A Sequent Calculus and a Theorem Prover for Standard Conditional Logics

NICOLA OLIVETTI
LSIS - UMR CNRS 6168 Université Paul Cézanne (Aix-Marseille 3)
GIAN LUCA POZZATO
Università diegli Studi di Torino
and
CAMILLA B. SCHWIND
École d'Architecture de Marseille

In this paper we present a cut-free sequent calculus, called SeqS, for some standard conditional logics. The calculus uses labels and transition formulas and can be used to prove decidability and space complexity bounds for the respective logics. We also show that these calculi can be the base for uniform proof systems. Moreover, we present CondLean, a theorem prover in Prolog for these calculi.

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1. INTRODUCTION

Conditional logics have a long history. They have been studied first by Lewis [Lewis 1973; Nute 1980; Chellas 1975; Stalnaker 1968] in order to formalize a kind of hypothetical reasoning (if A were the case then B), that cannot be captured by classical logic with material implication. In particular they were introduced to capture *counterfactual sentences*, i.e. conditionals of the form "if A were the case

Authors' addresses: N. Olivetti, LSIS - UMR CNRS 6168 Universit Paul Czanne (Aix-Marseille 3) Avenue Escadrille Normandie-Niemen 13397 Marseille Cedex 20 - France - e-mail: nicola.olivetti@univ.u-3mrs.fr.
G.L. Pozzato, Dipartimento di Informatica - Università degli Studi di Torino, corso Svizzera 185 - 10149 Turin - Italy, e-mail: pozzato@di.unito.it.
C.B. Schwind, École d'Architecture de Marseille - Luminy, 184 avenue de Luminy - 13288 Marseille Cedex 9 - France,
e-mail: Camilla.Schwind@map.archi.fr

then B would be the case", where A is false. If we interpret the *if...then* in the above sentence as a classical implication, we obtain that all counterfactuals are trivially true. Nonetheless one may want to reason about counterfactuals sentences and hence be capable of distinguishing among true and false ones (for a broader discussion we refer to [Costello and McCarthy 1999]).

In the last years, there has been a considerable amount of work on applications of conditional logics to various areas of artificial intelligence and knowledge representation such as non-monotonic reasoning, hypothetical reasoning, belief revision, and even diagnosis.

The application of conditional logics in the realm of nonmonotonic reasoning was firstly investigated by Delgrande [Delgrande 1987] who proposed a conditional logic for prototypical reasoning; the understanding of a conditional $A \Rightarrow B$ in his logic is "the A's have typically the property B". For instance, one could have:

$$\forall x(\mathit{Penguin(x)} \rightarrow \mathit{Bird(x)})$$
$$\forall x(\mathit{Penguin(x)} \rightarrow \neg \mathit{Fly(x)})$$
$$\forall x(\mathit{Bird(x)} \Rightarrow \mathit{Fly(x)})$$

The last sentence states that birds typically fly. Observe that replacing $\Rightarrow$ with the classical implication $\rightarrow$, the above knowledge base is consistent only if there are no penguins. The study of the relations between conditional logics and nonmonotonic reasoning has gone much further since the seminal work by Kraus, Lehmann, and Magidor [Kraus et al. 1990] (KLM framework), who proposed an axiomatization of the properties of a non-monotonic consequence relation: their system comprises non-monotonic assertions of the form $A \mid\sim B$, interpreted as "B is a plausible conclusion of A". It turns out that all forms of inference studied in KLM framework are particular cases of well-known conditional axioms [Crocco and Lamarre 1992]. In this respect the KLM language is just a fragment of conditional logics (a deep study of conditional logics for default reasoning is contained in [Friedman and Halpern 2001]).

Conditional logics have also been used to formalize knowledge update and revision. For instance, Grahne presents a conditional logic (a variant of Lewis' VCU) to formalize knowledge-update as defined by Katsuno and Mendelzon [Grahne 1998]. More recently in [Giordano et al. 2005] and [Giordano et al. 2002], it has been shown a tight correspondence between AGM revision systems and a specific conditional logic, called BCR. The connection between revision/update and conditional logics can be intuitively explained in terms of the so-called Ramsey Test (RT): the idea is that $A \Rightarrow B$ "holds" in a knowledge base K if and only if B "holds" in the knowledge base K revised/updated with A; this can be expressed by

$$\text{(RT)} \quad K \vdash A \Rightarrow B \text{ iff } K \circ A \vdash B,$$

where $\circ$ denotes a revision/update operator. Observe that on the one hand (RT) gives a definition of the conditional $\Rightarrow$ in terms of a belief-change operator; on the other hand, the conditional logic resulting from the (RT) allows one to describe (and to reason on) the effects of revision/update within the language of conditional logics.

Conditional logics have also been used to model hypothetical queries in deductive databases and logic programming; the conditional logic CK+ID is the basis of the logic programming language CondLP defined in [Gabbay et al. 2000]. In that language one can have hypothetical goals of the form (quoting the old Yale's Shooting problem)

$$Load\_gun \Rightarrow (Shoot \Rightarrow Dead)$$

and the idea is that the hypothetical goal succeeds if *Dead* succeeds in the state "revised" first by *Load_gun* and then by *Shoot*[1].

In a related context, conditional logics have been used to model causal inference and reasoning about action execution in planning [Schwind 1999; Giordano and Schwind 2004]. In the causal interpretation the conditional $A \Rightarrow B$ is interpreted as "A causes B"; observe that identity (i.e. $A \Rightarrow A$) is not assumed to hold.

Moreover, conditional logics have found some applications in diagnosis, where they can be used to reason counterfactually about the expected functioning of system components in face of the observed faults [Obeid 2001].

In spite of their significance, very few proof systems have been proposed for conditional logics: we just mention [Lamarre 1993; Delgrande and Groeneboer 1990; Crocco and del Cerro 1995; Artosi et al. 2002; Gent 1992; de Swart 1983; Giordano et al. 2003]. One possible reason of the underdevelopment of proof-methods for conditional logics is the lack of a universally accepted semantics for them. This is in sharp contrast to modal and temporal logics which have a consolidated semantics based on a standard kind of Kripke structures.

Similarly to modal logics, the semantics of conditional logics can be defined in terms of possible world structures. In this respect, conditional logics can be seen as a generalization of modal logics (or a type of multi-modal logic) where the conditional operator is a sort of modality indexed by a formula of the same language.
The two most popular semantics for conditional logics are the so-called *sphere semantics* [Lewis 1973] and the *selection function semantics* [Nute 1980]. Both are possible-world semantics, but are based on different (though related) algebraic notions. Here we adopt the selection function semantics, which is more general and considerably simpler than the sphere semantics.

With the selection function semantics, truth values are assigned to formulas depending on a world; intuitively, the selection function f selects, for a world w and a formula A, the set of worlds $f(w, A)$ which are "most-similar to w" or "closer to w" given the information A. In *normal* conditional logics, the function f depends on the set of worlds satisfying A rather than on A itself, so that $f(w, A) = f(w, A')$ whenever A and A' are true in the same worlds (normality condition). A conditional sentence $A \Rightarrow B$ is true in w whenever B is true in every world selected by f for A and w. It is the normality condition which marks essentially the difference between conditional logics on the one hand, and multimodal logic, on the other (where one might well have a family of $\Box$ indexed by formulas). We believe that it is the very condition of normality what makes difficult to develop proof systems for conditional logics with the selection function semantics.

Since we adopt the selection function semantics, CK is the fundamental system; it has the same role as the system K (from which it derives its name) in modal

[1]The language CondLP comprises a nonmonotonic mechanism of revision to preserve the consistency of a program potentially violated by an hypothetical assumption.

logic: CK-valid formulas are exactly those ones that are valid in every selection function model.

In this work we present a sequent calculus for CK and for some of its standard extensions, namely CK+{ID, MP, CS, CEM} including most of the combinations of these extensions. To the best of our knowledge, the presented calculi are the first ones for these logics. Our calculi make use of labels, following the line of [Viganò 2000] and [Gabbay 1996]. Two types of formulas are involved in the rules of the calculi: world formulas of the form $x : A$ representing that A holds at world x and transition formulas of the form $x \xrightarrow{A} y$ representing that $y \in f(x, A)$. The rules manipulate both kinds of formulas.

We are able to give cut-free calculi for CK and all its extensions in the set {ID, MP, CS, CEM}, except those including *both* CEM and MP. The completeness of the calculi is an immediate consequence of the admissibility of cut.

We show that one can derive a decision procedure from the cut-free calculi. Whereas the decidability of these systems was already proved by Nute (by a finite-model property argument), our calculi give the first *constructive* proof of decidability. As usual, we obtain a terminating proof search mechanism by controlling the backward application of some critical rules. By estimating the size of the finite derivations of a given sequent, we also obtain a polynomial space complexity bound for these logics.

We can also obtain a tighter complexity bound for the logics CK{+ID}, as they satisfy a kind of *disjunction property*.

Our calculi can be the starting point to develop goal-oriented proof procedures, according to the paradigm of Uniform Proofs by Miller and others [Miller et al. 1991; Gabbay and Olivetti 2000]. Calculi of these kind are suitable for logic programming applications. As a preliminary result we present a goal-directed calculus for CK, where the "clauses" are a sort of conditional Harrop formulas.

We finally present a simple implementation of our calculi, called **CondLean**; it is a Prolog program which is inspired to the *lean* methodology [Beckert and Posegga 1995; Fitting 1998], in which every clause of a predicate `prove` implements an axiom or rule of the calculus and the proof search is provided for free by the mere depth-first search mechanism of Prolog, without any ad hoc mechanism.

The plan of the paper is as follows: in section 2 we introduce the conditional systems we consider, in section 3 we present the sequent calculi for conditional systems above. In section 4 we analyze the calculi in order to obtain a decision procedure and an explicit space complexity bound for the basic conditional system, CK, and for the mentioned extensions of it. In section 5 we give a more detailed analysis of CK and the extensions MP and ID. In section 6 we present a goal-directed proof procedure based on SeqCK. In section 7 we present the theorem prover CondLean. In section 8 we discuss some related work and possible future research.

2. CONDITIONAL LOGICS

Conditional logics are extensions of classical logic obtained by adding the conditional operator $\Rightarrow$. In this paper, we only consider propositional conditional logics.

A propositional conditional language $\mathcal{L}$ contains the following items:

- a set of propositional variables ATM;
- the symbol of *false* $\bot$;
- a set of connectives[2] $\rightarrow, \Rightarrow$.

We define formulas of $\mathcal{L}$ as follows:

- $\bot$ and the propositional variables of ATM are *atomic formulas*;
- if A and B are formulas, $A \rightarrow B$ and $A \Rightarrow B$ are *complex formulas*.

We adopt the *selection function semantics*. We consider a non-empty set of possible worlds $\mathcal{W}$. Intuitively, the selection function f selects, for a world w and a formula A, the set of worlds of $\mathcal{W}$ which are *closer* to w given the information A. A conditional formula $A \Rightarrow B$ holds in a world w if the formula B holds in *all the worlds selected by f for w and A*.

A model is a triple $\mathcal{M} = \langle \mathcal{W}, f, [\ ] \rangle$ where:

- $\mathcal{W}$ is a non empty set of items called *worlds*;
- f is the so-called *selection function* and has the following type:

$$f\colon \mathcal{W} \times 2^{\mathcal{W}} \longrightarrow 2^{\mathcal{W}}$$

- $[\ ]$ is the *evaluation function*, which assigns to an atom $P \in ATM$ the set of worlds where P is true, and is extended to the other formulas as follows:
  - $\star$ $[\bot] = \emptyset$
  - $\star$ $[A \rightarrow B]$=$(\mathcal{W}$ - $[A]) \cup [B]$
  - $\star$ $[A \Rightarrow B]$=$\{w \in \mathcal{W} \mid f(w, [A]) \subseteq [B]\}$

Observe that we have defined f taking $[A]$ rather than A (i.e. $f(w,[A])$ rather than $f(w,A)$) as an argument; this is equivalent to define f on formulas, i.e. $f(w,A)$ but imposing that if $[A]$=$[A']$ in the model, then $f(w, A)$=$f(w, A')$. This condition is called *normality*.

The semantics above characterizes the *basic conditional system*, called CK. An axiomatization of the CK system is given by:

- all tautologies of classical propositional logic.
- (Modus Ponens) $\dfrac{A \quad A \rightarrow B}{B}$
- (RCEA) $\dfrac{A \leftrightarrow B}{(A \Rightarrow C) \leftrightarrow (B \Rightarrow C)}$
- (RCK) $\dfrac{(A_1 \wedge \ldots \wedge A_n) \rightarrow B}{(C \Rightarrow A_1 \wedge \ldots \wedge C \Rightarrow A_n) \rightarrow (C \Rightarrow B)}$

Other conditional systems are obtained by assuming further properties on the selection function; we consider the following standard extensions of the basic system CK:

[2]The usual connectives $\top$, $\wedge$, $\vee$ and $\neg$ can be defined in terms of $\bot$ and $\rightarrow$.

System	Axiom	Model condition
ID	$A \Rightarrow A$	$f(w,[A]) \subseteq [A]$
MP	$(A \Rightarrow B) \rightarrow (A \rightarrow B)$	$w \in [A] \rightarrow w \in f(w,[A])$
CS	$(A \wedge B) \rightarrow (A \Rightarrow B)$	$w \in [A] \rightarrow f(w,[A]) \subseteq \{w\}$
CEM	$(A \Rightarrow B) \vee (A \Rightarrow \neg B)$	$\mid f(w,[A]) \mid \leq 1$

From now on we use the following notation: AX is the set of axioms considered, i.e. AX={CEM, CS, ID, MP}. S stands for any subset of AX, i.e. S $\subseteq$ AX. We also denote with S* each S' such that S $\subseteq$ S' $\subseteq$ AX; for instance, CS+ID* is used to represent any of the following systems: CK+CS+ID, CK+CEM+CS+ID, CK+CS+ID+MP and CK+CEM+CS+ID+MP.

The above axiomatization is complete with respect to the semantics. To denote that A is valid in the axiomatization S we write $\vdash_S A$.

Theorem 2.1 Completeness of axiomatization, [Nute 1980]. *If a formula A is valid in S then it is valid in the respective axiomatization, i.e.* $\vdash_S A$.

Observe that the condition (CS) is derivable in systems characterized by conditions (CEM) and (MP). Indeed, for (CEM) we have that $(*) \mid f(w,[A]) \mid \leq 1$; if $w \in [A]$, then we have that $w \in f(w,[A])$ by (MP), but by $(*)$ we have that $f(w,[A]) = \{w\}$, satisfying the (CS) condition.

3. A SEQUENT CALCULUS FOR CONDITIONAL LOGICS

In this section we present **SeqS**, a sequent calculs for the conditional systems introduced above. The calculi make use of labels to represent possible worlds. We consider a language $\mathcal{L}$ and a denumerable alphabet of labels $\mathcal{A}$, whose elements are denoted by $x, y, z, \ldots$.
There are two kinds of labelled formulas:

(1) *world formulas*, denoted by x: A, where $x \in \mathcal{A}$ and $A \in \mathcal{L}$, used to represent that A holds in a world x;

(2) *transition formulas*, denoted by $x \xrightarrow{A} y$, where $x, y \in \mathcal{A}$ and $A \in \mathcal{L}$. A transition formula $x \xrightarrow{A} y$ represents that $y \in f(x, [A])$.

A **sequent** is a pair $\langle \Gamma, \Delta \rangle$, usually denoted with $\Gamma \vdash \Delta$, where Γ and Δ are multisets of labelled formulas.

For technical reasons we introduce the notion of positive/negative occurrences of a world formula:

Definition 3.1 Positive and negative occurrences of a world formula. *Given a world formula $x : A$, we say that:*

- *$x : A$ occurs positively in $x : A$;*
- *if a world formula $x : B \rightarrow C$ occurs positively (negatively) in $x : A$, then $x : C$ occurs positively (negatively) in $x : A$ and $x : B$ occurs negatively (positively) in $x : A$;*
- *if a formula $x : B \Rightarrow C$ occurs positively (negatively) in $x : A$, then $x : C$ occurs positively (negatively) in $x : A$.*

A world formula $x : A$ occurs positively (negatively) in a multiset Γ if $x : A$ occurs positively (negatively) in some world formula $x : G \in \Gamma$. Given $\Gamma \vdash \Delta$, we say that $x : A$ occurs positively (negatively) in $\Gamma \vdash \Delta$ if $x : A$ occurs positively (negatively) in Γ or $x : A$ occurs negatively (positively) in Δ.

