A Modal Logic for Hypothesis Theory*

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Abstract. Hypothesis theory for nonmonotonic reasoning expresses notions of hypotheses and of known information. In this paper, we define these notions in the framework of a new modal system with two modalities, one for exprressing known information and the other for expressing possible hypotheses. A complete characterization of the new logic is given in terms of Kripke semantics. Moreover, our logic allows to characterize completely default logic, including a necessary and sufficient criterium for the existence and the non-existence of extensions. We also present a notion of nonmonotonic inference which is cumulative.

1 Introduction

Modal logics for nonmonotonic reasonings have been considered by sereval authors [13, 14, 6, 11, 12, 10, 22, 9, 21, 18, 19]. All these approaches (with the exception of [19]) are fixed-point approaches. [21] introduces a modal formalism with a notion of hypothesis. His formalism is a further development of the logic of suppositions [2, 3]. This logic has been the first preferential model approach using a language with modalities. One important property is that a consistent supposition theory has always a preferred model. In [21] a hypothesis is a formula Hp (p is any formula) where H is a modal operator definied by $H = L\neg L\neg$ and L is the modal operator of the modal logic T[4]. Given a consistent set of formulae F of this logic and a set of hypotheses HY, an extension of F in HY is defined by adding to F a maximal set of hypotheses HY' such that $F \cup HY'$ is consistent. Given this first definition of H in terms of L, in $\mathcal{T}$ it is possible to derive $Lp \rightarrow Hp$ meaning "if we know p then we make the hypotheses p". Since this property is not desirable and in order to get the weakest possible definition of H, [18] give a modified weakened definition of the original theory, where the Hp are propositional constants. The link between Lp and Hp is given by the only nonlogical axiom $Hp \rightarrow \neg L\neg p$. The purpose of this present

*Fundamenta Informaticae, , vol. 21, No. 1-2, pages 89-102, July Augustus 1994

Hypothesis H is then defined by the dual of $[H]$ ($Hp \equiv \neg[H]\neg p$). For that, we define a new modal logic $\mathcal{H}$ with two modal operators L and $[H]$. The operator L is defined as for the standard reflexive modal system $\mathcal{T}$ (is is also possible to use $\mathcal{S}4$ or $\mathcal{S}5$, but we want to use the weakest reflexive system). The operator $[H]$ is defined as for the weakest standard non reflexive modal system $\mathcal{K}$ (non reflexivity is fundamental). For the syntactic aspect, the only link between L and $[H]$ is the logical axiom $Lp \rightarrow [H]p$ (equivalent to our original nonlogical axiom $Hp \rightarrow \neg L\neg p$). For the semantic aspect, in term of Kripke type models, the accessibility relation of $[H]$ is included in the accessibility relation of L. We prove the completeness and the soundness of this system. Every standard modal logic [4], uses the logical inference rule of necessitation : if $\vdash p$ then $\vdash Lp$. It is important to understand that according to standard modal logic necessitation applies <u>exclusively</u> to logical theorems (tautologies) and <u>not</u> to nonlogical theorems, i.e. formulae implied by some set of formulae F. Our system is based on this use of necessitation rule. On the other hand, [13, 12, 22], who study modal interpretations of defaults, apply necessitation also to arbitrary formulae of a theory. This use implies that satisfiability of a modal formula by a model is defined as satisfiability within <u>all</u> worlds of that model. But then the deduction theorem does not hold as can easily been seen by considering the following example: they have always $p \vdash Lp$, but they have not $p \rightarrow Lp$. [10] present an interesting preferential model approach using two modal operators, which is very close to our theory. One important difference between to our approach is that in [10], no axioms or inference rules are given for $\mathcal{A}$. The definition of extensions is based on the identity of assumed and believed formulae. But we cannot see how the set of assumed formulae could be determined. Therefore it seems impossible to get a compactness theorem (and consequently, it seems impossible to get proof procedures). The important properties of our approach are :

(i) We have both, syntactic and semantic definition and we prove the equivalence (we have a complete and sound logic).

(ii) We have a compactness result and therefore proof procedures exist.

(iii) Every consistent hypothesis theory admits extensions.

(iv) We have a complete characterization of default logic and a simple criterium for the existence of extensions of a default theory [19]. We therefore have a correspondance between a preferential model approach and a fixed-point approach.

The organization of this paper is as follows. In the second chapter, the modal logic $\mathcal{H}$ is introduced together with its semantics and the completeness theorem is given. In the third chapter, the notion of hypothesis theory is introduced and correspondance results are recalled. In the fourth chapter, cumulativity is studied. Finally, we duscuss some issues on non-monotonic reasonning and give sample theories.

