\phi) = \top$ and
(2) $v(s, \Box\Diamond\phi) = \bot$.
From this follows that there is $s' \in R(s)$ such that $v(s', \phi) = \top$, and there is $s'' \in R(s)$ such that $v(s'', \Diamond\phi) = \bot$. But this means that for all $s''' \in R(s'')$ $v(s''', \phi) = \bot$. Since R is eucleadian, we have $s' \in R(s'')$, from which we conclude $v(s', \phi) = \bot$, contradiction!

*On the other hand, axiom (**Eu**) is not valid in non-eucleadian structures: Let $M_0 = (S, R)$ a frame where R is not eucleadian. Then there are $s, s', s'' \in S$ and $s' \in R(s)$, $s'' \in R(s)$ but $s' \notin R(s'')$. Let us define a valuation v by $v(s', p) = \top$ and $v(s''', p) = \bot$ for all $s''' \in R(s'')$. Then we have $v(s'', \Box\neg p) = \top$ and hence $v(s, \Diamond\Box\neg p) = \top$. $\Diamond\Box\neg p \equiv \neg\Box\Diamond p$ hence we have $v(s, \neg\Box\Diamond p) = \top$ i.e. $v(s, \Box\Diamond p) = \bot$.* Q.E.D.

Corollary 4 *The class of eucleadian structures can be expressed in modal logic.*

We know that for a relation R on a set M we have the following relation between properties:

If R is reflexive and symmetric, then it is transitive iff it is eucleadian. It is worth noting that this relationship can be found on the level of modal logic:

From (**K**), (**T**) and (**S5**) and (**S4**) we can derive (**Eu**) and
from (**K**), (**T**) and (**S5**) and (**Eu**) we can derive (**S4**):

7.5 Table of Correspondence

Axiom	Property of R
$\Box\phi \to \phi$	reflexive
$\Box\phi \to \Box\Box\phi$	transitive
$\neg\Box\phi \to \Box\neg\Box\phi$	Euclidean
$\neg\Box\bot$	serial
$\neg\Box\neg\Box\phi \to \phi$	symmetric

7.6 Non expressible properties

Not all properties of a tranisition relation are expressibly by axioms. There are classes of modal structures which cannot be represented in modal logic.

Corollary 5 *The class of non-reflexive structures cannot be expressed by modal logic.*

Proof: Suppose there is an axiom ϕ which characterizes this class. But then ϕ is false in the structure $M = (\{s\}, \{(s, s)\})$, since this is a reflexive structure. But then ϕ is also false in $M' = (\{s, s', s''\}, \{(s, s), (s', s'')\})$, which is not reflexive, hence ϕ cannot characterize the class of reflexive structures (we have found a structure, which is not reflexive and where ϕ is true). Q.E.D.

From corollaries 1 and 5, we obtain straightforwardly

Corollary 6 *The class of structures that can be expressed in modal logic is not closed under complementation.*

On the other hand, it is easy to see that it is closed under intersection:

Theorem 7 *The class of structures expressible in modal logic is closed under intersection.*

Proof: If ϕ expresses the class $\mathcal{C}_A$ and ψ expresses the class $\mathcal{C}_B$, then $\phi \wedge \psi$ expresses the class $\mathcal{C}_A \cap \mathcal{C}_B$. Q.E.D.

Corollary 7 *The class of connected structures cannot be expressed in modal logic.*

Proof: Suppose ϕ characterizes this class. Then ϕ is false in $M = (\{s, s'\}, \{(s, s), (s', s')\})$. But then either $v(s, \phi) = \bot$ or $v(s', \phi) = \bot$. Hence ϕ is false either in $(\{s\}, \{(s, s)\})$ or in $(\{s'\}, \{(s', s')\})$ which are both connected structures, hence ϕ cannot characterize this class. Q.E.D.

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