The intuitive meaning of $\Gamma \vdash \Delta$ is: every model that satisfies all labelled formulas of Γ in the respective worlds (specified by the labels) satisfies at least one of the labelled formulas of Δ (in those worlds). This is made precise by the notion of *validity* of a sequent given in the next definition:

Definition 3.2 Sequent validity. *Given a model*

$$\mathcal{M} = \langle \mathcal{W}, f, [\ ] \rangle$$

for $\mathcal{L}$, and a label alphabet $\mathcal{A}$, we consider any mapping

$$I : \mathcal{A} \rightarrow \mathcal{W}$$

Let F be a labelled formula, we define $\mathcal{M} \models_I F$ as follows:

- $\mathcal{M} \models_I x{:}\ A$ iff $I(x) \in [A]$
- $\mathcal{M} \models_I x \xrightarrow{A} y$ iff $I(y) \in f(I(x), [A])$

We say that $\Gamma \vdash \Delta$ is valid *in $\mathcal{M}$ if for every mapping $I : \mathcal{A} \rightarrow \mathcal{W}$, if $\mathcal{M} \models_I F$ for every $F \in \Gamma$, then $\mathcal{M} \models_I G$ for some $G \in \Delta$. We say that $\Gamma \vdash \Delta$ is valid in a system (CK or one of its extensions) if it is valid in every $\mathcal{M}$ satisfying the specific conditions for that system (if any).*

In Figures 1 and 2 we present the calculi for CK and its mentioned extensions. Observe that the restrictions $x \neq y$ on (CS) and $y \neq z$ on (CEM) have a different nature than the restriction on ($\Rightarrow$ R). The former ones are needed to avoid a looping application of the rules, as the right premise would be identical to the conclusion modulo label substitution. The latter one is necessary to preserve the soundness of the calculus.

Example 3.3. *We show a derivation of an instance of the (ID) axiom ($P \in ATM$).*

$$\dfrac{\dfrac{x \xrightarrow{P} y, y : P \vdash y : P}{x \xrightarrow{P} y \vdash y : P}\,(ID)}{\vdash x : P \Rightarrow P}\,(\Rightarrow R)$$

Example 3.4. *We show a derivation of an instance of the (MP) axiom ($P, Q \in ATM$).*

$$\dfrac{\dfrac{\dfrac{\dfrac{x : P \Rightarrow Q, x : P \vdash x \xrightarrow{P} x, x : P, x : Q}{x : P \Rightarrow Q, x : P \vdash x \xrightarrow{P} x, x : Q}\,(MP) \qquad x : P \Rightarrow Q, x : P, x : Q \vdash x : Q}{x : P \Rightarrow Q, x : P \vdash x : Q}\,(\Rightarrow L)}{x : P \Rightarrow Q \vdash x : P \rightarrow Q}\,(\rightarrow R)}{\vdash x : (P \Rightarrow Q) \rightarrow (P \rightarrow Q)}\,(\rightarrow R)$$

$$(\mathbf{AX})\quad \Gamma, x : P \vdash \Delta, x : P \quad (P \in ATM) \qquad\qquad (\mathbf{A}\bot)\quad \Gamma, x : \bot \vdash \Delta$$

$$(\rightarrow \mathbf{L})\ \frac{\Gamma \vdash x : A, \Delta \qquad \Gamma, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B \vdash \Delta} \qquad\qquad (\rightarrow \mathbf{R})\ \frac{\Gamma, x : A \vdash x : B, \Delta}{\Gamma \vdash x : A \rightarrow B, \Delta}$$

$$(\Rightarrow \mathbf{L})\ \frac{\begin{array}{c}\Gamma, x : A \Rightarrow B \vdash x \xrightarrow{A} y, \Delta \\ \Gamma, x : A \Rightarrow B, y : B \vdash \Delta\end{array}}{\Gamma, x : A \Rightarrow B \vdash \Delta} \qquad\qquad (\Rightarrow \mathbf{R})\ \frac{\Gamma, x \xrightarrow{A} y \vdash y : B, \Delta}{\Gamma \vdash x : A \Rightarrow B, \Delta}\ (y \notin \Gamma, \Delta)$$

$$(\mathbf{EQ})\ \frac{u : A \vdash u : B \qquad u : B \vdash u : A}{\Gamma, x \xrightarrow{A} y \vdash x \xrightarrow{B} y, \Delta}$$

Fig. 1. Sequent calculus SeqCK.

$$(\mathbf{ID})\ \frac{\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y \vdash \Delta}$$

$$(\mathbf{MP})\ \frac{\Gamma \vdash x \xrightarrow{A} x, x : A, \Delta}{\Gamma \vdash x \xrightarrow{A} x, \Delta}$$

$$(\mathbf{CS})\ \frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A \qquad \Gamma[x/u, y/u], u \xrightarrow{A} u \vdash \Delta[x/u, y/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\ (x \neq y, u \notin \Gamma, \Delta)$$

$$(\mathbf{CEM})\ \frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \qquad (\Gamma, x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\ (y \neq z, u \notin \Gamma, \Delta)$$

Fig. 2. SeqS's rules for systems extending CK. We denote with $\Sigma[x/u]$ the multiset obtained from Σ by replacing the label x by u wherever it occurs.

EXAMPLE 3.5. *We show a derivation of an instance of the (CS) axiom ($P, Q \in$*

$$(\wedge\mathbf{L})\ \frac{\Gamma, x:A, x:B \vdash \Delta}{\Gamma, x:A\wedge B \vdash \Delta} \qquad (\wedge\mathbf{R})\ \frac{\Gamma \vdash \Delta, x:A \qquad \Gamma \vdash \Delta, x:B}{\Gamma \vdash \Delta, x:A\wedge B}$$

$$(\vee\mathbf{L})\ \frac{\Gamma, x:A \vdash \Delta \qquad \Gamma, x:B \vdash \Delta}{\Gamma, x:A\vee B \vdash \Delta} \qquad (\vee\mathbf{R})\ \frac{\Gamma \vdash \Delta, x:A, x:B}{\Gamma \vdash \Delta, x:A\vee B}$$

$$(\neg\mathbf{L})\ \frac{\Gamma \vdash \Delta, x:A}{\Gamma, x:\neg A \vdash \Delta} \qquad (\neg\mathbf{R})\ \frac{\Gamma, x:A \vdash \Delta}{\Gamma \vdash \Delta, x:\neg A}$$

$$(\mathbf{A}\top)\ \Gamma \vdash \Delta, x:\top$$

Fig. 3. Additional axioms and rules in SeqS for the other boolean operators, derived from the rules in Figure 1 by the usual equivalences.

ATM).

$$\cfrac{\cfrac{\cfrac{\cfrac{x:P, x:Q, x \xrightarrow{P} y \vdash y:Q, x:P \qquad u:P, u:Q, u \xrightarrow{P} u \vdash u:Q}{x:P, x:Q, x \xrightarrow{P} y \vdash y:Q}\,(CS)}{x:P\wedge Q, x \xrightarrow{P} y \vdash y:Q}\,(\wedge L)}{x:P\wedge Q \vdash x:P \Rightarrow Q}\,(\Rightarrow R)}{\vdash x:(P\wedge Q) \rightarrow (P \Rightarrow Q)}\,(\rightarrow R)$$

Example 3.6. *We show a derivation of an instance of the (CEM) axiom ($P, Q \in ATM$).*

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x \xrightarrow{P} y, x \xrightarrow{P} z, z:Q \vdash y:Q, x \xrightarrow{P} y \qquad x \xrightarrow{P} u, x \xrightarrow{P} u, u:Q \vdash u:Q}{x \xrightarrow{P} y, x \xrightarrow{P} z, z:Q \vdash y:Q}\,(CEM)}{x \xrightarrow{P} y, x \xrightarrow{P} z \vdash y:Q, z:\neg Q}\,(\neg R)}{x \xrightarrow{P} y \vdash y:Q, x:P \Rightarrow \neg Q}\,(\Rightarrow R)}{\vdash x:P \Rightarrow Q, x:P \Rightarrow \neg Q}\,(\Rightarrow R)}{\vdash x:(P \Rightarrow Q) \vee (P \Rightarrow \neg Q)}\,(\vee R)$$

3.1 Basic Structural Properties of SeqS

In order to prove that the sequent calculus SeqS is sound and complete with respect to the semantics, we introduce some structural properties.

First of all we define the complexity of a labelled formula:

Definition 3.7 Complexity of a labelled formula cp(F). *We define the complexity of a labelled formula F as follows:*

(1) $cp\ (x:A) = 2 * \mid A \mid$

(2) $cp\ (x \xrightarrow{A} y) = 2 * \mid A \mid +1$

where $\mid A \mid$ *is the number of symbols occurring in the string representing the formula* A.

We can prove the following Lemma:

LEMMA 3.8. *Given any multiset of formulas* Γ *and* Δ*, and a labelled formula* F*, we have that* $\Gamma, F \vdash \Delta, F$ *is derivable in SeqS.*

Proof. By induction on the complexity of the formula F. The proof is easy and left to the reader.

□

LEMMA 3.9 HEIGHT-PRESERVING LABEL SUBSTITUTION. *If a sequent* $\Gamma \vdash \Delta$ *has a derivation of height* h*, then* $\Gamma[x/y] \vdash \Delta[x/y]$ *has a derivation of height* $\leq h$*, where* $\Gamma[x/y] \vdash \Delta[x/y]$ *is the sequent obtained from* $\Gamma \vdash \Delta$ *by replacing a label* x *by a label* y *wherever it occurs.*

Proof. By induction on the height of a derivation of $\Gamma \vdash \Delta$. We show the most interesting cases, the other ones are easy and left to the reader. We begin with the case when (CS) is applied to $\Gamma \vdash \Delta$ with a derivation of height h of the form:

$$\frac{(1)\Gamma', y \xrightarrow{A} x \vdash y : A, \Delta \qquad (2)\Gamma'[y/u, x/u], u \xrightarrow{A} u \vdash \Delta[y/u, x/u]}{\Gamma', y \xrightarrow{A} x \vdash \Delta}\ (CS)$$

Our goal is to find a proof of height $\leq h$ of $\Gamma'[x/y], y \xrightarrow{A} y \vdash \Delta[x/y]$. Applying the inductive hypothesis to (2), we have a proof of height $\leq h-1$ of the sequent $(3)(\Gamma'[y/u, x/u])[u/y], y \xrightarrow{A} y \vdash (\Delta[y/u, x/u])[u/y]$, but (3) corresponds to $\Gamma'[x/y], y \xrightarrow{A} y \vdash \Delta[x/y]$, since labels x and y have both been replaced by a *new* label u in (2), and the proof is over.

The other interesting case is when ($\Rightarrow$ R) is the rule ending the derivation (i.e. the rule applied to $\Gamma \vdash \Delta$); the situation is as follows:

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta', y : B}{\Gamma \vdash \Delta', x : A \Rightarrow B}\ (\Rightarrow R)$$

We show that there is a derivation of $\Gamma[x/y] \vdash \Delta'[x/y], y : A \Rightarrow B$, of height less or equal than h. First, observe that the label y in the above derivation is a *new* label, not occurring in the conclusion of ($\Rightarrow$ R); therefore, we can rename it with another new label, for instance w:

$$\frac{\Gamma, x \xrightarrow{A} w \vdash \Delta', w : B}{\Gamma \vdash \Delta', x : A \Rightarrow B}\ (\Rightarrow R)$$

Applying the inductive hypothesis on the premise of ($\Rightarrow$ R), we obtain a proof of $\Gamma[x/y], y \xrightarrow{A} w \vdash \Delta'[x/y], w : B$ of height no greater than $h-1$. We conclude by an application of ($\Rightarrow$ R), obtaining a proof (height $\leq h$) of $\Gamma[x/y] \vdash \Delta'[x/y], y : A \Rightarrow B$.

□

Theorem 3.10 Height-preserving admissibility of weakening. *If a sequent $\Gamma \vdash \Delta$ has a derivation of height h, then $\Gamma \vdash \Delta, F$ and $\Gamma, F \vdash \Delta$ have a derivation of height $\leq h$.*

Proof. By induction on the height of a derivation of $\Gamma \vdash \Delta$. The proof is easy and left to the reader.

□

Theorem 3.11 Height-preserving invertibility of rules. *Let $\Gamma \vdash \Delta$ be the conclusion of an application of one of the SeqS's rules, say R, with R different from (EQ). If $\Gamma \vdash \Delta$ is derivable, then the premise(s) of R is (are) derivable with a derivation of (at most) the same height, i.e. SeqS's rules are height-preserving invertible.*

Proof. We consider each of the rules. We distinguish two groups of rules:

(i) $(\rightarrow L), (\rightarrow R)$, and $(\Rightarrow R)$: for these rules we proceed by an inductive argument on the height of a proof of their conclusions; the cases of $(\rightarrow L)$ and $(\rightarrow R)$ are easy and left to the reader. The proof for ($\Rightarrow$ R) is as follows: for any y, if $\Gamma \vdash \Delta, x : A \Rightarrow B$ is an axiom, then $\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B$ is an axiom too, since axioms are restricted to atomic formulas. If $h > 0$ and the proof of $\Gamma \vdash \Delta, x : A \Rightarrow B$ is concluded (looking forward) by any rule other than ($\Rightarrow$ R), we apply the inductive hypothesis to the premise(s), then we conclude by applying the same rule. If the derivation of $\Gamma \vdash \Delta, x : A \Rightarrow B$ is ended by ($\Rightarrow$ R) we have the following subcases:

⋆ $x : A \Rightarrow B$ is the principal formula of ($\Rightarrow$ R): the proof is ended as follows:

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B}{\Gamma \vdash \Delta, x : A \Rightarrow B} \ (\Rightarrow R)$$

We have a proof of $\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B$ of height $h-1$ and the proof is over;

⋆ $x : A \Rightarrow B$ is not the principal formula of ($\Rightarrow$ R): the proof is ended as follows:

$$\frac{\Gamma, w \xrightarrow{C} z \vdash \Delta, z : D, x : A \Rightarrow B}{\Gamma \vdash \Delta, x : A \Rightarrow B, w : C \Rightarrow D} \ (\Rightarrow R)$$

where z is a "new" label and then, without loss of generality, we can assume that z is not y, since we can apply the height-preserving label substitution. By inductive hypothesis on the premise we obtain a derivation of $\Gamma, w \xrightarrow{C} z, x \xrightarrow{A} y \vdash \Delta, z : D, y : B$ from which we conclude as follows:

$$\frac{\Gamma, w \xrightarrow{C} z, x \xrightarrow{A} y \vdash \Delta, z : D, y : B}{\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B, w : C \Rightarrow D} \ (\Rightarrow R)$$

(ii) $(\Rightarrow L), (ID), (MP), (CS)$, and (CEM): these rules are height-preserving invertible, since their premise(s) is (are) obtained by weakening from the conclusion, and weakening is height-preserving admissible (Theorem 3.10). We consider each of the rules:

- ($\Rightarrow$ L): its premises are obtained by weakening from the conclusion, i.e. given a proof (height h) of $\Gamma, x : A \Rightarrow B \vdash \Delta$, by weakening we have proofs of height $\leq h$ of $\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y$ and $\Gamma, x : A \Rightarrow B, y : B \vdash \Delta$;
- (CS): given a derivation of $\Gamma, x \xrightarrow{A} y \vdash \Delta$, we can obtain a proof of no greater height of $\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A$ since weakening is height-preserving admissible; we can also obtain a proof of $\Gamma[x/u, y/u], u \xrightarrow{A} u \vdash \Delta[x/u, y/u]$ since label substitution is height-preserving admissible (Lemma 3.9);
- (CEM): given a proof (height h) of $\Gamma, x \xrightarrow{A} y \vdash \Delta$, we can obtain a proof of at most the same height of $\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$ by weakening. There is also a proof (height $\leq h$) of $(\Gamma, x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]$ by the height-preserving label substitution.

The cases of (ID) and (MP) are similar and left to the reader.

□

It is worth noticing that the height-preserving invertibility also preserves the number of applications of the rules in a proof, that is to say: if $\Gamma_1 \vdash \Delta_1$ is derivable by Theorem 3.11 since it is the premise of a backward application of an invertible rule R to $\Gamma_2 \vdash \Delta_2$, then it has a derivation containing *the same rule applications* of the proof of $\Gamma_2 \vdash \Delta_2$. For instance, if (1) $\Gamma, x \xrightarrow{A} y \vdash \Delta$ is derivable with a proof Π, then (2) $\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta$ is derivable since (ID) is invertible; moreover, there exists a proof of (2) containing the same rules of Π, obtained by adding $y : A$ in each sequent of Π from which (1) descends. This fact will be systematically used throughout the paper, in the sense that we will assume that every proof transformation due to the invertibility preserves the number of rules applications in the initial proof.

Theorem 3.12 Height-preserving admissibility of contraction. *The rules of contraction are height-preserving admissible in SeqS, i.e. if a sequent* $\Gamma \vdash \Delta, F, F$ *is derivable in SeqS, then there is a derivation of no greater height of* $\Gamma \vdash \Delta, F$*, and if a sequent* $\Gamma, F, F \vdash \Delta$ *is derivable in SeqS, then there is a derivation of no greater height of* $\Gamma, F \vdash \Delta$*. Moreover, the proof of the contracted sequent does not add any rule application to the initial proof* [3].