2 The Modal Logic $\mathcal{H}$

We define a new modal logic $\mathcal{H}$ with two modal operators L and $[H]$. The operator L is defined as for the standard reflexive modal system $\mathcal{T}$. The operator $[H]$ is defined the only link between L and $[H]$ is the logical axiom $Lp \rightarrow [H]p$. For the semantic aspect, in term of Kripke type models, the accessibility relation of $[H]$ is included in the accessibility relation of L. We prove the completeness and the soundness of this system.

2.1 The language of hypothesis theory.

The language of $\mathcal{H}$, $\mathcal{L}(\mathcal{H})$, is an extension of the language of classical predicate logic by two modal operators L and $[H]$. Terms and formulae are defined in accordance with normal usage, and if A is a formula then LA and $[H]A$ are formulae. Hypothesis H is defined by the dual of $[H]$ ($Hp \equiv \neg[H]\neg p$). The intention behind this language is the following :

- LA expresses that A is known.
- HA expresses that A is a hypothesis and therefore $[H]p$ expresses that $\neg p$ is not a hypothesis.
- A expresses that A holds.

2.2 Axioms and inference rules of $\mathcal{H}$.

A0 all valid formulae of classical predicate logic

A1 $L(A \rightarrow B) \rightarrow LA \rightarrow LB$

A2 $LA \rightarrow A$

A3 $\forall x(LP(x)) \rightarrow L(\forall x P(x))$

A4 $[H](A \rightarrow B) \rightarrow [H]A \rightarrow [H]B$

A5 $\forall x([H]P(x)) \rightarrow [H](\forall x P(x))$

A6 $LA \rightarrow [H]A$

R0 all valid inference rules of classical predicate logic

R1 Generalization from $A \rightarrow B$ infer $A \rightarrow \forall x B$, provided x is not free in A.

R2 L-introduction if $\vdash A$ then $\vdash LA$

Clearly, the rule of $[H]$-introduction if $\vdash A$ then $\vdash [H]A$ can be proven from the above calculus by (A6, R2). We write $\vdash_{\mathcal{H}} A$ or $\vdash A$ if A is a theorem of $\mathcal{H}$. When we introduce a set of non-logical axioms $\mathcal{M}$ from which A can be deduced within the logic $\mathcal{H}$ then we write $\mathcal{M} \vdash_{\mathcal{H}} A$. We have three types of facts represented by three types of formulae : known formulae, hypotheses and true formulae. For these modalities, the axioms of $\mathcal{H}$ gives the intuitive meaning: For the operator L, the reflexive axioms means that "every known formula is true" . On the other hand, this principle does not hold for $[H]$, and it is possible to have Hp and $\neg p$. If A is a tautology, then A is known, but merely true facts are not necessarily known. If hypothesis A is made hypothesis p does not allow know p; not to make the hypothesis p does not allow to know its negation. The definition of hypotheses by $Hp = \neg L \neg p$, as proposed by (McDermott 1982), implies $\neg Hp = \neg\neg L \neg p = L \neg p$, i.e. if p is not hypothesized then $\neg p$ is known. This is very strong and gives problems with contraposition (see example 4). It is interesting to note that this logic is about the same as the logic for action theories which has been proposed by [16] with the exception that L in [16] is $\mathcal{S}5$ whereas in our system is is $\mathcal{T}$.

2.3 Semantical characterization of $\mathcal{H}$, Kripke-type-structures.

$\mathcal{H}$-structures are Kripke-structures [8], i.e. sets of classical structures. $(\{\mathcal{A}_s : s \in \mathcal{S}\}, R_1, R_2)$ is called $\mathcal{H}$-structure, if

1. $\mathcal{S} \neq \emptyset$ is the set of (possible) worlds
2. $R_1 \subset \mathcal{S} \times \mathcal{S}$ is a binary relation on $\mathcal{S}$, called hypothesis relation
3. $R_2 \subset \mathcal{S} \times \mathcal{S}$ is a binary, reflexive relation on $\mathcal{S}$, called knowledge relation
4. for every $s \in \mathcal{S}$, $\mathcal{A}_s$ is a classical structure.
5. $R_1 \subset R_2$

A truth function T assigns truth values to formulae depending on a world s. $T(s, A)$ is defined for a classical formula A as usual according to the classical interpretation $\mathcal{A}_s$.
$T(s, [H]A) = t$ iff for all s' with sR_1s', $T(s', A) = t$
$T(s, LA) = t$ iff for all s' with sR_2s', $T(s', A) = t$

A formula A is called valid in state $s \in S$ iff $T(s, A') = t$ for every O-instance A' of A. This is denoted by $\models_s$. A formula A is called valid in a $\mathcal{H}$-structure $\mathcal{M}$ with the set of states $\mathcal{S}$, iff A is valid in every $s \in S$. We denote that by $\mathcal{M} \models A$. $\mathcal{M}$ is then called a model for A. A formula A is called $\mathcal{H}$-valid iff A is valid in every $\mathcal{H}$-structure.This is denoted by $\models_{\mathcal{H}} A$.

$\mathcal{H}$ is sound and complete.