Proof. By simultaneous induction on the height of derivation for left and right contraction. If $h = 0$, i.e. $\Gamma \vdash \Delta, F, F$ is an axiom, then we have to consider the following subcases:

- $w : \bot \in \Gamma$: in this case, obviously $\Gamma \vdash \Delta, F$ is an axiom too;
- an atom $G \in \Gamma \cap \Delta$: we conclude, since $\Gamma \vdash \Delta, F$ is an axiom too;
- F is an atom and $F \in \Gamma$: the proof is over, observing that $\Gamma \vdash \Delta, F$ is an axiom too.

The proof of the case where $\Gamma, F, F \vdash \Delta$ is an axiom is symmetric.

If $h > 0$, consider the last rule applied (looking forward) to derive the premise of contraction. We distinguish two cases:

[3] In this case we say that contractions are rule-preserving admissible.

- the contracted formula F is not principal in it: in this case, both occurrences of F are in the premise(s) of the rule, which have a smaller derivation height. By the inductive hypothesis, they can be contracted and the conclusion is obtained by applying the rule to the contracted premise(s). As an example, consider a proof ended by an application of (CS) as follows:

$$\frac{\Gamma', x \xrightarrow{A} y, F, F \vdash \Delta, x : A \quad \Gamma'[x/u, y/u], u \xrightarrow{A} u, F[x/u, y/u], F[x/u, y/u] \vdash \Delta[x/u, y/u]}{\Gamma', x \xrightarrow{A} y, F, F \vdash \Delta} (CS)$$

By the inductive hypothesis, we have a proof of no greater height than the respective premise of the sequents $\Gamma', x \xrightarrow{A} y, F \vdash \Delta, x : A$ and $\Gamma'[x/u, y/u], u \xrightarrow{A} u, F[x/u, y/u] \vdash \Delta[x/u, y/u]$, from which we conclude as follows, obtaining a proof of (at most) the same height as the initial proof:

$$\frac{\Gamma', x \xrightarrow{A} y, F \vdash \Delta, x : A \qquad \Gamma'[x/u, y/u], u \xrightarrow{A} u, F[x/u, y/u] \vdash \Delta[x/u, y/u]}{\Gamma', x \xrightarrow{A} y, F \vdash \Delta} (CS)$$

- the contracted formula F is principal in it: we consider all the rules:
  - ⋆ (→ L): the proof is ended as follows:

$$\frac{(1)\Gamma, x : A \rightarrow B \vdash \Delta, x : A \qquad (2)\Gamma, x : A \rightarrow B, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B, x : A \rightarrow B \vdash \Delta} (\rightarrow L)$$

Since (→ L) is height-preserving invertible (see Theorem 3.11), there is a derivation of no greater height than (1) of $(1a)\Gamma \vdash \Delta, x : A, x : A$ and $(1b)\Gamma, x : B \vdash \Delta, x : A$ and of no greater height than (2) of $(2a)\Gamma, x : B \vdash \Delta, x : A$ and $(2b)\Gamma, x : B, x : B \vdash \Delta$. Applying the inductive hypothesis on $(1a)$ and $(2b)$ and applying (→ L) to the contracted sequents, we obtain a derivation of no greater height ending with (be $(1a')$ and $(2b')$ the contracted sequents):

$$\frac{(1a')\Gamma \vdash \Delta, x : A \qquad (2b')\Gamma, x : B \vdash \Delta}{\Gamma, x : A \rightarrow B \vdash \Delta} (\rightarrow L)$$

  - ⋆ (→ R): we proceed as in the previous case, since (→ R) is height-preserving invertible;
  - ⋆ (⇒ L): we have a proof ending with:

$$\frac{\Gamma, x : A \Rightarrow B, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \quad \Gamma, x : A \Rightarrow B, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B, x : A \Rightarrow B \vdash \Delta} (\Rightarrow L)$$

Applying the inductive hypothesis to both premises we can immediately conclude as follows:

$$\frac{\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad \Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta} (\Rightarrow L)$$

- ⋆ ($\Rightarrow$ R): the proof is ended by:

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A \Rightarrow B, y : B}{\Gamma \vdash \Delta, x : A \Rightarrow B, x : A \Rightarrow B} (\Rightarrow R)$$

Applying the height-preserving invertibility of ($\Rightarrow$ R), we have a proof of $(1)\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} z \vdash \Delta, y : B, z : B$. Applying the height-preserving label substitution (Lemma 3.9) to (1), replacing z with y, we obtain a derivation of the sequent $(2)\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta, y : B, y : B$, since y and z are new labels not occurring in Γ, Δ. We can then apply the inductive hypothesis on (2), obtaining a proof of $(3)\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B$, from which we conclude by an application of ($\Rightarrow$ R):

$$\frac{(3)\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B}{\Gamma \vdash \Delta, x : A \Rightarrow B} (\Rightarrow R)$$

- ⋆ (EQ): the proof is ended as follows:

$$\frac{u : A \vdash u : A' \qquad u : A' \vdash u : A}{\Gamma', x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A'} y} (EQ)$$

It is easy to observe that (EQ) does not require the second occurrence of $x \xrightarrow{A} y$; thus we obtain the following proof:

$$\frac{u : A \vdash u : A' \qquad u : A' \vdash u : A}{\Gamma', x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A'} y} (EQ)$$

The other half is symmetric ($\Gamma, x \xrightarrow{A'} y \vdash \Delta', x \xrightarrow{A} y, x \xrightarrow{A} y$ derives from (EQ));

- ⋆ (ID): given a proof ending with:

$$\frac{\Gamma, y : A, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta} (ID)$$

we can apply the inductive hypothesis on the premise, obtaining a proof of no greater height of $\Gamma, y : A, x \xrightarrow{A} y \vdash \Delta$, from which we conclude by an application of (ID);

- ⋆ (MP): the proof is ended as follows:

$$\frac{\Gamma \vdash \Delta, x \xrightarrow{A} x, x \xrightarrow{A} x, x : A}{\Gamma \vdash \Delta, x \xrightarrow{A} x, x \xrightarrow{A} x} (MP)$$

We can apply the inductive hypothesis on the premise, obtaining a proof of no greater height of $\Gamma \vdash \Delta, x \xrightarrow{A} x, x : A$, from which we conclude by an application of (MP);

- ⋆ (CEM): given a proof ending with:

$$\frac{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \qquad (\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta}\ (CEM)$$

  we can apply the inductive hypothesis on the two premises, obtaining proofs of $\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$ and $(\Gamma, x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]$, from which we conclude by an application of (CEM);
- ⋆ (CS): given a derivation ending as follows:

$$\frac{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta, x : A \qquad (\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta)[x/u, y/u]}{\Gamma, x \xrightarrow{A} y, x \xrightarrow{A} y \vdash \Delta}\ (CS)$$

  by the inductive hypothesis on the premises we can find derivations of $\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A$ and $(\Gamma, x \xrightarrow{A} y \vdash \Delta)[x/u, y/u]$, then we conclude by an application of (CS).

Notice that, in each case, the proof is concluded by applying the same rule under consideration, i.e. the derivation of the contracted sequent does not add any rule application to the proof of the initial sequent.

□

We now consider the cut rule:

$$\frac{\Gamma \vdash \Delta, F \quad F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (cut)$$

where F is any labelled formula. We prove that this rule is admissible in all SeqS calculi, except those which contain *both* CEM and MP. For the systems without CEM the proof follows the standard pattern. For systems with CEM it is more complicated. For systems with both CEM and MP the admissibility of cut is open at present: we are unable to prove it, and we do not have a counterexample (see Appendix A in [Olivetti et al. 2005] for details).

Now we prove that cut is admissible in all SeqS systems, except for systems allowing both (CEM) and (MP). From now on, we restrict our concern to all other systems.

THEOREM 3.13 ADMISSIBILITY OF CUT. *In systems SeqS - SeqCEM+MP* if* $\Gamma \vdash \Delta, F$ *and* $F, \Gamma \vdash \Delta$ *are derivable, so* $\Gamma \vdash \Delta$.

Proof. As usual, the proof proceeds by a double induction over the complexity of the cut formula and the sum of the heights of the derivations of the two premises of cut, in the sense that we replace one cut by one or several cuts on formulas of smaller complexity, or on sequents derived by shorter derivations. We first consider the case of systems without (CEM). We have several cases: (i) one of the two premises is an axiom, (ii) the last step of *one* of the two premises is obtained by a rule in which F is *not* the principal formula, (iii) F is the principal formula in the last step of *both* derivations.

(i). If one of the two premises is an axiom then either $\Gamma \vdash \Delta$ is an axiom, or the premise which is not an axiom contains two copies of F and $\Gamma \vdash \Delta$ can be obtained by contraction, which is admissible (see Theorem 3.12 above).

(ii). We distinguish two cases:

(1) the sequent where F is not principal is derived by any rule (R), except the (EQ) rule. This case is standard, we can permute (R) over the cut: i.e. we cut the premise(s) of (R) and then we apply (R) to the result of cut. As an example, consider the case when $F = x \xrightarrow{A} y$ and it is the principal formula of an application of (CS) in the right derivation, and (CS) is also the last rule in the left derivation; the situation is as follows (we denote the substitution $\Sigma[x/u, y/u]$ with $\Sigma(u)$):

$$\dfrac{\dfrac{(1)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y, x : A' \qquad (2)\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u), u \xrightarrow{A} u}{(5)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y}\,(CS) \qquad \dfrac{(3)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta, x : A \qquad (4)\Gamma'(u), u \xrightarrow{A'} u, u \xrightarrow{A} u \vdash \Delta(u)}{(6)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta}\,(CS)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}\,(cut)$$

We can apply the inductive hypothesis on the height to replace the following cut[4]:

$$\dfrac{(2)\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u), u \xrightarrow{A} u \qquad (4)\Gamma'(u), u \xrightarrow{A'} u, u \xrightarrow{A} u \vdash \Delta(u)}{(7)\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u)}\,(cut)$$

We replace the initial cut as follows:

$$\dfrac{\dfrac{(1)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x : A', x \xrightarrow{A} y \qquad (6')\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y \vdash \Delta, x : A'}{\Gamma', x \xrightarrow{A'} y \vdash \Delta, x : A'}\,(cut) \qquad (7)\Gamma'(u), u \xrightarrow{A'} u \vdash \Delta(u)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}\,(CS)$$

where $(6')$ is obtained by weakening from (6).

(2) if one of the sequents, say $\Gamma \vdash \Delta, F$ is obtained by the (EQ) rule, where F is not principal, then also $\Gamma \vdash \Delta$ is derivable by the (EQ) rule and we are done.

(iii). F is the principal formula in both the inferences steps leading to the two cut premises. There are seven subcases: F is introduced (a) by ($\rightarrow$ L), ($\rightarrow$ R), (b) by $(\Rightarrow L), (\Rightarrow R)$, (c) by (EQ), (d) by (ID) on the left and by (EQ) on the right, (e) by (EQ) on the left and by (MP) on the right, (f) by (ID) on the left and by

[4]Notice that cutting (4) and (2) corresponds to cutting (2) and (6) with the necessary label substitutions, i.e. permuting (CS) over the cut.

(MP) on the right and (g) by (CS) on the left and by (EQ) on the right. The list is exhaustive[5].

(a). This case is standard and left to the reader.

(b). $F = x : A \Rightarrow B$ is introduced by ($\Rightarrow R$) and ($\Rightarrow L$). Then we have

$$\dfrac{\dfrac{(1)\Gamma, x \xrightarrow{A} z \vdash \Delta, z : B}{(2)\Gamma \vdash \Delta, x : A \Rightarrow B}(\Rightarrow R) \quad \dfrac{(3)\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \quad (4)\Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{(5)\Gamma, x : A \Rightarrow B \vdash \Delta}(\Rightarrow L)}{\Gamma \vdash \Delta}(cut)$$

where z does not occur in Γ, Δ and $z \neq x$; By Lemma 3.9, we obtain that $(1^{'})\Gamma, x \xrightarrow{A} y \vdash y : B, \Delta$ is derivable by a derivation of no greater height than (1); moreover, we can obtain a proof of no greater height of $(2^{'})\Gamma \vdash \Delta, x : A \Rightarrow B, x \xrightarrow{A} y$ and of $(2^{''})\Gamma, y : B \vdash \Delta, x : A \Rightarrow B$, both by weakening from (2) (Theorem 3.10).

First, we can make the following cut, which uses the inductive hypothesis on the height:

$$\dfrac{(2^{'})\Gamma \vdash \Delta, x : A \Rightarrow B, x \xrightarrow{A} y \qquad (3)\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y}{(6)\Gamma \vdash \Delta, x \xrightarrow{A} y}(cut)$$

By weakening, we have a proof of no greater height than (6) of $(6^{'})\Gamma \vdash \Delta, x \xrightarrow{A} y, y : B$. Thus we can replace the initial cut as follows:

$$\dfrac{\dfrac{(2^{''})\Gamma, y : B \vdash \Delta, x : A \Rightarrow B \quad (4)\Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, y : B \vdash \Delta}(cut) \quad \dfrac{(1^{'})\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B \quad (6^{'})\Gamma \vdash \Delta, x \xrightarrow{A} y, y : B}{\Gamma \vdash \Delta, y : B}(cut)}{\Gamma \vdash \Delta}(cut)$$

The upper cut on the left uses the induction hypothesis on the height, the others the induction hypothesis on the complexity of the cut formula.

(c). $F = x \xrightarrow{A} y$ is introduced by (EQ) in both premises, we have

$$\dfrac{\dfrac{(1)\, u : A^{'} \vdash u : A \;\; (2)\, u : A \vdash u : A^{'}}{\Gamma', x \xrightarrow{A^{'}} y \vdash x \xrightarrow{A} y, \Delta}(EQ) \quad \dfrac{(3)\, u : A \vdash u : A^{''} \;\; (4)\, u : A^{''} \vdash u : A}{\Gamma, x \xrightarrow{A} y \vdash x \xrightarrow{A^{''}} y, \Delta'}(EQ)}{\Gamma', x \xrightarrow{A^{'}} y \vdash x \xrightarrow{A^{''}} y, \Delta'}(cut)$$

where $\Gamma = \Gamma', x \xrightarrow{A^{'}} y$, $\Delta = x \xrightarrow{A^{''}} y, \Delta'$. (1)-(4) have been derived by a shorter derivation; thus we can replace the cut by cutting (1) and (3) on the one hand, and (4) and (2) on the other, which give respectively

$$(5)\ u : A^{'} \vdash u : A^{''} \text{ and } (6)\ u : A^{''} \vdash u : A^{'}.$$

Using (EQ) we obtain $\Gamma', x \xrightarrow{A^{'}} y \vdash \Delta', x \xrightarrow{A^{''}} y$.

[5]Notice that the case when $F = x \xrightarrow{A} x$ is introduced by (CS) on the left and (MP) on the right has not to be considered, since (CS) introduces a transition $x \xrightarrow{A} y$ (looking forward) only if $x \neq y$.

(d). $F = x \xrightarrow{A} y$ is introduced on the left by (ID) rule, and it is introduced on the right by (EQ). Thus we have

$$\dfrac{\dfrac{u : A' \vdash u : A \quad u : A \vdash u : A'}{(2)\Gamma', x \xrightarrow{A'} y \vdash \Delta, x \xrightarrow{A} y}(EQ) \quad \dfrac{(1)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y, y : A \vdash \Delta}{x \xrightarrow{A} y, \Gamma', x \xrightarrow{A'} y \vdash \Delta}(\mathrm{ID})}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(\mathrm{cut})$$

From (2) we can obtain a proof of no greater height of $(2')\Gamma', x \xrightarrow{A'} y, y : A \vdash \Delta, x \xrightarrow{A} y$ by weakening (Theorem 3.10); moreover, by label substitution (Lemma 3.9) and weakening we can find a derivation of no greater height than $u : A' \vdash u : A$'s of $(3)\Gamma', x \xrightarrow{A'} y, y : A' \vdash y : A, \Delta$. First, we replace the following cut by inductive hypothesis on the height:

$$\dfrac{(2')\Gamma', x \xrightarrow{A'} y, y : A \vdash \Delta, x \xrightarrow{A} y \qquad (1)\Gamma', x \xrightarrow{A'} y, x \xrightarrow{A} y, y : A \vdash \Delta}{(4)\Gamma', x \xrightarrow{A'} y, y : A \vdash \Delta}(cut)$$

From (4), we can find a proof of $(4')\Gamma', x \xrightarrow{A'} y, y : A, y : A' \vdash \Delta$ by weakening, thus we replace the initial cut as follows:

$$\dfrac{\dfrac{(3)\Gamma', x \xrightarrow{A'} y, y : A' \vdash y : A, \Delta \qquad (4')\Gamma', x \xrightarrow{A'} y, y : A, y : A' \vdash \Delta}{\Gamma', x \xrightarrow{A'} y, y : A' \vdash \Delta}(cut)}{\Gamma', x \xrightarrow{A'} y \vdash \Delta}(ID)$$

The above cut can be replaced by inductive hypothesis on the complexity of the cut formula;

(e). $F = x \xrightarrow{A} x$ is introduced on the left by (EQ) rule, and it is introduced on the right by (MP). Thus we have

$$\dfrac{\dfrac{(1)\Gamma \vdash \Delta', x \xrightarrow{A'} x, x \xrightarrow{A} x, x : A}{\Gamma \vdash \Delta', x \xrightarrow{A'} x, x \xrightarrow{A} x}(\mathrm{MP}) \quad \dfrac{u : A \vdash u : A' \quad u : A' \vdash u : A}{(2)\Gamma, x \xrightarrow{A} x \vdash \Delta', x \xrightarrow{A'} x}(EQ)}{\Gamma \vdash \Delta', x \xrightarrow{A'} x}(\mathrm{cut})$$

From (2) we obtain a proof of no greater height of $(2')\Gamma, x \xrightarrow{A} x \vdash \Delta', x \xrightarrow{A'} x, x : A$ by weakening, then we first replace the following cut by inductive hypothesis on the height:

$$\dfrac{(1)\Gamma \vdash \Delta', x \xrightarrow{A'} x, x \xrightarrow{A} x, x : A \qquad (2')\Gamma, x \xrightarrow{A} x \vdash \Delta', x \xrightarrow{A'} x, x : A}{(3)\Gamma \vdash \Delta', x \xrightarrow{A'} x, x : A}(cut)$$

from which we obtain a derivation of $(3^{'})\Gamma \vdash \Delta^{'}, x \xrightarrow{A^{'}} x, x : A, x : A^{'}$ by weakening. Moreover, by label substitution and weakening we can find a proof of (at most) the same height of $u : A \vdash u : A^{'}$ of $(4)\Gamma, x : A \vdash \Delta^{'}, x \xrightarrow{A^{'}} x, x : A^{'}$. The initial cut can be replaced as follows (the cut below can be eliminated by inductive hypothesis on the complexity of the cut formula):

$$\dfrac{\dfrac{(3^{'})\Gamma \vdash \Delta^{'}, x \xrightarrow{A^{'}} x, x : A, x : A^{'} \qquad (4)\Gamma, x : A \vdash \Delta^{'}, x \xrightarrow{A^{'}} x, x : A^{'}}{\Gamma \vdash \Delta^{'}, x \xrightarrow{A^{'}} x, x : A^{'}}\,(cut)}{\Gamma \vdash \Delta^{'}, x \xrightarrow{A^{'}} x}\,(MP)$$

(f). $F = x \xrightarrow{A} x$ is introduced on the right by (MP) rule and on the left by (ID). Thus we have

$$\dfrac{\dfrac{(1)\Gamma \vdash \Delta, x \xrightarrow{A} x, x : A}{(3)\Gamma \vdash \Delta, x \xrightarrow{A} x}\,(MP) \qquad \dfrac{(2)\Gamma, x \xrightarrow{A} x, x : A \vdash \Delta}{(4)\Gamma, x \xrightarrow{A} x \vdash \Delta}\,(ID)}{\Gamma \vdash \Delta}\,(\text{cut})$$

By weakening, we can find derivations of $(3^{'})\Gamma, x : A \vdash \Delta, x \xrightarrow{A} x$ and $(4^{'})\Gamma, x \xrightarrow{A} x \vdash \Delta, x : A$ of no greater height than (3) and (4), respectively. Thus we replace the initial cut as follows:

$$\dfrac{\dfrac{(1)\Gamma \vdash \Delta, x \xrightarrow{A} x, x : A \quad (4^{'})\Gamma, x \xrightarrow{A} x \vdash \Delta, x : A}{\Gamma \vdash \Delta, x : A}\,(cut) \qquad \dfrac{(3^{'})\Gamma, x : A \vdash \Delta, x \xrightarrow{A} x \quad (2)\Gamma, x \xrightarrow{A} x, x : A \vdash \Delta}{\Gamma, x : A \vdash \Delta}\,(cut)}{\Gamma \vdash \Delta}\,(cut)$$

The lower cut can be replaced by inductive hypothesis on the complexity of the cut formula, the other ones by inductive hypothesis on the height.

(g). $F = x \xrightarrow{A} y$ is derived on the left by (CS) and on the right by (EQ). Thus we have (we denote with $\Sigma(u)$ the substitution $\Sigma[x/u, y/u]$):

$$\dfrac{\dfrac{(1)u : A \vdash u : A^{'} \quad (2)u : A^{'} \vdash u : A}{(5)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y}\,(EQ) \qquad \dfrac{(3)\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta, x : A \quad (4)\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A} u \vdash \Delta(u)}{\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta}\,(CS)}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta}\,(cut)$$

First, we can replace the cut below:

$$\dfrac{(5^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x \xrightarrow{A} y, x : A \qquad (3)\Gamma^{'}, x \xrightarrow{A^{'}} y, x \xrightarrow{A} y \vdash \Delta, x : A}{(6)\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A}\,(cut)$$

by inductive hypothesis on the height. $(5^{'})$ is obtained from (5) since weakening is height-preserving admissible. We replace the initial cut as follows:

$$\dfrac{\dfrac{(6^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A, x : A^{'} \qquad (1^{'})\Gamma^{'}, x \xrightarrow{A^{'}} y, x : A \vdash \Delta, x : A^{'}}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta, x : A^{'}}\,(cut) \qquad \dfrac{(5^{''})\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u), u \xrightarrow{A} u \qquad (4)\Gamma^{'}(u), u \xrightarrow{A^{'}} u, u \xrightarrow{A} u \vdash \Delta(u)}{\Gamma^{'}(u), u \xrightarrow{A^{'}} u \vdash \Delta(u)}\,(cut)}{\Gamma^{'}, x \xrightarrow{A^{'}} y \vdash \Delta}\,(CS)$$

where $(6^{'})$ is obtained from (6) by weakening, $(5^{''})$ from (5) by label substitution and $(1^{'})$ from (1) by weakening and label substitution. The cut on the left can be replaced by inductive hypothesis on the complexity of the cut formula; the cut on the right can be replaced by inductive hypothesis on the height.

In the systems containing (CEM) the standard proof does not work in one case: when a transition $x \xrightarrow{A} y$ is the cut formula and it is introduced by (EQ) in one premise of cut and by (CEM) in the other, one needs to apply cut *two times on the same formula* $x \xrightarrow{A} y$ to replace the initial cut. Therefore, in order to prove the admissibility of cut for systems with (CEM) we prove by mutual induction the following propositions:

(A) if $\Gamma \vdash \Delta, F$ and $\Gamma, F \vdash \Delta$ are derivable, so $\Gamma \vdash \Delta$ (cut);

(B) given a derivable sequent $\Gamma \vdash \Delta$, if $u : A \vdash u : A^{'}$ and $u : A^{'} \vdash u : A$ are derivable, then the sequent obtained by replacing in $\Gamma \vdash \Delta$ *any* transition $x \xrightarrow{A} y$ with $x \xrightarrow{A^{'}} y$ is derivable too.

The detailed proof is presented in the Appendix A in [Olivetti et al. 2005].

□

3.2 Soundness and completeness of SeqS

SeqS calculi are sound and complete with respect to the semantics.

THEOREM 3.14 SOUNDNESS. *If* $\Gamma \vdash \Delta$ *is derivable in SeqS then it is valid in the corresponding system.*

Proof. By induction on the height of a derivation of $\Gamma \vdash \Delta$. As an example, we examine the cases of ($\Rightarrow$ R), (MP), (CS) and (CEM). The other cases are left to the reader.

- ($\Rightarrow$ R) Let $\Gamma \vdash \Delta, x : A \Rightarrow B$ be derived from (1) $\Gamma, x \xrightarrow{A} y \vdash \Delta, y : B$, where y does not occur in Γ, Δ and it is different from x. By induction hypothesis we know that the latter sequent is valid. Suppose the former is not, and that it is not valid in a model $\mathcal{M} = \langle \mathcal{W}, f, [\,] \rangle$, via a mapping I, so that we have:

  $\mathcal{M} \models_I F$ for every $F \in \Gamma$, $\mathcal{M} \not\models_I F$ for any $F \in \Delta$ and $M \not\models_I x : A \Rightarrow B$.

  As $\mathcal{M} \not\models_I x : A \Rightarrow B$ there exists $w \in f(I(x), [A]) - [B]$. We can define an interpretation $I'(z) = I(z)$ for $z \neq y$ and $I'(y) = w$. Since y does not occur in

Γ, Δ and is different from x, we have that $\mathcal{M} \models_{I'} F$ for every $F \in \Gamma$, $\mathcal{M} \not\models_{I'} F$ for any $F \in \Delta$, $\mathcal{M} \not\models_{I'} y : B$ and $\mathcal{M} \models_{I'} x \xrightarrow{A} y$, against the validity of (1).

- (MP) Let $\Gamma \vdash \Delta, x \xrightarrow{A} x$ be derived from (2) $\Gamma \vdash \Delta, x \xrightarrow{A} x, x : A$. Let (2) be valid and let $\mathcal{M} = \langle \mathcal{W}, f, [\,] \rangle$ be a model satisfying the MP condition. Suppose that for one mapping I, $\mathcal{M} \models_I F$ for every $F \in \Gamma$, then by the validity of (2) either $\mathcal{M} \models_I G$ for some $G \in \Delta$, or $\mathcal{M} \models_I x \xrightarrow{A} x$, or $\mathcal{M} \models_I x : A$. In the latter case, we have $I(x) \in [A]$, thus $I(x) \in f(I(x), [A])$, by MP, this means that $\mathcal{M} \models_I x \xrightarrow{A} x$.
- (CS) Let (3)$\Gamma, x \xrightarrow{A} y \vdash \Delta$, with $x \neq y$, be derived from (4)$\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A$ and (5)$\Gamma[x/u, y/u], u \xrightarrow{A} u \vdash \Delta[x/u, y/u]$, where u does not occur in Γ, Δ. Suppose that (4) and (5) are valid, whereas (3) is not, considering a model $\mathcal{M} = \langle \mathcal{W}, f, [\ ] \rangle$ satisfying the CS condition. Therefore, there is a mapping I such that $\mathcal{M} \models_I F$ for every $F \in \Gamma$, $\mathcal{M} \models_I x \xrightarrow{A} y$ (i.e. $I(y) \in f(I(x), [A])$) and $\mathcal{M} \not\models_I G$ for every $G \in \Delta$. We distinguish two cases:
  - ⋆ $I(x) \notin [A]$: in this case, we have that $\mathcal{M} \not\models_I x : A$, against the validity of (4);
  - ⋆ $I(x) \in [A]$: we observe that $f(I(x), [A]) \subseteq \{I(x)\}$, since $\mathcal{M}$ respects the CS condition. We have also that $I(y) \in f(I(x), [A])$, then it can be only $I(x) = I(y)$: say $w = I(x) = I(y)$. We introduce another mapping $I^{'}$ as follows: $I^{'}(u) = w$, $I^{'}(v) = I(v)$ for every label different from u. Obviously, $\mathcal{M} \models_{I'} F$ for every $F \in \Gamma[x/u, y/u]$, and $\mathcal{M} \not\models_{I'} G$ for every $G \in \Delta[x/u, y/u]$, but $\mathcal{M} \models_{I'} u \xrightarrow{A} u$, since $I^{'}(u) = w \in f(w, [A])$, against the validity of (5).
- (CEM) Let (8)$\Gamma, x \xrightarrow{A} y \vdash \Delta$ be derived from (6)$\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$ and (7)$(\Gamma, x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]$, with $z \neq y$. Suppose (6) and (7) are valid, whereas (8) is not. $\mathcal{M} = \langle \mathcal{W}, f, [\ ] \rangle$ respects the CEM condition. Then, one can find a mapping I such that $\mathcal{M} \models_I F$ for every $F \in \Gamma$, $\mathcal{M} \not\models_I G$ for every $G \in \Delta$ and $\mathcal{M} \models_I x \xrightarrow{A} y$, thus $I(y) \in f(I(x), [A])$. We distinguish two cases:
  - ⋆ $I(y) \neq I(z)$: since $\mathcal{M}$ respects CEM, we have that $\mid f(I(x), [A]) \mid \leq 1$. In this case, $f(I(x), [A]) = \{I(y)\}$, then $I(z) \notin f(I(x), [A])$. We can conclude $\mathcal{M} \not\models_I x \xrightarrow{A} z$, against the validity of (6);
  - ⋆ $I(z) = I(y) = w$, therefore $f(I(x), [A]) = \{w\}$. We introduce another mapping $I^{'}$ in this way: $I^{'}(u) = w$; $I^{'}(v) = I(v)$ for every label v different from u. Obviously, $\mathcal{M} \models_{I'} F$ for every $F \in \Gamma[y/u, z/u]$, and $\mathcal{M} \not\models_{I'} G$ for every $G \in \Delta[y/u, z/u]$. However, $\mathcal{M} \models_{I'} x \xrightarrow{A} y[y/u, z/u]$ since $I^{'}(u) = w \in f(I^{'}(x), [A])$, against the validity of (7).

□

Completeness is an easy consequence of the admissibility of cut[6].

[6]One can give a semantic proof of completeness, however as a difference with modal logics, the proof is considerably more complex and require nonetheless the cut rule (see [Olivetti and Schwind 2000] for a semantic completeness proof of a tableau calculus for CK). We explain intuitively the difficulty. The usual way to prove completeness semantically is by contraposition, that is to

THEOREM 3.15 COMPLETENESS. *If A is valid in CK or in one of its mentioned extensions, then $\vdash x : A$ is derivable in the respective SeqS system.*

Proof. If A is valid in CK or in one of its mentioned extensions, then $\vdash_S A$ is valid in the corresponding axiomatization by the completeness of the axioms (Theorem 2.1). We show that if $\vdash_S A$ is valid then $\vdash x : A$ is derivable in SeqS. We must show that the axioms are derivable and that the set of derivable formulas is closed under (Modus Ponens), (RCEA), and (RCK). A derivation of axioms (ID), (MP), (CS) and (CEM) can be obtained from examples 3.3, 3.4, 3.5 and 3.6 respectively; indeed, by Lemma 3.8, one can generalize these proofs to the case in which a propositional variable P is replaced by any formula A. Let us examine the other rules.
For (Modus Ponens), suppose that $\vdash x : A \rightarrow B$ and $\vdash x : A$ are derivable. We easily have that $x : A \rightarrow B, x : A \vdash x : B$ is derivable too. Since cut is admissible, by two cuts we obtain $\vdash x : B$.
For (RCEA), we have to show that if $A \leftrightarrow B$ is derivable, then also $(A \Rightarrow C) \leftrightarrow (B \Rightarrow C)$ is so. The formula $A \leftrightarrow B$ is an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$. Suppose that $\vdash x : A \rightarrow B$ and $\vdash x : B \rightarrow A$ are derivable, we can derive $x : B \Rightarrow C \vdash x : A \Rightarrow C$ as follows: (the other half is symmetrical).

$$\dfrac{\dfrac{\dfrac{x : A \vdash x : B \quad x : B \vdash x : A}{x : A \Rightarrow C, x \xrightarrow{B} y \vdash x \xrightarrow{A} y, y : C}\,(EQ) \qquad x : A \Rightarrow C, x \xrightarrow{B} y, y : C \vdash y : C}{x \xrightarrow{B} y, x : A \Rightarrow C \vdash y : C}\,(\Rightarrow L)}{x : A \Rightarrow C \vdash x : B \Rightarrow C}\,(\Rightarrow R)$$

For (RCK), supposed that (1) $\vdash x : B_1 \wedge B_2 \ldots \wedge B_n \rightarrow C$ is derivable, it must be derivable also $y : B_1, \ldots, y : B_n \vdash y : C$. Then we have (we omit side formulas in $x \xrightarrow{A} y \vdash x \xrightarrow{A} y$):

$$\dfrac{x \xrightarrow{A} y \vdash x \xrightarrow{A} y \qquad x : A \Rightarrow B_1, \ldots, x : A \Rightarrow B_n, y : B_1, \ldots, y : B_n \vdash y : C}{x \xrightarrow{A} y, x : A \Rightarrow B_1, \ldots, x : A \Rightarrow B_n, y : B_1, \ldots, y : B_{n-1} \vdash y : C}\,(\Rightarrow L)$$

$$\vdots$$

$$\dfrac{\dfrac{x \xrightarrow{A} y \vdash x \xrightarrow{A} y \qquad x \xrightarrow{A} y, x : A \Rightarrow B_1, \ldots, x : A \Rightarrow B_n, y : B_1 \vdash y : C}{x \xrightarrow{A} y, x : A \Rightarrow B_1, \ldots, x : A \Rightarrow B_n \vdash y : C}\,(\Rightarrow L)}{x : A \Rightarrow B_1, \ldots, x : A \Rightarrow B_n \vdash x : A \Rightarrow C}\,(\Rightarrow R)$$

□

say to extract a counter model from a failed branch of a (suitable) proof tree. To this purpose one needs to "saturate" a branch by applying the rules as much as possible. However the model being constructed must satisfy the normality condition, i.e. if $[A] = [A']$ then it must be $f(A, x) = f(A', x)$, or equivalently, the selection function must be well-defined on arbitrary subsets of worlds; to ensure this property, a simple branch saturation is not enough. One has to consider in the saturation process other formulas not occurring in the branch and use inevitably the cut rule to make the whole construction work, the latter being a kind of Henkin construction. For this reason we prefer the much simpler syntactic proof.