Theorem 2.1 *A formula A is $\mathcal{H}$-valid iff it is deducible in $\mathcal{H}$.*

The proof, which is along the same lines as the completeness proof for ZK [17], may be found in the appendix.

3 Hypothesis Theories

Hypothesis theories [21, 18] are formulated within the language of $\mathcal{H}$ containing additional formulae of the form Hp where p is any formula. If p is closed, Hp is a called a hypothesis. A hypothesis theory is a set of formulae formulated in this language. Clearly, the system $\mathcal{H}$ is a <u>monotonic</u> logic. To obtain nonmonotonicity, the notion of extension is defined by a preferential model approach.

The following definitions are from [18, 19] and the proofs of the theorems may be found there.

of formulae and HY is some set of hypotheses. Extensions of F in HY are obtained by adding to F a subset HY' of HY such that $F \cup HY'$ is maximal consistent according to HY (maximal consistent according to HY means that if any other hypothesis of HY is added, the resulting theory is inconsistent).

Definition 3.2 *Let $\mathcal{HT} = (F, HY)$ be a hypothesis theory. An extension of F in HY is a set $Th_{\mathcal{H}}(F \cup HY')$, where HY' is a maximal (in the sense of set inclusion) subset of HY, consistent with F. We say that the hypotheses of HY are maximized. As we have seen, an extension E of a hypotheses theory (F, HY) contains formulae which are true (the theorems of E), formulae which are known (formulae f, such that Lf is a theorem of E) and hypotheses are made by adding formulae Hp to F. Nonmonotonicity comes from the fact that an extension is defined by adding hypotheses. Therefore, the addition of new formulae to F can prevent the addition of previously admissible hypotheses.*

Example 3.1 *The propositional default $\frac{p:q}{r}$ is translated to $Lp \wedge Hq \rightarrow Lr$. If p is known and if the hypothesis q is made, then r is known. Let be $F = \{Lp, Lp \wedge Hq \rightarrow Lr\}$ representing the default theory $\Delta = (\{p\}; \{\frac{p:q}{r}\})$. F has one extension obtained by adding to F the hypothesis Hq. This extension contains Lp, Hq and Lr. Therefore, the known formulae are p and r and the theorems of $\{p, r\}$. $Th(\{p, r\})$ is the only extension of Δ.*

An important property of hypothesis theories is the existence of extensions.

Theorem 3.1 *A hypotheses theory $HT = (F, HY)$ has an extension whenever F is consistent. (Proof see [18]*

Our definition of extension is not a fixed point definition. Nevertheless, it is possible to give a recursive characterization as the following theorem shows.

Theorem 3.2 *Let be $E = Th_{\mathcal{H}}(F \cup HY')$ an extension of (F, HY). Then E is a solution of the recursive equation : $E = Th_{\mathcal{H}}(F \cup \{h : h \in HY and \neg h \notin E\})$*

Corollary 3.3 *If E is an extension of F in HY then for all $h \in HY$, either $h \in E$ or $\neg h \in E$.*

As illustrated by example 1, default logic can be expressed by hypothesis theory. Our translation from default theory to hypothesis theory preserves extensions and moreover gives an interesting criterium for the existence of extensions. A propositional default theory $\Delta = (W, D)$, is translated to the hypothesis theory $(LW \cup LD, HY)$, where $LW = \{Lw \mid w \in W\}$, $LD = \{L\alpha \wedge H\beta \rightarrow L\gamma \mid \frac{\alpha:\beta}{\gamma} \in \Delta\}$ and $HY = \{H\beta \mid \beta \in JUST(D)\}$. Given an extension E of Δ, the set of hypotheses corresponding to E is noted: $H(E) = \{H\beta \mid \beta \in JUST(D) and \neg\beta \notin E\}$. On the other hand, any hypotheses theory of the above form, i.e. containing uniquely formulae Lw and $L\alpha \wedge H\beta \rightarrow L\gamma$ together with the set of hypotheses $HY = \{H\beta : L\alpha \wedge H\beta \rightarrow L\gamma \in F\}$ for classical formulae w, α, β and γ corresponds to a default theory (W, D) where $D = \{\frac{\alpha:\beta}{\gamma} : L\alpha \wedge H\beta \rightarrow L\gamma \in F\}$ and $W = \{w : Lw \in F\}$. For this translation, we precisely the formulae p such that Lp belongs to an extension of the translation of Δ 3.4. Moreover this extension contains $L\neg p$ whenever it contains $\neg Hp$. Conversely, a certain class of extensions of a hypotheses theory, namely those which contain $L\neg p$ whenever they contain $\neg Hp$ correspond to extensions of the corresponding default theory 3.5. This is a simple criterion for the existence of extensions : Given a hypotheses theory HT which is a translation of a default theory Δ, Δ has an extension iff HT has at least one extension which contains $L\neg p$ whenever it contains $\neg Hp$. This means that for a default theory, we know the negation of every hypothesis which has not been made.