4. DECIDABILITY AND COMPLEXITY

In this section we analyze SeqS calculi in order to obtain a decision procedure for all conditional systems under consideration[7]. We first present some common properties, then we analyze separately systems CK{+ID}{+MP}, systems CEM{+ID} and systems CS*.

In general, cut-freeness alone does not ensure termination of proof search in a sequent calculus; the presence of labels and of the ($\Rightarrow$ L) rule, which increases the complexity of the sequent in a backward proof search, are potential causes of a non-terminating proof search. In this section we show that SeqS's rules introduce only a finite number of labels in a backward proof search, and that ($\Rightarrow$ L) can be applied in a controlled way: these conditions allow to describe a decision procedure for the corresponding logics. We also give explicit complexity bounds for our systems.

As a first step, we show that it is useless to apply ($\Rightarrow$ L) on $x : A \Rightarrow B$ by introducing (looking backward) the same transition formula $x \xrightarrow{A} y$ *more than once in each branch of a proof tree.* More in detail, we have the following:

Lemma 4.1 Controlled use of ($\Rightarrow$ L). *If $\Gamma \vdash \Delta$ is derivable, then there is a proof of it which does not contain more than one application of ($\Rightarrow$ L) (looking backward) on $x : A \Rightarrow B$ introducing the same transition formula $x \xrightarrow{A} y$ in each branch.*

Proof. Consider a derivation of $\Gamma \vdash \Delta$ in which ($\Rightarrow$ L) is applied to $x : A \Rightarrow B$ with a transition $x \xrightarrow{A} y$ more than once in a branch; in particular, consider the two highest[8] applications. We have the following situation:

$$\dfrac{\begin{matrix}\Pi_a \\ \Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y\end{matrix} \qquad \begin{matrix}\Pi_b \\ \Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1\end{matrix}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}\ (\Rightarrow L)$$
$$\vdots$$
$$\Gamma \vdash \Delta$$

and in Π_a or in Π_b the rule ($\Rightarrow$ L) is applied (looking backward) to $x : A \Rightarrow B$ by using $x \xrightarrow{A} y$. If the highest application is in Π_a we have:

$$\dfrac{\dfrac{\dfrac{\Gamma_2, x : A \Rightarrow B \vdash \Delta_2, x \xrightarrow{A} y \qquad \Gamma_2, x : A \Rightarrow B, y : B \vdash \Delta_2}{\Gamma_2, x : A \Rightarrow B \vdash \Delta_2}\ (\Rightarrow L)}{\begin{matrix}\vdots \\ \Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y\end{matrix}} \qquad \begin{matrix}\Pi_b \\ \Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1\end{matrix}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}\ (\Rightarrow L)$$

The application of ($\Rightarrow$ L) in the left branch can be permuted over the other rules in Π_a (remember that ($\Rightarrow$ L) is invertible, see Theorem 3.11); we have the following proof tree ($\Pi_a^{'}$ is Π_a after the permutation):

[7]Obviously, we restrict our concern to all cut-free systems, i.e. all SeqS systems except SeqCEM+MP*.
[8]The applications having the greatest distance from the root $\Gamma \vdash \Delta$.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_a'\\ \Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y, x \xrightarrow{A} y \quad \ldots y : B \vdash \ldots\end{array}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y}(\Rightarrow L) \qquad \begin{array}{c}\Pi_b\\ \Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1\end{array}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}(\Rightarrow L)$$

By contraction (Theorem 3.12), we have a proof Π_a'' of $\Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y$, which does not contain any application of ($\Rightarrow$ L) on $x : A \Rightarrow B$ introducing the same transition $x \xrightarrow{A} y$ (remember that contraction is rule-preserving admissible) and then we have the following proof:

$$\dfrac{\begin{array}{c}\Pi_a''\\ \Gamma_1, x : A \Rightarrow B \vdash \Delta_1, x \xrightarrow{A} y\end{array} \quad \begin{array}{c}\Pi_b\\ \Gamma_1, x : A \Rightarrow B, y : B \vdash \Delta_1\end{array}}{\Gamma_1, x : A \Rightarrow B \vdash \Delta_1}(\Rightarrow L)$$

$$\vdots$$

$$\Gamma \vdash \Delta$$

If the highest application is in Π_b the proof is similar and left to the reader.

□

From now on, we analyze separately the decidability of systems CK{+ID}{+MP}, CEM{+ID} and CS*.

4.1 Termination and complexity for CK{+ID}{+MP}

In this subsection we prove some properties characterizing calculi SeqS for conditional logics CK{+ID}{+MP}, in order to give a decision procedure for these systems. From now on we only refer to calculi for these systems unless stated otherwise.

First of all, observe that the calculi are characterized by the following property:

THEOREM 4.2 PROPERTY OF RIGHT-TRANSITION FORMULAS. *Let the sequent*

$$\Gamma \vdash \Delta, \; x \xrightarrow{A} y$$

with $x \neq y$, be derivable, then one of the following sequents:

(1) $\Gamma \vdash \Delta$

(2) $x \xrightarrow{F} y \vdash x \xrightarrow{A} y$, *where* $x \xrightarrow{F} y \in \Gamma$

is also derivable.

Proof. (EQ) is the only rule which operates, considering a backward proof search, on a transition formula on the right hand side (consequent) of a sequent. Thus, we have to consider three cases, analyzing the proof tree of $\Gamma \vdash \Delta, \; x \xrightarrow{A} y$:

- $\Gamma \vdash \Delta, x \xrightarrow{A} y$ is an axiom: we have to consider two subcases:
  - ⋆ an atom $u : P$ occurs in both Γ and Δ, therefore $\Gamma \vdash \Delta$ is derivable;
  - ⋆ $w : \bot \in \Gamma$, then $\Gamma \vdash \Delta$ is derivable;

- $x \xrightarrow{A} y$ is never principal in the derivation. In this case, $\Gamma \vdash \Delta$ is derivable, since we can remove an occurrence of $x \xrightarrow{A} y$ from every sequent descending (looking forward) from $\Gamma_1 \vdash \Delta_1, x \xrightarrow{A} y$;
- $x \xrightarrow{A} y$ is introduced (looking forward) by the (EQ) rule: in this case, another transition $x \xrightarrow{F} y$ *must* be in Γ, in order to apply (EQ). To see this, observe that the only rule that could introduce a transition formula (looking backward) in the antecedent of a sequent is ($\Rightarrow$ R), but it can only introduce a transition of the form $x \xrightarrow{F} z$, where *z does not occur in that sequent* (it is a *new* label), thus it cannot introduce the transition $x \xrightarrow{F} y$.
The (EQ) rule is only applied to transition formulas:
$$\frac{u : F \vdash u : A \qquad u : A \vdash u : F}{x \xrightarrow{F} y \vdash x \xrightarrow{A} y}\ (EQ)$$
therefore we can conclude that $x \xrightarrow{F} y \vdash x \xrightarrow{A} y$ is derivable.

□

Notice that this theorem holds for all the systems under consideration, but only if $x \neq y$. In systems with MP, considering a backward proof search, the (MP) rule operates on transitions in the consequent, although only on transitions of the form $x \xrightarrow{A} x$. In this case the theorem does not hold, as shown by the following counterexample:

$$\frac{x : A \vdash x \xrightarrow{A} x, x : A, x : B}{x : A \vdash x \xrightarrow{A} x, x : B}\ (MP)$$

for A and B arbitrary. The sequent $x : A \vdash x \xrightarrow{A} x, x : B$ is derivable in SeqMP, but $x : A \vdash x : B$ is not derivable in this system and the second condition is not applicable (no transition formula occurs in the antecedent). The first hypothesis of the theorem ($x \neq y$) excludes this situation.

In order to control the application of ($\Rightarrow$ L) we show that it is useless to apply (backward) the ($\Rightarrow$ L) rule on $x : A \Rightarrow B$ by introducing a transition $x \xrightarrow{A} y$ if no $x \xrightarrow{A'} y$ belongs to the left-hand side of the sequent, since there will be no way to prove that transition. This property is stated by the following:

Lemma 4.3 Controlled use of ($\Rightarrow$ L) for CK{+ID}{+MP}. *SeqS calculi for CK{+ID}{+MP} are complete even if the ($\Rightarrow$ L) rule is applied as follows:*

$$\frac{\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad \Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta}\ (\Rightarrow L)$$

by choosing a label y such that:

- *(for CK{+ID}) there is a transition* $x \xrightarrow{A'} y \in \Gamma$*;*
- *(for CK+MP{+ID}) there is a transition* $x \xrightarrow{A'} y \in \Gamma$ *or* $y = x$.

Proof. Let us consider a derivation where ($\Rightarrow$ L) is applied (backward) to $\Gamma, x : A \Rightarrow B \vdash \Delta$ by introducing a transition $x \xrightarrow{A} y$ such that no transitions of the form $x \xrightarrow{A'} y$ belong to Γ and $y \neq x$. The left premise of the rule is $\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y$: by Theorem 4.2 we have that $\Gamma, x : A \Rightarrow B \vdash \Delta$ is derivable, then the application of ($\Rightarrow$ L) under consideration is useless.

□

In order to get a decision procedure for these logics we need to control the application of (ID) and (MP), whose premises have a higher complexity than the respective conclusions. We prove that it is useless to apply (ID) and (MP) more than once on the same transition in each derivation branch, as stated by the following lemmas:

LEMMA 4.4 CONTROLLED USE OF (ID). *It is useless to apply (ID) on the same transition $x \xrightarrow{A} y$ more than once in a backward proof search in each branch of a derivation.*

Proof. Consider a proof where (ID) is applied more than once on the same transition in a derivation and consider the two highest applications: since (ID) is invertible (Theorem 3.11), we can consider, without loss of generality, that the two applications of (ID) are consecutive, as follows:

$$\dfrac{\dfrac{(1)\Gamma, x \xrightarrow{A} y, y : A, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta}\,(ID)}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\,(ID)$$

From (1) we can find a derivation of $(1^{'})\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta$ by contraction, and this derivation does not have any application of (ID) having $x \xrightarrow{A} y$ as a principal formula (remember that contraction is rule-preserving admissible). Thus, we can remove one application of (ID) as follows:

$$\dfrac{(1^{'})\Gamma, x \xrightarrow{A} y, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\,(ID)$$

□

LEMMA 4.5 CONTROLLED USE OF (MP). *It is useless to apply (MP) on the same transition $x \xrightarrow{A} x$ more than once in a backward proof search in each branch of a derivation.*

Proof. The proof is similar to the proof of Lemma 4.4 and left to the reader.

□

Now we have all the elements to prove the decidability of systems for CK{+ID}{+MP}:

THEOREM 4.6 TERMINATION FOR CK{+MP}{+ID}. *Systems SeqCK, SeqID, SeqMP and SeqID+MP ensure termination.*

Proof. In all rules the premises have a smaller complexity than the conclusion, except ($\Rightarrow$ L), (ID) and (MP). However, Lemma 4.1 guarantees that ($\Rightarrow$ L) can be applied in a controlled way, i.e. one needs to apply ($\Rightarrow$ L) only once on a formula $x : A \Rightarrow B$ with the same transition $x \xrightarrow{A} y$ in each branch. Moreover, considering a derivation of a sequent $\Gamma, x : A \Rightarrow B \vdash \Delta$, the number of applications of ($\Rightarrow$ L) on $x : A \Rightarrow B$ is bounded by the cardinality of $\mathcal{B} = \{x \xrightarrow{A} y \mid x \xrightarrow{A'} y \in \Gamma\}$[9] by Lemma 4.3. The number of different y such that $x \xrightarrow{A} y \in \mathcal{B}$ is finite, since labels are only introduced by conditional formulas occurring negatively in the initial sequent, which are finite.

Lemmas 4.4 and 4.5 guarantee that we only need a finite number of applications of (ID) and (MP) in a backward proof search. Moreover, observe that the rules are analytic, so that the premises contains only (labelled) subformulas of the formulas in the conclusion. In the search of a proof of $\vdash x_0 : D$, with $\mid D \mid= n$, new labels are introduced only by conditional subformulas occurring negatively in D.

The number of different labels occurring in a proof is $O(n)$, and the length of each branch of a proof tree is bounded by $O(n^2)$.

□

This itself gives decidability:

Theorem 4.7 CK{+ID}{+MP} decidability. *Logic CK{+ID} is decidable.*

Proof. We just observe that there is only a finite number of derivations to check of a given sequent $\vdash x_0 : D$, as both the length of a proof and the number of labelled formulas which may occur in it is finite.

□

We conclude this subsection by giving an explicit space complexity bound for CK{+ID}{+MP}. As usual, a proof may have an exponential size because of the branching introduced by the rules. However we can obtain a much sharper space complexity bound since we do not need to store the whole proof, but only a sequent at a time plus additional information to carry on the proof search; this standard technique in similar to [Hudelmaier 1993] and [Viganò 2000] (more details are given in Theorem 4.6 in [Olivetti et al. 2005]).

Theorem 4.8 Space complexity of CK{+ID}{+MP}. *Provability in CK{+ID}{+MP} is decidable in $O(n^2 \log n)$ space.*

4.2 Termination and complexity for CK+CEM{+ID}

In this subsection we analyze sequent calculi containing (CEM). In order to show that SeqCEM{+ID} ensure termination we proceed in a similar manner as we made in the previous subsection; in particular, we have to show that both (CEM) and ($\Rightarrow$ L) rules can be applied in a controlled way. The principal formula of these rules is maintained in their premises, and this is a potential cause of non termination in a

[9] In systems allowing the (MP) rule the number of applications of ($\Rightarrow$ L) is bounded by the cardinality of $\mathcal{B} = \{x \xrightarrow{A} y \mid x \xrightarrow{A'} y \in \Gamma\} \cup \{x \xrightarrow{A} x\}$.

backward proof search. However, we prove that the number of applications of both (CEM) and ($\Rightarrow$ L) is finite, and this gives the decidability.

LEMMA 4.9 CONTROLLED USE OF (CEM) (PART 1). *SeqCEM{+ID} are complete even if the (CEM) rule is applied with the following restrictions:*

$$\frac{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z \qquad (\Gamma, x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\ (CEM)$$

(1) $y \neq z$;

(2) there exists a transition $x \xrightarrow{A'} z \in \Gamma$.

Proof. Consider the transition $x \xrightarrow{A} z$ introduced (backward) by the (CEM) rule in its left premise. If it is introduced by weakening, it can be removed and we are done (the application of (CEM) is useless). Otherwise, it derives (forward) from an application of (EQ) or it is used in the derivation of the right premise of another application of (CEM) by the identification of labels (for instance (CEM) is also applied to $x \xrightarrow{B} y$ and in the right premise y and z are identified with a new label u). In the first case, a transition $x \xrightarrow{A'} z$ must belong to Γ and the restriction 2. is satisfied. In the second case, we observe that $x \xrightarrow{A} z$ *still appears in the right-hand side of the left premise of the rule*, therefore it can be derived by (EQ) or used by (CEM) in the derivation of the right premise, and so on. Obviously, given a proof tree of $\Gamma' \vdash \Delta, x \xrightarrow{A} z$, we can repeat this reasoning on each left premise of an application of (CEM) using $x \xrightarrow{A} z$ in its right derivation, until we find that $x \xrightarrow{A} z$ is derived by an application of (EQ) or by weakening, since the proof tree is finite, as shown below (we can assume, without loss of generality, that all the applications of (CEM) are consecutive, since (CEM) is invertible and then we can permute it over the other rules):

$$\cfrac{\cfrac{\begin{array}{c}\Pi \\ \Gamma' \vdash \Delta_n, x \xrightarrow{A} z \qquad \ldots \\ \vdots \\ \Gamma' \vdash \Delta_2, x \xrightarrow{A} z \qquad \ldots\end{array}}{\Gamma' \vdash \Delta_1, x \xrightarrow{A} z \qquad \ldots}\ (CEM)}{\Gamma' \vdash \Delta, x \xrightarrow{A} z}\ (CEM)$$

In Π the transition $x \xrightarrow{A} z$ can only be introduced by weakening or by an application of (EQ) with a transition $x \xrightarrow{A'} z$ in the left-hand side of a sequent. In the first case, all instances of $x \xrightarrow{A} z$ can be removed; in the second case, we can conclude that the transition $x \xrightarrow{A'} z$ belongs to Γ', since we can reason as in the proof of Theorem 4.2: ($\Rightarrow$ R) is the only rule introducing (looking backward) a transition $x \xrightarrow{A'} z$ in the left-hand side of a sequent; moreover, z is a new label, then it is not possible that $x \xrightarrow{A'} z$ is introduced in Π, since z already occurs in all sequents; thus, $x \xrightarrow{A'} z \in \Gamma$.