Theorem 3.4 *Let $\Delta = (W, D)$ be a closed default theory, and (F, HY) its translation. Let E be an extension of Δ. Then*

1. *$E = \{p : Lp \in E'\}$, where $E' = Th_{\mathcal{T}}(F \cup H(E))$ is an extension of F in HY and*
2. *If $\neg Hp \in E'$ then $L\neg p \in E'$*

Theorem 3.5 *Let $\Delta = (W, D)$ be a closed default theory, and (F, HY) its translation. Let E' be an extension of F in HY such that if $\neg Hb \in E'$ then $L\neg b \in E'$. Then there exists an extension E of Δ with $E = \{p : Lp \in E'\}$. The correspondence theorems do not give a bijection between extensions of Δ and extensions of F (some extensions of F do not correspond to extensions of Δ). Particularly, even if Δ has no extension, F has an extension (since F always has an extension). But every extension of F which does not correspond to an extension of Δ has the property of containing at least one negation of hypotheses $\neg Hf$ without $\neg f$ being known. This property gives a simple characterization of default theories with no extensions.*

Example 3.2 *The default theory $(\{p\}, \{\frac{p:q}{r}, \frac{p:s}{\neg r}\})$ is represented by the hypothesis theory $HT = (\{Lp, Lp \wedge Hq \rightarrow Lr, Lp \wedge Hs \rightarrow L\neg r\}, \{Hq, Hs\})$. Maximizing the hypotheses $\{Hq, Hs\}$ corresponding to the justifications of the defaults yields two extensions for HT:*
$E1 = Th_{\mathcal{H}}(\{Lp, Lp \wedge Hq \rightarrow Lr, Lp \wedge Hs \rightarrow L\neg r\} \cup Hq)$
$E2 = Th_{\mathcal{H}}(\{Lp, Lp \wedge Hq \rightarrow Lr, Lp \wedge Hs \rightarrow L\neg r\} \cup \{Hs\})$,
$E1$ containing Lr and $E2$ containing $L\neg r$ and therefore, $\neg Lr$ (by reflexivity of L).

Example 3.3 *Let $F = \{L\neg r, Lp, Lp \wedge Hq \rightarrow Lr\}$ The corresponding default theory: $\Delta = (\{\neg r, p\}, \{\frac{p:q}{r}\})$ has no extension. By axiom A2, $\neg Lr$ is a theorem of F (since $L\neg r$ is in F). As F contains also Lp, we deduce that $\neg Hq$ is a theorem of F. Therefore, an extension of F cannot contain Hq and there exists only one extension E which contains Lp, $\neg Hq$ and $L\neg r$. The known formulae are p and $\neg r$. E does not contain $L\neg q$. E is an example of an extension which has no correspondence in the default theory. It contains the negation $\neg Hq$ of a hypothesis without containing $L\neg q$. Let us remark that, if we choose $H = \neg L\neg$ as definition of hypotheses, we would obtain here $\neg Hq = \neg\neg L\neg q = L\neg q$, and therefore, $\neg q$ would be a known formula in the extension. This does not correspond to our intuition that it should be possible not to make a hypothesis without knowing its negation. With the above definition however, not to make the hypothesis q entails that $\neg q$ is known.*

$(\emptyset, \{\frac{:p}{\neg p}\})$. *The translation in our formalism gives* $F = \{Hp \rightarrow L\neg p, Hp \rightarrow \neg L\neg p\}$. *Hence* $\neg Hp$ *is a theorem of* F. F *has one extension whose only known formulae are tautologies (*F *is "empty"). As every hypothesis theory with consistent* F *has an extension and since the modal logic* $\mathcal{H}$ *is compact [17], we can prove compactness of hypothesis theories.*

Theorem 3.6 *Compactness property. Let* f *be a closed formula and* (F, HY) *a hypotheses theory.*

1. *Let* $E = Th_{\mathcal{H}}(F \cup HY')$ *be an extension of* F *in* HY. *Then* f *is a theorem of* E *iff there exists a finite set* $h_1,..,h_n$ *of hypotheses in HY' such that* f *is a theorem of* $F \cup \{h_1, .., h_n\}$.

2. f *is a theorem of some extension* E *of* (F, HY) *iff there exists a finite set* $h_1,..,h_n$ *of hypotheses such that* f *is a theorem of* $F \cup \{h_1, .., h_n\}$ *and* $F \cup \{h_1, .., h_n\}$ *is consistent.*

Proof:

1. Since H is compact and the hypotheses are closed formulae, f is a theorem of E iff there exists a finite subset S of E such that f is a theorem of S. This subset contains a finite subset $\{h_1, .., h_n\}$ of HY' and therefore f is a theorem of $F \cup \{h_1, .., h_n\}$. On the other hand if f is a theorem of $F \cup \{h_1, .., h_n\}$ f is a theorem of E.

2. Let $E = F \cup HY'$ be an extension containing f. By (1), there exists a finite subset $\{h_1, .., h_n\}$ of HY' such that f is a theorem of $F \cup \{h_1, .., h_n\}$. Since E is an extension, E is consistent and so is $F \cup \{h_1, .., h_n\}$. Conversely, if $F \cup \{h_1, .., h_n\}$ is consistent, by theorem 1, there exists an extension E' of $F \cup \{h_1, .., h_n\}$ in HY. Obviously, E' is also an extension of F. That f is a theorem of E' follows from f is a theorem of$F \cup \{h_1, .., h_n\}$. Q.E.D.

The compactness property yields proof procedures, not complete in the general case, that can be based on the production of clauses whose literals are negations of hypotheses. To prove that a formula f is true in an extension, it is enough to exhibit a clause $\neg h_1 \vee .. \vee \neg h_p$, which is theorem of $F \cup \{\neg f\}$ and is not a theorem of F. For the propositional logic, such production algorithms are defined in [20] and [5]. This technique has first been introduced for supposition based logic by [2].

Nonmonotonic inference may be defined by means af extensions. The following definition provides a notion of inference for which we can show cumulativity [7].

Definition 3.3 *Let be* (F, HY) *a hypothesis theory and* E *the intersection of all extensions of* (F, HY). *Then we set* $(F, HY) \mid\!\sim p$ *iff* $p \in E$.

This definition expresses a sceptical attitude to nonmonotonic inference.

We will show that $\mid\!\sim$ is cumulative, i.e. If $(F, HY) \mid\!\sim p$ and $(F, HY) \mid\!\sim q$ then $(F \cup \{p\}, HY) \mid\!\sim q$.

This property holds, because, in this case, (F, HY) and $(F \cup \{p\}, HY)$ have the same extensions, as is shown in the following lemma.

sions.

Proof: Let be E_0 an extension of (F, HY). Then $E_0 = Th_{\mathcal{H}}(F \cup HY')$, where $F \cup HY'$ is maximal consistent according to HY. Since $p \in E_0$, we have $E_0 = E_0 \cup \{p\}$ $= Th_{\mathcal{H}}(F \cup HY') \cup \{p\} = Th_{\mathcal{H}}(Th_{\mathcal{H}}(F \cup HY') \cup \{p\}) = Th_{\mathcal{H}}(F \cup \{p\} \cup HY')$

On the other hand, let be E' an extension of $(F \cup \{p\}, HY)$. Then $E' = Th_{\mathcal{H}}(F \cup \{p\} \cup HY')$, where $F \cup \{p\} \cup HY'$ is maximal consistent according to HY. Now, consider $E'' = Th_{\mathcal{H}}(F \cup HY')$. Then $F \cup HY'$ must be maximal consistent : Clearly, $F \cup HY'$ is consistent. If it were not maximal, it would be contained in $F \cup HY''$, extension of (F, HY). Since this extension contains p, it is an extension of $(F \cup \{p\}, HY)$. But this contredicts the maximality of $F \cup \{p\} \cup HY'$

4 Examples and Applications

Example 4.1 *Let* $F = \{L\neg q, Lp, Lp \wedge Hq \rightarrow Lr\}$, *F is a translation of the default theory :* $\Delta = (\{\neg q, p\}; \{\frac{p : q}{r}\})$. *Since* $L\neg q$ *is in F,* $\neg Hq$ *is in F (by* $Lf \rightarrow \neg H \neg f$*). Therefore, an extension of F cannot contain* Hq *and there exists only one extension which contains* Lp, $L\neg q$ *and* $\neg Hq$. *The known formulae are the theorems of* $\{p, \neg q\}$, $Th(\{p, \neg q\})$ *being the corresponding extension of* Δ, Δ *has no other extension.*

Example 4.2 *Case analysis. The default theory* $\Delta = (\{p1 \vee p2\}; \{\frac{p1 : q}{r}, \frac{p2 : q}{r}\})$ *has one extension which does not contain r. It is the same for the translation* $F = (\{L(p1 \vee p2)\}, \{Lp1 \wedge Hq \rightarrow Lr, Lp2 \wedge Hq \rightarrow Lr\})$ *because disjunction does not distribute over L. Neither the default theory nor our translation supports case analysis. It is possible to obtain another translation which allows case analysis, by supressing the modality for the prerequisites.* $F = \{L(p1 \vee p2)\}, \{p1 \wedge Hq \rightarrow Lr, p2 \wedge Hq \rightarrow Lr\}$. *Since* $L(p1 \vee p2) \rightarrow p1 \vee p2$ *(Axiom A2), the extension obtained by adding* Hq *contains* Lr.