Notice that the restriction 1. is the initial restriction given in SeqCEM{+ID} in order to avoid a looping application of (CEM).

□

Similarly to (CEM) we can control the application of (⇒ L) as stated by the following:

Lemma 4.10 Controlled use of (⇒ L) for CEM{+ID}. *SeqCEM{+ID} is complete even if the (⇒ L) rule is applied as follows:*

$$\frac{\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad \Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta} \ (\Rightarrow L)$$

by choosing a label y such that there is a transition $x \xrightarrow{A'} y \in \Gamma$.

Proof. The proof is similar to the proof of Lemma 4.9.

□

To prove that SeqCEM{+ID} ensure termination we also need the following:

Lemma 4.11 Controlled use of (CEM) (Part 2). *It is useless to apply (CEM) to $x \xrightarrow{A} y$ by introducing the same transition $x \xrightarrow{A} z$ in the left premise of the rule more than once in each branch of a backward proof search.*

Proof. Consider a proof where (CEM) is applied to $x \xrightarrow{A} y$ more than once by introducing the same transition $x \xrightarrow{A} z$ in a branch; consider the two highest applications: since (CEM) is invertible, it permutes over the other rules, then we can consider, without loss of generality, the following proof:

$$\cfrac{\cfrac{\overset{\Pi_1}{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z, x \xrightarrow{A} z} \quad (\Gamma, x \xrightarrow{A} y \vdash \ldots)[y, z/u]}{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z} \ (CEM) \qquad \overset{\Pi_2}{(\Gamma, x \xrightarrow{A} y \vdash \Delta)[y, z/u]}}{\Gamma, x \xrightarrow{A} y \vdash \Delta} \ (CEM)$$

By contraction, one can find a proof Π_1' of the sequent $\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$, then we can conclude as follows, obtaining a proof where the upper application of (CEM) has been removed:

$$\frac{\overset{\Pi_1'}{\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z} \qquad \overset{\Pi_2}{(\Gamma, x \xrightarrow{A} y \vdash \Delta)[y, z/u]}}{\Gamma, x \xrightarrow{A} y \vdash \Delta} \ (CEM)$$

□

By the above Lemmas 4.9, 4.10 and 4.11 we prove the decidability of CK+CEM{+ID}:

Theorem 4.12 Termination for CK+CEM{+ID}. *Systems SeqCEM{+ID} ensure termination.*

Proof. We proceed as we made for systems CK{+ID}{+MP}. In particular, one can control the application of ($\Rightarrow$ L) and (CEM) by Lemmas 4.9, 4.11, 4.10 and 4.1.

□

As in the case of CK{+ID}{+MP}, this itself gives decidability:

Theorem 4.13 Decidability of CK+CEM{+ID}. *Logic CK+CEM{+ID} is decidable.*

We can easily extend results for space complexity given in the previous subsection to systems allowing (CEM):

Theorem 4.14 Space complexity of CK+CEM{+ID}. *Provability in CK+ +CEM{+ID} is decidable in $O(n^2 \log n)$ space.*

4.3 Termination and complexity for CK+CS*

As we made in the previous subsections, we have to show that one can apply the ($\Rightarrow$ L) rule in a controlled way. We have also to show that (CS) can be controlled too.

However, in these systems we have a problem. Consider the following derivation:

Example 4.15.

$$\dfrac{x \xrightarrow{\top} y \vdash x \xrightarrow{\top} y \qquad \dfrac{\dfrac{\dfrac{\dots x \xrightarrow{\top} y, y \xrightarrow{\top} z \vdash \dots, x \xrightarrow{\top} z \qquad \begin{array}{c}\Pi\\ x : \top \Rightarrow (\neg(\top \Rightarrow A)), \dots, z : \neg(\top \Rightarrow A) \vdash y : A, z : A\end{array}}{x : \top \Rightarrow (\neg(\top \Rightarrow A)), x \xrightarrow{\top} y, y \xrightarrow{\top} z \vdash y : A, z : A}\,(\Rightarrow L)}{x : \top \Rightarrow (\neg(\top \Rightarrow A)), x \xrightarrow{\top} y, \vdash y : A, y : \top \Rightarrow A}\,(\Rightarrow R)}{x : \top \Rightarrow (\neg(\top \Rightarrow A)), x \xrightarrow{\top} y, y : \neg(\top \Rightarrow A) \vdash y : A}\,(\neg L)}{\dfrac{x : \top \Rightarrow (\neg(\top \Rightarrow A)), x \xrightarrow{\top} y \vdash y : A}{x : \top \Rightarrow (\neg(\top \Rightarrow A)) \vdash x : \top \Rightarrow A}\,(\Rightarrow R)}\,(\Rightarrow L)$$

In the above derivation there is a loop by the combination of the following facts:

(1) the conditional formulas $x : \top \Rightarrow A$, $y : \top \Rightarrow A$, ..., generate *new* labels in the proof tree;

(2) one may have *transitive transitions*, as $x \xrightarrow{\top} y, y \xrightarrow{\top} z \vdash \dots, x \xrightarrow{\top} z$.

More generally, we can have sequents of the form $\Gamma, x_0 \xrightarrow{A_1} x_1, x_1 \xrightarrow{A_2} x_2, \dots, x_{n-1} \xrightarrow{A_n} x_n \vdash \Delta, x_0 \xrightarrow{A} x_n$ derived by applying the (CS) rule. Intuitively, the reason is that an application of (CS) on $x_{i-1} \xrightarrow{A_i} x_i$ has the effect of identifying labels x_{i-1} and x_i, therefore several backwards applications of this rule lead to a sequent of the form $\Gamma', u \xrightarrow{A_n} x_n \vdash \Delta', u \xrightarrow{A} x_n$, which can be derived for instance by (EQ).

We can remedy to this problem by restricting the application of ($\Rightarrow$ L) on $\Gamma, x : A \Rightarrow B \vdash \Delta$ by using only transitions $x \xrightarrow{A} y$ such that $x \xrightarrow{A'} y \in \Gamma$. The reason why we can control the application of ($\Rightarrow$ L) can be explained as follows: one may have transitive transitions *since* (CS) *identifies two labels in its right premise.* Indeed, to prove $x \xrightarrow{A} y, y \xrightarrow{A} z \vdash x \xrightarrow{A} z$ one can apply (CS) on $x \xrightarrow{A} y$: the right premise $u \xrightarrow{A} u, u \xrightarrow{A} z \vdash u \xrightarrow{A} z$ is derivable by the identification of labels x and y with u. However, the transition $x \xrightarrow{A} z$ is maintained in the left premise, where it can only be introduced by an application of (EQ) or by weakening. The intuition is that if one needs to propagate a conditional $x : A \Rightarrow B$ from x to y, and then to z by an application of (CS), where (CS) has the effect of identifying x and y, then one can *first* identify labels x and y with u by (CS), and *then* propagate the conditional $u : A \Rightarrow B$ from u to z by an application of ($\Rightarrow$ L).

Lemma 4.16 Controlled use of ($\Rightarrow$ L) for CS*. *SeqCS* are complete even if the ($\Rightarrow$ L) rule is applied as follows:*

$$\frac{\Gamma, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} y \qquad \Gamma, x : A \Rightarrow B, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta} \ (\Rightarrow L)$$

by choosing a label y such that:

- *(for systems without MP) there is a transition* $x \xrightarrow{A'} y \in \Gamma$*;*
- *(for systems with MP) there is a transition* $x \xrightarrow{A'} y \in \Gamma$ *or* $y = x$*.*

Proof. Consider a proof tree where ($\Rightarrow$ L) is applied introducing transitions by transitivity; the situation is as follows (as usual, we denote with $\Sigma(u)$ the substitution $\Sigma[x/u, y/u]$):

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} z, x : A'\end{matrix} \quad \begin{matrix}\Pi_2 \\ \Gamma(u), u \xrightarrow{A'} u ... \vdash \Delta, u \xrightarrow{A} z\end{matrix}}{\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, x \xrightarrow{A} z} (CS) \qquad \begin{matrix}\Pi_3 \\ \Gamma, x \xrightarrow{A'} y, ..., z : B \vdash \Delta\end{matrix}}{\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta} (\Rightarrow L)$$

We show that there exists a proof tree of $\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta$ where ($\Rightarrow$ L) is applied without introducing transitions by transitivity. If $x \xrightarrow{A} z$ derives in Π_1 by an application of (EQ), then a transition $x \xrightarrow{C} z$ belongs to the left-hand side of the sequent, therefore ($\Rightarrow$ L) is applied respecting the restriction stated by this Lemma. Otherwise, $x \xrightarrow{A} z$ can be removed in Π_1, since it is introduced by weakening[10]. Therefore, there is a proof $\Pi_1^{'}$ of the sequent $\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, x : A'$. Moreover, by applying the label substitution (Lemma 3.9) to the sequent derived by proof Π_3 we can obtain a proof $\Pi_3^{'}$ of $\Gamma(u), u \xrightarrow{A'} u, u \xrightarrow{A''} z, u : A \Rightarrow B, z : B \vdash \Delta(u)$. We can conclude by applying first the (CS) rule (to identify labels x and y with u) and then by propagating the conditional formula from u to z, as follows:

[10]We can have several consecutive applications of (CS). However, as in the case of (CEM), $x \xrightarrow{A} z$ can be derived either by (EQ) or by weakening since it is maintained in the consequent of the left premise of (CS).

$$\dfrac{\begin{matrix}\Pi_1^{'} \\ \Gamma, x \xrightarrow{A^{'}} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta, x : A'\end{matrix} \quad \dfrac{\begin{matrix}\Pi_2 \\ \Gamma(u), u \xrightarrow{A'} u... \vdash \Delta, u \xrightarrow{A} z\end{matrix} \quad \begin{matrix}\Pi_3^{'} \\ \Gamma(u), u \xrightarrow{A'} u, ..., z : B \vdash \Delta(u)\end{matrix}}{\Gamma(u), u \xrightarrow{A'} u, u \xrightarrow{A''} z, u : A \Rightarrow B \vdash \Delta(u)}\,(\Rightarrow L)}{\Gamma, x \xrightarrow{A'} y, y \xrightarrow{A''} z, x : A \Rightarrow B \vdash \Delta}\,(CS)$$

where the (⇒ L) rule has been applied respecting the restriction of this lemma, i.e. by introducing (backward) a transition $u \xrightarrow{A} z$ such that $u \xrightarrow{A''} z$ belongs to the left-hand side of the sequent. We proceed in the same manner if we have a proof where (CS) is applied on $y \xrightarrow{A''} z$ and also when the transition $x_0 \xrightarrow{A} x_n$ is used to apply (⇒ L) on $\Gamma, x_0 \xrightarrow{A_1} x_1, x_1 \xrightarrow{A_2} x_2, ..., x_{n-1} \xrightarrow{A_n} x_n$.

□

As in the case of (CEM) we need to show that (CS) can be applied in a controlled way, in order to show that SeqCS* ensure termination.

Lemma 4.17 Controlled use of (CS). *It is useless to apply (CS) on the same transition $x \xrightarrow{A} y$ more than once in each branch of a backward proof search.*

Proof. Consider a branch where (CS) is applied more than once on $x \xrightarrow{A} y$ and consider the two highest applications; without loss of generality, we can consider the following proof since (CS) is invertible (see Theorem 3.11, as usual we denote $\Sigma(u) = \Sigma[x/u, y/u]$):

$$\dfrac{\dfrac{(1)\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A, x : A \qquad \Gamma(u), u \xrightarrow{A} u \vdash \Delta(u), u : A}{\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A}\,(CS) \qquad (2)\Gamma(u), u \xrightarrow{A} u \vdash \Delta(u)}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\,(CS)$$

By Theorem 3.12 we can find a proof of $(1^{'})\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A$, thus we conclude as follows:

$$\dfrac{(1^{'})\Gamma, x \xrightarrow{A} y \vdash \Delta, x : A \qquad (2)\Gamma(u), u \xrightarrow{A} u \vdash \Delta(u)}{\Gamma, x \xrightarrow{A} y \vdash \Delta}\,(CS)$$

We have found a derivation of the initial sequent in which a useless application of (CS) has been removed.

□

Now we have all the elements to prove the following:

Theorem 4.18 Termination for CK+CS*. *Systems SeqCS* ensure termination.*

Proof. One can control the application of (⇒ L) by Lemma 4.16 and of (CS) by Lemma 4.17. For systems with (ID) and/or (MP), one can control the application of these rules since Lemmas 4.4 and 4.5 hold in systems with (CS) too. In the

$$(\mathbf{ID})\ \frac{\Gamma, y : A \vdash \Delta}{\Gamma, x \xrightarrow{A} y \vdash \Delta} \qquad\qquad (\mathbf{MP})\frac{\Gamma \vdash x : A, \Delta}{\Gamma \vdash x \xrightarrow{A} x, \Delta}$$

Fig. 4. Rules (ID) and (MP) reformulated for CK{+ID}{+MP}.

case of SeqCEM+CS* just observe that one can control the application of (CEM) in this way: one needs to apply (CEM) on $\Gamma, x \xrightarrow{A} y \vdash \Delta$ by using a transition $x \xrightarrow{A} z$ (i.e. premises are $\Gamma, x \xrightarrow{A} y \vdash \Delta, x \xrightarrow{A} z$ and $(\Gamma, x \xrightarrow{A} y \vdash \Delta)[y/u, z/u]$) such that $x \xrightarrow{A'} z \in \Gamma$, since Lemma 4.9 holds in these systems too. Moreover, one needs to apply (CEM) at most once by using the same transition $x \xrightarrow{A} z$ in each branch, since we can easily observe that Lemma 4.11 holds in these systems[11]. The number of applications of ($\Rightarrow$ L) and (CEM) is finite, since the number of transitions introduced by conditionals occurring negatively in the initial sequent of a backward proof search is finite.

□

As in the previous cases, this itself gives decidability:

Theorem 4.19 Decidability of CK+CS*. *Logics CK+CS* are decidable.*

We conclude by giving an explicit space complexity bound:

Theorem 4.20 Space complexity of CK+CS*. *Provability in CK+CS* is decidable in* $O(n^2 \log n)$ *space.*

5. REFINEMENTS AND OTHER PROPERTIES FOR CK{+ID}{+MP}

In this section we restrict our concern to the calculi for CK{+ID}{+MP} and we get a better terminating calculus and complexity bound for CK{+ID}. Intuitively, we can do this because these systems enjoy a sort of *disjunction property* for conditional formulas: if $(A_1 \Rightarrow B_1) \vee (A_2 \Rightarrow B_2)$ is valid, then either $(A_1 \Rightarrow B_1)$ or $(A_2 \Rightarrow B_2)$ is valid too.

First of all we observe that we can reformulate the rules (ID) and (MP) with the non invertible ones presented in Figure 4 (details are given in [Pozzato 2003]).

Let us introduce the notion of *regular sequent*. Intuitively, regular sequents are those sequents whose set of transitions in the antecedent forms a *forest*. We assume that trees do not contain cycles in the vertexes and that a forest is a set of trees. As we show in Proposition 5.3 below, any sequent in a proof beginning with a sequent of the form $\vdash x_0 : D$, for an arbitrary formula D, is regular. For this reason, we will restrict our concern to regular sequents.

We define the multigraph $\mathcal{G}$ of the transition formulas in the antecedent of a sequent:

[11] We can repeat the same proof of the theorem even if we have (CS), since (CEM) and (CS) are both invertible.

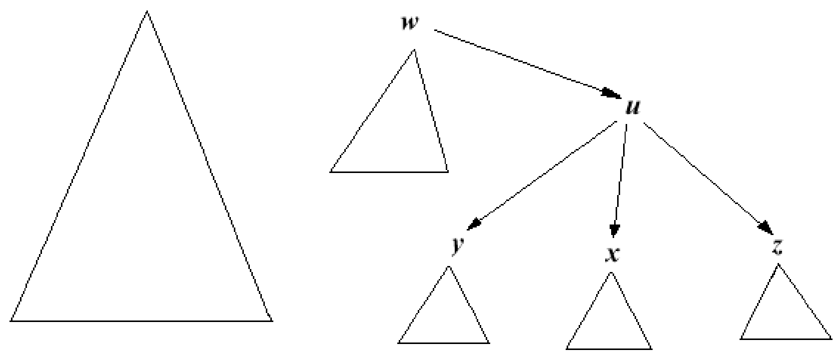

Fig. 5. The forest $\mathcal{G}$ of a regular sequent.

Definition 5.1 Multigraph of transitions $\mathcal{G}$. *Given a sequent* $\Gamma \vdash \Delta$*, where* $\Gamma = \Gamma', T$ *and* T *is the multiset of transition formulas and* Γ' *does not contain transition formulas, we define the multigraph* $\mathcal{G} =< V, E >$ *associated to* $\Gamma \vdash \Delta$ *with vertexes* V *and edges* E*.* V *is the set of labels occurring in* $\Gamma \vdash \Delta$ *and* $< x, y >\in E$ *whenever* $x \xrightarrow{F} y \in T$*.*

Definition 5.2 Regular sequent. *A sequent* $\Gamma \vdash \Delta$ *is called regular if its associated multigraph of transitions* $\mathcal{G}$ *is a forest.*

The graph of transitions of regular sequents forms a forest, as shown in Figure 5.