Let us remark that this translation allows case analysis without contraposition: if $W = \{\neg q\}$, we obtain an extension which does not contain $L\neg p$ and $\neg p$. This representation gives case analysis in a very natural way. It is possible to capture this phenomenon because we can express information at the three levels of truth, knowledge and hypothesis. Neither default logic nor autoepistemic logic have this expressive power. On the other hand, circumscription allows case analysis but admits also contraposition.

5 Conclusion

We presented a theory for nonmonotonic reasoning, the theory of hypotheses, based on a new multimodal logic Hwith two modalities. This is the first result where default logic can be completely characterized (i.e. including a result about the existence of extensions) by standard modal logic (with the standard use of necessitation). Moreover, it is the second result, after (Besnard and Siegel 1988) about the relationship between a fixed-point definition of extension, which not always admits extensions and a minimal (preferential) model based definition which always has extensions. By using logic by translating Moore's $\Diamond$ to H and every classical formula p of an autoepistemic theory to Lp.

By choosing as the modal logic base the system S4, it seems possible to capture also a large part of circumscription. To circumscribe a predicate $p(x_1, .., x_n)$ in a non modal theory F, we add to F the formula : $\forall x_1, .., x_n H(\neg p(x_1, .., x_n)) \rightarrow \neg p(x_1, .., x_n)$ and maximize the set of hypotheses $\{H(\neg p(t_1, .., t_n))\}$ (t_i's are terms).

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Appendix: Proof of 2.1

The proof of the soundness is trivial. The idea of the completeness proof is the following: if A is not an $\mathcal{H}$ - theorem, then $\neg A$ is a consistent set of formulae. We can then define a (canonical) model for $\neg A$, in which $\neg A$ is true. Then A cannot be $\mathcal{H}$ - valid.

Definition .1 *A set of formulae, s, is called $\mathcal{H}$ -inconsistent if it has a finite subset $\{A_1, ..., A_k\}$ with $\vdash_{\mathcal{H}} \neg A_1 \vee ... \vee \neg A_k$. Otherwise s is called $\mathcal{H}$ - consistent (or consistent). Let s be a set of formulae. Then we denote by $V(s)$ the set of all terms occurring in formulae of s; we denote by $P(s)$ the set of all formulae of $L(H)$ containing only nonlogical symbols (terms, function and predicate symbols) of formulae of s.*

Definition .2 *A set of formulae s is called complete (or $\mathcal{H}$ - complete) if*

1. *s is $\mathcal{H}$-consistent.*
2. *s is maximal, i.e. for all $A \in P(s)$ if $A \notin s$ then $s \cup \{A\}$ is inconsistent.*

$A_x[c] \in s$ *for some constant* $c \in V(s)$.

Lemma .1 *Every consistent set of formulae can be extended to a complete set of formulae.*

Proof: Analogous to the proof by Henkin-Hasenjäger. Let s be a consistent set of formulae and let a_1,$a - 2$... be a sequence of "new" object variables not in $V(s)$. We define $P^*(s) = P(s) \cup \{j : jisaformulawithvariablesfroma_1, ...\}$. Let $\{\exists x_i A(x_i)\} i = 1, ...$ be an enumeration of the existential formulae of $P^*(s)$. Then we form a set of formulae $s^\sim$, by adding to s a set of formulae $\exists x_i A(x_i) \to A(a_i)$ for any existential formula of $P(s)$. $s^\sim$ is consistent : assume for the contrary that there are formulae A_1,... A_k in s, such that $\vdash \neg A_1 \vee \ldots \vee \neg A_k \vee \neg(\neg \exists x_{i_0} A_{i_0}(x_{i_0}) \vee A_{i_0}(a_{i_0}))$ This gives

1. $\vdash \neg A_1 \vee \ldots \vee \neg A_k \vee \exists x_{i_0} A_{i_0}(x_{i_0})$
2. $\vdash \neg A_1 \vee \ldots \neg A_k \vee \neg A_{i_0}(a_{i_0})$

   From 2, we obtain.
3. $\vdash \neg A_1 \vee \ldots \vee \neg A_k \vee \forall x \neg A_{i_0}(x)$ and hence
4. $\vdash \neg A_1 \vee \ldots \vee \neg A_k \vee \neg \exists x A_{i_0}(x)$

1 and 4 imply $\vdash \neg A_1 \vee \ldots \vee \neg A_k$, which contradicts the consistency of s. $s^\sim$ is extended to s^* in the following way : let Θ be the set of all consistent formula sets, containing $s^\sim$: $\Theta = \{s' : s \subset s' ands' consistent\}$ Let H be a chain in Θ , i.e. $H \subset \Theta$ and $s_1, s_2 \in H$ implies $s_1 \subset s_2$ or $s_2 \subset s_1$. Then $\bigcup H$ is an upper bound of H in Θ, since $\forall s \in H$, $s \subset \bigcup H$. $\bigcup H$ is consistent: if not, assume that there are formulae $A_1, \ldots A_k \in \bigcup H$ such that $\vdash \neg A_1 \vee \ldots \vee \neg A_k$. Then there are $s_1, \ldots s_k \in H$ and $A_i \in s_i$ for $1 \leq i \leq k$. There is $s_{k_0} \in \{s_1, \ldots s_k\}$, such that $s_i \subset s_{k_0}$ for $1 \leq i \leq k$, hence $A_i \in s_{k_0}$ for $1 \leq i \leq k$, which contradicts the consistency of s_{k_0}. Therefore $\bigcup H \in \Theta$ (since $\bigcup H$ is consistent). $\bigcup H$ is an upper bound of H and $\bigcup H \in \Theta$. By Zorn's Lemma Θ has a maximal element s^*, i.e. $\forall s' \in \Theta : s^* \subset s'$ entails $s' = s^*$. s^* is complete:

- i s^* is consistent, because $s^* \in \Theta$
- ii s^* is maximal: let $B \notin s^*$. Then $s^* \cup \{B\} \notin \Theta$ because s^* is a maximal element in Θ. Therefore, from the definition of Θ, $s^* \cup \{B\}$ is not consistent.
- iii s^* is saturated: let $\exists x A \in s^*$. From the definition of $s^\sim$ and because of $s^\sim \subset s^*$, we have $(\exists x A(x) \to A(a)) \in s^*$. Hence $\neg \exists x A \notin s^*$ and $\neg(\exists x A(x) \to A(a)) \notin s^*$ because s^* is consistent. Assume now that $A(a) \notin s^*$. Then $s^* \cup \{A(a)\}$ is inconsistent, i.e. there are formulae $A_1, \ldots A_k \in s^*$, such that

  1. $\vdash \neg A_1 \vee \ldots \vee \neg A_k \vee \neg A(a)$.

     By the same reasoning, there are formulae $B_1, \ldots B_m \in s^*$ and $C_1, \ldots C_n \in s^*$, such that
  2. $\vdash \neg B_1 \vee \ldots \vee \neg B_m \vee \neg\neg \exists x A$ and
  3. $\vdash \neg C_1 \vee \ldots \vee \neg C_n \vee \neg\neg(\exists x A(x) \to A(a))$.

     From 1 and 2 and 3 it follows easily

the consistency of ${}^s*$.

Lemma .2 *Let s be a complete set of formulae. Then:*

1. *If $A \in s$ then $\neg A \notin s$*
2. *If $A \in P(s)$ and $A \notin s$ then $\neg A \in s$*
3. *If $A \in s$ and $B \in P(s)$ and $\vdash A \rightarrow B$ then $B \in s$*
4. *If $A \vee B \in s$ then $A \in s$ or $B \in s$ 5*
5. *If $A \vee B \in P(s)$ and $A \in s$ or $B \in s$ then $A \vee B \in s$*
6. *$A_x[a] \in s$ iff $\exists x A \in s$*
7. *If $LA \in s$ then $[H]A \in s$*
8. *If $LA \in s$ then $A \in s$*
9. *$t = t \in s$ for every term $t \in V(s)$*
10. *If $t_1 = t'_1 \in s$,$\ldots t_n = t'_n \in s$ and $F(t_1, \ldots, t_n) \in V(s)$ and $F(t'_1, \ldots, t'_n) \in V(s)$ then $F(t_1, \ldots t_n) = F(t'_1, \ldots, t'_n) \in s$*
11. *If $t_1 = t'_1 \in s, \ldots t_n = t'_n \in s$ and $P(t_1, \ldots, t_n) \in s$ and $P(t'_1, \ldots, t'_n) \in P(s)$ then $P(t'_1, \ldots, t'_n) \in s$*

Proof: 1 Since s is consistent and $\vdash \neg A \vee \neg\neg A$. 2 $s \cup \{A\}$ is inconsistent. Hence there are formulae $A_1, \ldots A_k \in s$ and $\vdash \neg A_1 \vee \ldots \vee \neg A_k \vee \neg A$. Assume $\neg A \notin s$. Then $s \cup \{\neg A\}$ is inconsistent and there are formulae $B_1, \ldots B_m \in s$ such that $\vdash \neg B_1 \vee \ldots \vee \neg B_m \vee \neg\neg A$. It follows that $\vdash \neg A_1 \vee \ldots \vee \neg A_k \vee \neg B_1 \vee \ldots \vee B_m$, which contradicts the consistency of s. 3, 4, 5 are proved along the same lines as 2. 6 holds because s is saturated and from $\vdash A[a] \rightarrow \exists x A[x]$ and from 3. 7 from 3 and A6. 8 from 3 and A1. 9: Assume for the contrary that $t = t \notin s$. Then there are formulae $A_1, \ldots A_k \in s$ such that $\vdash \neg A_1 \vee \ldots \neg A_k \vee \neg(t = t)$. But $\vdash t = t$ by AO (identity axiom). Hence $\vdash \neg A_1 \vee \ldots \vee \neg A_k$ from the cut rule contradicting the consistency of s. 10 is proven as 9 by the equality axiom $t_1 = t'_1 \ldots t_n = t'_n \rightarrow F(t_1, \ldots t_n) = F(t'_1, \ldots t'_n)$ of classical logic. 11 is proven as 9 by the equality axiom $t_1 = t'_1 \ldots t_n = t'_n \rightarrow P(t_1, \ldots t_n) \rightarrow P(t'_1, \ldots t'_n)$.

Definition .3 *Let s be a set of formulae. Then we define $s^{[H]} := \{A : [H]A \in s\}$ and $s^L := \{A : LA \in s\}$*

Lemma .3 *Let s be a complete set of formulae. Then $s^{[H]}$ and s^L are consistent and $s^L \subseteq s^{[H]}$.*

Proof: The consistency is obvious. $A \in s^L$ iff $LA \in s$. Therefore $[H]A \in s$ by axiom A6 and since s is complete. Hence $A \in s^{[H]}$. s^L is consistent because it is a subset of a consistent set of formulae.

Definition .4 *A system of sets S is called complete if*

2. *If $s \in S$, $A \in P(s)$ and $s^{[H]} \cup \{A\}$ is consistent, then there is $s' \in S$ such that $s^{[H]} \cup \{A\} \subset s'$.*

Lemma .4 *For every complete set of formulae s, there is a complete system of sets S with $s \in S$.*

Proof: For every $A \in P(s)$ such that $s^{[H]} \cup \{A\}$ is consistent, we extend $s^{[H]} \cup \{A\}$ to a complete set for formulae s' (which exists by .1).

Definition .5 *Given a complete system of sets S, we define two binary relations R_1 and R_2 on S: $sR_1s' iff_{def} s^{[H]} \subset s'$ and $sR_2s' iff_{def} s^L \subset s'$*

Lemma .5 *$R_1 \subset R_2$ (i.e. if $s^{[H]} \subset s'$ then $s^L \subset s'$)*

Proof: Let $s^{[H]} \subset s'$ and $A \in s^L$, hence $A \in s^{[H]}$ by .2 and therefore $A \in s'$.

Lemma .6 *Let S be a complete system of sets, let $s \in S$. Then*

1. *$[H]A \in s$ iff $A \in s'$ for every s' with sR_1s'*
2. *$LA \in s$ iff $A \in s'$ for every s' with sR_2s'*

Proof: 1 ($\Rightarrow$): $[H]A \in s$ iff $A \in s^{[H]}$ and therefore $A \in s'$ for all s' such that sR_1s' by the definition of R_1.

2 ($\Rightarrow$): $LA \in s$, hence $[H]A \in s$ by axiom A6 and the completeness of s, and hence for all s' if sR_1s' then $A \in s'$ by 1. ($\Rightarrow$) and therefore for all s' such that sR_2s', $A \in s'$ by .5.

2 ($\Leftarrow$): We first observe that $s^L \cup \{\neg A\}$ is inconsistent. Hence, there exist formulae $A_1, \dots A_k \in s^L$ such that $\vdash \neg A_1 \vee \dots \vee \neg A_k \vee A$. But then we have also $\vdash \neg LA_1 \vee \dots \vee \neg LA_k \vee LA$. $LA_1, \dots LA_k \in s$ therefore $\neg LA \notin s$, since s is consistent and hence $LA \in s$ by the maximality of s.

1 ($\Leftarrow$): Let be $A \in s'$ for all s' such that sR_1s'. From this, it follows that for all s' such that sR_2s', $A \in s'$ by .5. Hence $LA \in s$ by 2. ($\Leftarrow$) (this lemma).Therefore $[H]A \in s$ since s is complete by .2.

We are now in a position to define an $\mathcal{H}$ - model. The objects will be the equivalence classes according to the identity, =. Then $t_0 \cong t_1\ iff_{def}\ (t_0 = t_1) \in s$ for all $s \in S$. Clearly, $\cong$ is an equivalence relation, as can be easily verified by using the identity and equaliy axioms of classical predicate logic. The truth value function T is defined recursively: For every term t, $T(s,t) = [t]$; the truth value for a formula is given inductively by the definition of functions and predicates. It is then easy to show by induction over formulae (and by using .6):

Lemma .7 *For every closed formula $A \in P(s)$ $T(s, A) = true$ if and only if $A \in s$.*

Proof of the completeness theorem: Every $\mathcal{H}$ - valid formula is deducible in $\mathcal{H}$. Assume for the contrary that there is a valid formula A which is not deducible in $\mathcal{H}$. Then $\{\neg A\}$ is a consistent set of formulae from .1. $\{\neg A\}$ can be extended to a complete set of formulae s, by .1, where $\neg A \in s$. By .4 there is a complete system of sets S with $s \in S$. By .7, $T(s, \neg A) = true$ and hence $T(s, A) = false$, which contradicts the validity of A.