As mentioned above, we can always restrict our concern to regular sequents, since we have the following Proposition[12]:

Proposition 5.3 Proofs with regular sequents. *Every proof tree with a sequent* $\vdash x_0 : D$ *as root and obtained by applying backward SeqS's rules contains only regular sequents.*

For technical reasons we introduce the following definition:

Definition 5.4. *Given a forest* $\mathcal{G}$ *of transitions, let:*

- $\mathcal{G}(k)$ *be the tree of* $\mathcal{G}$ *with root* k*;*
- r_k *be the root of unique tree in* $\mathcal{G}$ *containing* k*.*

Observe that r_k *may be the root of* $\mathcal{G}(k)$*.*

Given a multiset of formulas Σ *we define:*

- Σ°_k *as the multiset of labelled formulas of* Σ *contained in* $\mathcal{G}(k)$*:*

$$\Sigma^{\circ}_k = \{u : F \in \Sigma \mid u \text{ is a vertex of } \mathcal{G}(k)\} \cup$$
$$\cup \{u \xrightarrow{F} v \in \Sigma \mid \ v \text{ is a vertex of } \mathcal{G}(k)\}$$

[12]Notice that this theorem does not hold in systems with CEM or CS: indeed, if (CEM) or (CS) is applied (looking backward) to $\Gamma, x \xrightarrow{A} y \vdash \Delta$, then two labels are identified in the right premise, thus the resulting graph of transitions is not a forest. The proof of the proposition can be found in [Olivetti et al. 2005], Theorem 5.3.

- Σ_k^p *as the multiset of labelled formulas of* Σ *contained on the path from* r_k *to* k*:*

$$\Sigma_k^p = \{u : F \in \Sigma \mid u \textit{ is on the path between } r_k \textit{ and } k\} \cup$$
$$\cup \; \{u \xrightarrow{F} v \in \Sigma \mid v \textit{ is on the path between } r_k \textit{ and } k\}$$

- Σ_k^* *as the multiset of labelled formulas of* Σ *contained in* $\mathcal{G}(k)$ *or on the path from* r_k *to* k*:*

$$\Sigma_k^* = \Sigma_k^\circ \;\cup\; \Sigma_k^p$$

Now we introduce the definition of x-branching formula. Intuitively, $\mathcal{B}(x,T)$ contains formulas that create a branching in x or in a predecessor of x according to T in a derivation of a sequent. For instance, consider the sequent $x : A, x : A \rightarrow B \vdash x : B$, obviously valid in CK. $x : A \rightarrow B$ is an x-branching formula, since it creates a branching in x in a derivation of the sequent:

$$\frac{x : A \vdash x : A, x : B \qquad x : A, x : B \vdash x : B}{x : A, x : A \rightarrow B \vdash x : B}\,(\rightarrow L)$$

$\mathcal{B}(x,T)$ also contains the conditionals $u : A \Rightarrow B$ such that $T \vdash u \xrightarrow{A} v$ and B creates a branching in x (i.e. $v = x$) or in a predecessor v of x.

Definition 5.5 x-branching formulas. *Given a multiset of transition formulas T, we define the set of x-branching formulas, denoted with $\mathcal{B}(x,\, T)$, as follows:*

- $x : A \rightarrow B \in \mathcal{B}(x,T)$*;*
- $u : A \rightarrow B \in \mathcal{B}(x,T)$ *if* $T \vdash u \xrightarrow{C} x$ *for some formula* C*;*
- $u : A \Rightarrow B \in \mathcal{B}(x,T)$ *if* $T \vdash u \xrightarrow{A} v$ *and* $v : B \in \mathcal{B}(x,T)$*.*

We also introduce the notion of x-branching sequent. Intuitively, we say that $\Gamma \vdash \Delta$ is x-branching if it contains an x-branching formula occurring positively in Γ or if it contains an x-branching formula occurring negatively in Δ. As an example, consider the following proof of $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : A \Rightarrow B, x : C \Rightarrow D$, valid in CK:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \xrightarrow{A} y \vdash x \xrightarrow{A} y \qquad y : B \vdash y : B}{(*)x \xrightarrow{A} y, x \xrightarrow{C} z, x : A \Rightarrow B \vdash y : B, z : D, x : \bot}\,(\Rightarrow L)}{x \xrightarrow{A} y, x \xrightarrow{C} z \vdash y : B, z : D, x : (A \Rightarrow B) \rightarrow \bot}\,(\rightarrow R) \qquad \dfrac{x \xrightarrow{C} z \vdash x \xrightarrow{C} z \qquad z : D \vdash z : D}{x \xrightarrow{A} y, x \xrightarrow{C} z, x : C \Rightarrow D \vdash y : B, z : D}\,(\Rightarrow L)}{x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D), x \xrightarrow{A} y, x \xrightarrow{C} z \vdash y : B, z : D}\,(\rightarrow L)}{x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D), x \xrightarrow{A} y \vdash y : B, x : C \Rightarrow D}\,(\Rightarrow R)}{x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : A \Rightarrow B, x : C \Rightarrow D}\,(\Rightarrow R)$$

The initial sequent $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : A \Rightarrow B, x : C \Rightarrow D$ is x-branching by the formula $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D)$, which creates a branching on x in the derivation. The sequent $(*)x \xrightarrow{A} y, x \xrightarrow{C} z, x : A \Rightarrow B \vdash y : B, z : D, x : \bot$ is *not* x-branching, since no formula creates a branching in the backward proof search on x or on a path to x.

Since in systems containing (ID) a transition $u \xrightarrow{F} v$ in the antecedent can be derived, looking forward, from $v : F$ and $v : F$ can be x-branching, we impose that a sequent $\Gamma', u \xrightarrow{F} v \vdash \Delta$ is x-branching if $\Gamma', v : F \vdash \Delta$ is x-branching; by the same reason, in systems containing (MP) we impose that a sequent $\Gamma \vdash \Delta', u \xrightarrow{F} u$ is x-branching if $\Gamma \vdash \Delta', u : F$ is x-branching.
In systems containing (MP) we also impose that a sequent $\Gamma', w : A \Rightarrow B \vdash \Delta$ is x-branching if the sequent $\Gamma' \vdash \Delta, w \xrightarrow{A} w$ is derivable and w is a predecessor of x (or $w = x$), since $w : A$ can introduce x-branching formula(s) in the sequent.

Definition 5.6 x-branching sequents. *Given a sequent $\Gamma \vdash \Delta$, we denote by Γ' the world formulas in Γ and by T the transition formulas in Γ, so that $\Gamma = \Gamma', T$. To define when a sequent $\Gamma \vdash \Delta$ is x-branching according to each system, we consider the following conditions:*

(1) *a world formula $u : F \in \mathcal{B}(x, T)$ occurs positively in Γ;*

(2) *a world formula $u : F \in \mathcal{B}(x, T)$ occurs negatively in Δ.*

(3) *$T = T', u \xrightarrow{F} v$ and the sequent $\Gamma', T', v : F \vdash \Delta$ is x-branching;*

(4) *$u \xrightarrow{F} u \in \Delta$ and the sequent $\Gamma \vdash \Delta', u : F$ is x-branching ($\Delta = \Delta', u \xrightarrow{F} u$);*

(5) *a formula $w : A \Rightarrow B \in \Gamma$, w is a predecessor of x in the forest $\mathcal{G}$ of transitions or $w = x$ and $\Gamma'' \vdash \Delta, w \xrightarrow{A} w$ is derivable, where $\Gamma = \Gamma'', w : A \Rightarrow B$.*

We say that $\Gamma \vdash \Delta$ is x-branching for each system if the following combinations of the previous conditions hold:

- *CK: 1, 2*
- *CK+ID: 1, 2, 3*
- *CK+MP: 1, 2, 4, 5*
- *CK+MP+ID: 1, 2, 3, 4, 5*

As anticipated at the beginning of this section, the disjunction property *only* holds for sequents that are not x-branching. The reason is twofold: on the one hand, only the formulas on the path from x going backwards through the transition formulas (i.e. on the worlds $u_1 \xrightarrow{A_1} u_2 \xrightarrow{A_2} \ldots \xrightarrow{A_n} x$) can contribute to a proof of a formula with label x. This is proved by Proposition 5.7 below. On the other hand, no formula on that path can create a branching in the derivation. As an example, consider the x-branching sequent $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : A \Rightarrow B, x : C \Rightarrow D$; it is valid in CK, but neither $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : A \Rightarrow B$ nor $x : ((A \Rightarrow B) \rightarrow \bot) \rightarrow (C \Rightarrow D) \vdash x : C \Rightarrow D$ are valid.

To prove the disjunction property, we need to consider a more general setting; namely, we shall consider a sequent of the form $\Gamma \vdash \Delta, y : A, z : B$, whose forest of transitions has the form represented in Figure 6, i.e. it has one subtree with root u and another subtree with root v, with $u \neq v$; y is a member of the tree with root u and z is a member of the tree with root v; x is the father of u and v and the tree containing x has root r.

Now we have all the elements to prove the following proposition; intuitively, it says that if $\Gamma \vdash \Delta, y : A, z : B$ is derivable and its forest $\mathcal{G}$ has form as in Figure

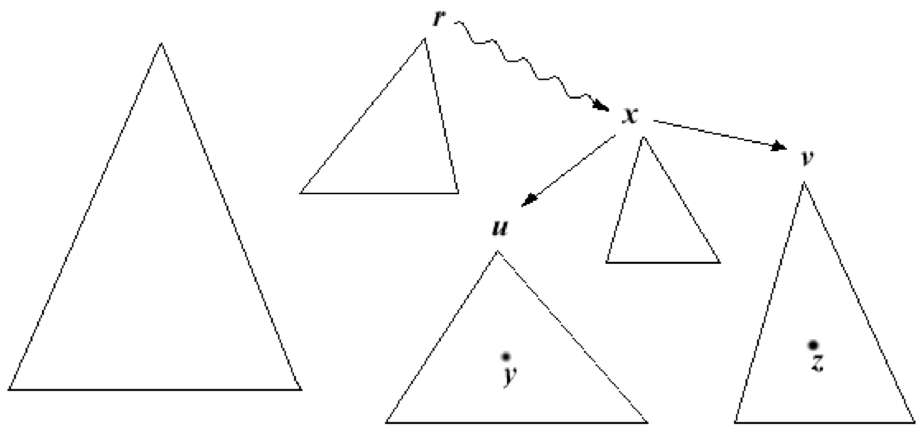

Fig. 6. The forest $\mathcal{G}$ of transitions used to prove the disjunction property.

6, then there is a derivation which involves (i) only the formulas whose labels are in $\mathcal{G}(u)$ or (ii) only the formulas whose labels are in $\mathcal{G}(v)$ or (iii) only the formulas whose labels are in the rest of the forest. This proposition is also crucial to prove the Lemma 5.10 below, which leads to a better space complexity bound for CK{+ID}. More details are given in Proposition 5.12 in [Olivetti et al. 2005].

PROPOSITION 5.7. *Given a sequent $\Gamma \vdash \Delta, y : A, z : B$ and its forest of transitions $\mathcal{G}$, if it is derivable and has the following features:*

(1) $\mathcal{G}$ is a forest of the form as shown in Figure 6 (thus y is a member of $\mathcal{G}(u)$ and z is a member of $\mathcal{G}(v)$, with $u \neq v$; u and v are sons of x);
(2) $\Gamma \vdash \Delta, y : A, z : B$ is not x-branching

then one of the following sequents is derivable:

(i) $\Gamma^*_u \vdash \Delta^*_u, y : A$
(ii) $\Gamma^*_v \vdash \Delta^*_v, z : B$
(iii) Γ - $(\Gamma^\circ_u \cup \Gamma^\circ_v) \vdash \Delta$ - $(\Delta^\circ_u \cup \Delta^\circ_v)$

Moreover, the proofs of (i), (ii), and (iii) do not add any application of $(\Rightarrow L)$ to the proof of $\Gamma \vdash \Delta, y : A, z : B$.

THEOREM 5.8 DISJUNCTION PROPERTY. *Given a non x-branching sequent*

$$\Gamma \vdash \Delta, x : A_1 \Rightarrow B_1, x : A_2 \Rightarrow B_2$$

derivable with a derivation Π, one of the following sequents:

(1) $\Gamma \vdash \Delta,\ x : A_1 \Rightarrow B_1$
(2) $\Gamma \vdash \Delta,\ x : A_2 \Rightarrow B_2$

is derivable.

Proof sketch. If $x : A_1 \Rightarrow B_1$ is introduced by weakening, then we obtain a proof of $\Gamma \vdash \Delta, x : A_2 \Rightarrow B_2$ by removing that weakening, and the same for the symmetric case. Otherwise, both conditionals are introduced by ($\Rightarrow$ R); by the invertibility of ($\Rightarrow$ R), we can consider a proof ended as follows:

$$\dfrac{\dfrac{\Gamma, x \xrightarrow{A_1} y, x \xrightarrow{A_2} z \vdash \Delta, y : B_1, z : B_2}{\Gamma, x \xrightarrow{A_1} y \vdash \Delta, y : B_1, x : A_2 \Rightarrow B_2}\ (\Rightarrow R)}{\Gamma \vdash \Delta, x : A_1 \Rightarrow B_1, x : A_2 \Rightarrow B_2}\ (\Rightarrow R)$$

in which $\Gamma, x \xrightarrow{A_1} y, x \xrightarrow{A_2} z \vdash \Delta, y : B_1, z : B_2$ respects all the conditions to Apply the Proposition 5.7. Therefore, we apply the Proposition 5.7 and in each case we conclude that either $\Gamma \vdash \Delta,\ x : A_1 \Rightarrow B_1$ or $\Gamma \vdash \Delta,\ x : A_2 \Rightarrow B_2$ is derivable. For the entire proof, see Theorem 5.13 in [Olivetti et al. 2005].

□

By the correctness and completeness of SeqS, it is easy to prove the following corollary of the disjunction property:

COROLLARY 5.9. *If $(A \Rightarrow B) \vee (C \Rightarrow D)$ is valid in CK{+MP}{+ID}, then either $A \Rightarrow B$ or $C \Rightarrow D$ is valid in CK{+MP}{+ID}.*

5.1 Refinements for CK{+ID}

In this subsection we give a further refinement for the basic conditional logic CK and its extension CK+ID. The Proposition 5.7 suggests the following fact: in SeqCK and SeqID systems, it is useless to apply ($\Rightarrow$ L) on the same formula $x : A \Rightarrow B$ *by using more than one transition* $x \xrightarrow{A} y$, with a different y. Intuitively, if ($\Rightarrow$ L) is applied to $x : A \Rightarrow B$ by using two (or more) transitions $x \xrightarrow{A} y$ and $x \xrightarrow{A} z$, then the proof can be directed either on the subtree with root y (i.e. $\mathcal{G}(y)$) or on $\mathcal{G}(z)$. Therefore, to prove $\Gamma, x \xrightarrow{A} y_1, x \xrightarrow{A} y_2, ..., x \xrightarrow{A} y_n, x : A \Rightarrow B \vdash \Delta$ one needs *only one application of* ($\Rightarrow$ L) *in each branch*. This means that only one transition $x \xrightarrow{A} y_i$ will be need to apply ($\Rightarrow$ L) on $x : A \Rightarrow B$ in each branch[13].

This fact is formalized as follows (the proof can be found in Lemma 5.15 in [Olivetti et al. 2005]):

LEMMA 5.10 CONTROLLED USED OF ($\Rightarrow$ L) IN SEQCK AND SEQID. *If $\Gamma, x : A \Rightarrow B \vdash \Delta$ is derivable in SeqCK (SeqID), then it has a derivation with at most one application of ($\Rightarrow$ L) with $x : A \Rightarrow B$ as a principal formula in each branch.*

By the above Lemma 5.10 and Theorem 4.2 we can reformulate the ($\Rightarrow$ L) rule as shown in Figure 7.

As mentioned above, thanks to the reformulation shown in Figure 7, it is possible to give a better space complexity bound for CK{+ID}, whose proof can be found in [Olivetti et al. 2005], Theorem 5.16:

[13]Observe that the reformulation of the rule would not be complete for the other systems, where we potentially need to apply ($\Rightarrow$ L) on $x : A \Rightarrow B$ by using *all* the transitions $x \xrightarrow{A} y_i$ in the left-hand side of the sequent. In systems with (MP) $x \xrightarrow{A} x$ can also be used.

$$(\Rightarrow \mathbf{L})\ \frac{x \xrightarrow{A'} y \vdash x \xrightarrow{A} y \qquad \Gamma, y : B \vdash \Delta}{\Gamma, x : A \Rightarrow B \vdash \Delta}\ , x \xrightarrow{A'} y \in \Gamma$$

Fig. 7. ($\Rightarrow$ L) rule for SeqCK and SeqID systems.

Theorem 5.11 Space complexity of CK{+ID}. *Provability in CK{+ID} is decidable in $O(n\ \log n)$ space.*

6. UNIFORM PROOFS

In this section we briefly discuss how our calculi can be used to develop goal-directed proof procedures for conditional logics, following the paradigm of Uniform Proof by Miller and others [Miller et al. 1991][14]. A full investigation of this topic could lead to the development of extensions of logic programming based on conditional logics and will be addressed in future research. The paradigm of uniform proof is an abstraction, or a generalization, of conventional logic programming. We are given a sequent $\Gamma \vdash G$ where Γ represents the "program" or "database", and G is the "goal" whose proof is searched. Intuitively, the idea of Uniform Proofs is that the backward proof of $\Gamma \vdash G$ is driven by the goal G, that is to say G is stepwise decomposed according to its logical structure by the rules of the calculus, until its atomic constituents are reached. The connectives in G can be interpreted operationally as search instructions. To prove an atomic goal Q, one looks in Γ for one "clause" whose head matches with Q and tries to prove the "body" of the clause in its turn. This step can be understood as a step of "resolution" or "back-chaining", namely it is a condensed application of suitable R-rules of the calculus. A proof of this sort is called a *uniform proof*. Given a sequent calculus for a logic L, in general, not every provable sequent admits a uniform proof: one must identify a (significant) fragment of L that allows uniform proofs. Usually, this fragment (in the propositional case) is alike to the Harrop-fragment of intuitionistic logic (see [Miller et al. 1991; Miller and Hodas 1994; Pym and Harland 1994]). To specify this fragment one distinguish between the formulas which can occurs in the database (D-formulas) and the formulas that can be asked as goals (G-formulas).

As a preliminary result we present a simple goal-directed calculus for a fragment of CK.

Definition 6.1 Language for uniform proofs. *We consider the fragment of CK, called $\mathcal{LU}(CK)$, comprising:*

- database formulas*, denoted with D*
- goal formulas*, denoted with G*
- transition formulas *of the form $x \xrightarrow{A} y$*

[14] A related methodology for goal-directed provability has been proposed also in [Gabbay and Olivetti 2000], where goal-directed proof procedures for several families of nonclassical logics are presented.

defined as follows ($Q \in ATM$)*:*

$D = Q \mid G \rightarrow Q \mid A \Rightarrow D$
$G = Q \mid \top \mid G \wedge G \mid G \vee G \mid A \Rightarrow G$
$A = Q \mid A \wedge A$

We define a database Γ *as a set of D-formulas and transition formulas.*

The calculus $\mathcal{U}$CK for uniform proofs is grounded on the properties below of the calculus SeqCK, when derivations are restricted to the fragment $\mathcal{LU}$(CK). First of all, we can prove the *strong* disjunction property: if $\Gamma \vdash x : A, y : B$ is derivable, then either $\Gamma \vdash x : A$ or $\Gamma \vdash y : B$ is derivable ($x : A$ and $y : B$ are *not* necessarily conditional formulas). This property follows immediately by the following Lemma, related to Proposition 5.7:

LEMMA 6.2. *If* $\Gamma \vdash x_1 : G_1, x_2 : G_2, ..., x_n : G_n$ *is derivable with a proof of height* h*, there exists* $i, i = 1, 2, ..., n$*, such that* $\Gamma^p_{x_i} \vdash x_i : G_i$ *is derivable with a proof of height no greater than* h*.*

Proof. By induction on the height of a derivation of $\Gamma \vdash x_1 : G_1, x_2 : G_2, ..., x_n : G_n$. The proof is easy and left to the reader.

□

By the above Lemma 6.2 we can obtain the following:

PROPOSITION 6.3 HEIGHT-PRESERVING STRONG DISJUNCTION PROPERTY. *If* $\Gamma \vdash x_1 : G_1, x_2 : G_2, ..., x_n : G_n$ *is derivable with a proof of height* h*, there exists* $i, i = 1, 2, ..., n$*, such that* $\Gamma \vdash x_i : G_i$ *is derivable with a proof of no greater height than* h*.*

Proof. By Lemma 6.2 we can find a proof of no greater height of $\Gamma^p_{x_i} \vdash x_i : G_i$, thus we conclude by finding a proof of no greater height of $\Gamma \vdash x_i : G_i$ since weakening is height-preserving admissible (Theorem 3.10).

□

We define a goal-directed proof procedure called $\mathcal{U}$CK, whose rules are shown in Figure 8 and are used to query a goal $x : G$ given a database Γ. We write $\Gamma \vdash_u x : G \Rrightarrow \Gamma_i \vdash_u x_i : G_i$ to denote that the sequent $\Gamma \vdash_u x : G$ is reduced to sequents $\Gamma_i \vdash_u x_i : G_i$. The rule called ($\mathcal{U}$ **prop**) is used when D-formulas have the form $A_1 \Rightarrow A_2 \Rightarrow ... \Rightarrow A_n \Rightarrow (G \rightarrow Q)$, where G could be $\top$.

The calculus presented in Figure 8 is sound and complete with respect to the semantics; the proof of soundness is easy and left to the reader. To prove the completeness we need the following Lemma, which is also related to Proposition 5.7:

LEMMA 6.4. *If* $\Gamma \vdash x : F$ *is derivable in the selected fragment of SeqCK with a proof of height* h*, then* $\Gamma^*_x \vdash x : F$ *is derivable with a proof of no greater height than* h*.*

Proof. By induction on the height of $\Gamma \vdash x : F$. □

$$(\mathcal{U}\top) \quad \Gamma \vdash_u x : \top$$

$$(\mathcal{U}\ \mathbf{ax}) \quad \Gamma \vdash_u x : Q \quad \text{if} \quad x : Q \in \Gamma$$

$$(\mathcal{U}\ \mathbf{prop}) \quad \Gamma \vdash_u x : Q \quad \Rrightarrow \quad \Gamma \vdash_u y \xrightarrow{A_1} x_1,\ \Gamma \vdash_u x_1 \xrightarrow{A_2} x_2,\ ...,\ \Gamma \vdash_u x_{n-1} \xrightarrow{A_n} x$$
$$\text{and} \quad \Gamma \vdash_u x : G \quad \text{if} \quad y : A_1 \Rightarrow A_2 \Rightarrow ... \Rightarrow A_n \Rightarrow (G \rightarrow Q) \in \Gamma$$

$$(\mathcal{U}\wedge) \quad \Gamma \vdash_u x : G_1 \wedge G_2 \quad \Rrightarrow \quad \Gamma \vdash_u x : G_1 \quad \text{and} \quad \Gamma \vdash_u x : G_2$$

$$(\mathcal{U}\vee) \quad \Gamma \vdash_u x : G_1 \vee G_2 \quad \Rrightarrow \quad \Gamma \vdash_u x : G_1 \quad \text{or} \quad \Gamma \vdash_u x : G_2$$

$$(\mathcal{U} \Rightarrow) \quad \Gamma \vdash_u x : A \Rightarrow G \quad \Rrightarrow \quad \Gamma, x \xrightarrow{A} y \vdash_u y : G \ (y \text{ new})$$

$$(\mathcal{U}\ \mathbf{trans}) \quad \Gamma \vdash_u x \xrightarrow{A} y \quad \Rrightarrow \quad u : A' \vdash_u u : A \quad \text{and} \quad u : A \vdash_u u : A' \quad \text{if} \quad x \xrightarrow{A'} y \in \Gamma$$

Fig. 8. Rules of $\mathcal{U}$CK for uniform proofs.

THEOREM 6.5 COMPLETENESS OF $\mathcal{U}$CK. *If* Γ *is a database,* G *is a goal, and* $\Gamma \vdash x : G$ *is derivable in SeqCK, then* $\Gamma \vdash x : G$ *is derivable in* $\mathcal{U}$*CK.*

Proof. By induction on the height of the derivation of $\Gamma \vdash x : G$. If $G = \top$, then we are done by the rule $(\mathcal{U}\top)$. If G is an atom P, then $x : G$ must belong to Γ and we are done by applying the rule ($\mathcal{U}$ ax). For the inductive step, we consider all the cases:

- $\Gamma \vdash x : G_1 \wedge G_2$: since ($\wedge$ R) is invertible, there is a proof of the sequent ended as follows:
$$\frac{\Gamma \vdash x : G_1 \qquad \Gamma \vdash x : G_2}{\Gamma \vdash x : G_1 \wedge G_2}\ (\wedge R)$$
By inductive hypothesis, $\Gamma \vdash_u x : G_1$ and $\Gamma \vdash_u x : G_2$ are derivable in $\mathcal{U}$CK, then we conclude by an application of $(\mathcal{U}\wedge)$;
- $\Gamma \vdash x : G_1 \vee G_2$: since ($\vee$ R) is invertible, we can find a derivation ended with an application of ($\vee$ R) to $x : G_1 \vee G_2$ as follows:
$$\frac{\Gamma \vdash x : G_1, x : G_2}{\Gamma \vdash x : G_1 \vee G_2}\ (\vee R)$$
By Proposition 6.3, either $\Gamma \vdash x : G_1$ or $\Gamma \vdash x : G_2$ is derivable, thus we can apply the inductive hypothesis obtaining a derivation in $\mathcal{U}$CK of either $\Gamma \vdash_u x : G_1$ or $\Gamma \vdash_u x : G_2$, thus we can conclude by an application of $(\mathcal{U}\vee)$;
- $\Gamma \vdash x : A \Rightarrow G_1$: since ($\Rightarrow$ R) is invertible (Theorem 3.11), there is a proof of

the sequent ended as follows:

$$\frac{\Gamma, x \xrightarrow{A} y \vdash y : G_1}{\Gamma \vdash x : A \Rightarrow G_1} (\Rightarrow R)$$

By inductive hypothesis on the premise, $\Gamma, x \xrightarrow{A} y \vdash_u y : G_1$ is derivable, then we conclude by an application of $(\mathcal{U} \Rightarrow)$;

- $\Gamma \vdash x : Q, Q \in ATM$: by Lemma 6.4 we have that $\Gamma^*_x \vdash x : Q$ is derivable with at most the same height. Γ^*_x contains clauses of the form $y : A_1 \Rightarrow A_2 \Rightarrow ... \Rightarrow A_n \Rightarrow (G_j \rightarrow Q_j)$, with $Q_j \in ATM$, and transition formulas. One can observe the following fact:

  FACT 6.6. *If* $\Gamma^*_x \vdash x : Q$ *is derivable, then there exists* $x_0 : A_1 \Rightarrow A_2 \Rightarrow ... \Rightarrow A_k \Rightarrow (G' \rightarrow Q) \in \Gamma^*_x$ *such that* $\Gamma^*_x \vdash x_0 \xrightarrow{A_1} x_1$ *and* $\Gamma^*_x \vdash x_1 \xrightarrow{A_2} x_2$ *and ... and* $\Gamma^*_x \vdash x_{k-1} \xrightarrow{A_k} x_k$*, where* $x_k = x$

  whose proof is left to the reader. By the above Fact 6.6, we observe that there is a proof of $\Gamma^*_x \vdash x : Q$ ended by several applications of (⇒ L) (as reformulated in Figure 7) as follows (say $\Gamma^*_x = \Gamma'^*_x, x_0 : A_1 \Rightarrow A_2 \Rightarrow ... \Rightarrow A_k \Rightarrow (G' \rightarrow Q)$):

$$\Gamma^*_x \vdash x_{k-1} \xrightarrow{A_k} x \qquad (1)\Gamma'^*_x, x : G' \rightarrow Q \vdash x : Q$$
$$\vdots$$
$$\dfrac{\Gamma^*_x \vdash x_0 \xrightarrow{A_1} x_1 \quad \dfrac{\Gamma^*_x \vdash x_1 \xrightarrow{A_2} x_2 \quad \Gamma'^*_x, x_2 : A_3 \Rightarrow ... \Rightarrow A_k \Rightarrow (G' \rightarrow Q) \vdash x : Q}{\Gamma'^*_x, x_1 : A_2 \Rightarrow ... \Rightarrow A_k \Rightarrow (G' \rightarrow Q) \vdash x : Q} (\Rightarrow L)}{\Gamma'^*_x, x_0 : A_1 \Rightarrow ... \Rightarrow A_k \Rightarrow (G' \rightarrow Q) \vdash x : Q} (\Rightarrow L)$$

  Since (→ L) is invertible (Theorem 3.11), from (1) we can find a proof with at most the same height of $\Gamma'^*_x \vdash x : G', x : Q$, then of $(2)\Gamma \vdash x : G', x : Q$ since weakening is height preserving admissible (Theorem 3.10). By applying the height-preserving strong disjunction property (Proposition 6.3) to (2) we can find a proof of either $\Gamma \vdash x : Q$ or $\Gamma \vdash x : G'$, to which we can apply the inductive hypothesis obtaining a proof in $\mathcal{U}$CK of either $(i)\Gamma \vdash_u x : Q$ or $(ii)\Gamma \vdash_u x : G'$, respectively. In case (i) we are done. In case (ii), we can find proofs of $(3_1)\Gamma \vdash x_0 \xrightarrow{A_1} x_1, ..., (3_k)\Gamma \vdash x_{k-1} \xrightarrow{A_k} x$ from $\Gamma^*_x \vdash x_0 \xrightarrow{A_1} x_1, ..., \Gamma^*_x \vdash x_{k-1} \xrightarrow{A_k} x$ since weakening is admissible, thus we can apply the inductive hypothesis on (3_i) obtaining derivations of $\Gamma \vdash_u x_0 \xrightarrow{A_1} x_1, ..., \Gamma \vdash_u x_{k-1} \xrightarrow{A_k} x$: from these ones and from (ii) we conclude by an application of ($\mathcal{U}$ **prop**).

□

The following simple example illustrates the usage of the rules. Here the reading of a conditional $A \Rightarrow B$ would be something like “if the current state is updated with A then B holds”; A might well be an action thus, the conditional can also be read “as an effect of A, B holds”, or “having performed A, B holds”. However, we deliberately do not fix the exact interpretation of conditionals, being out of the scope of this work. We just observe that conditionals have been widely used to express update/action/causation (see for instance [Schwind 1999; Giordano and Schwind 2004; Gabbay et al. 2000]).

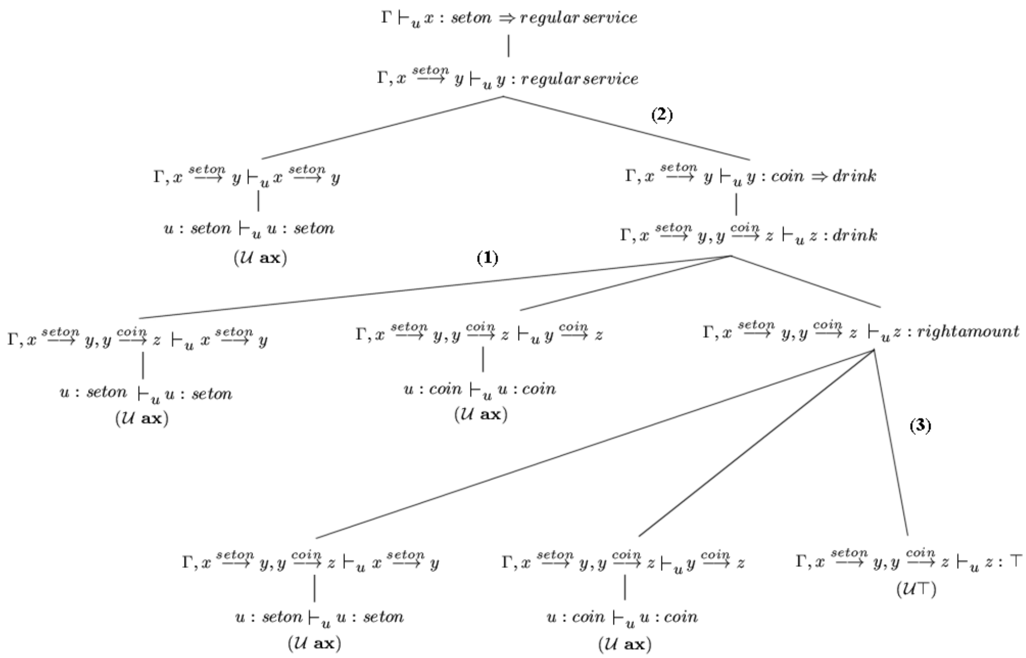

Fig. 9. A derivation of the goal $x : seton \Rightarrow regularservice$. When **($\mathcal{U}$ prop)** is applied the corresponding edge is labelled with the number of the formula of the initial database used.

EXAMPLE 6.7. *Our knowledge base describes a vending machine, the database contains:*

(1) $x : seton \Rightarrow coin \Rightarrow (rightamount \rightarrow drink)$
(2) $x : seton \Rightarrow ((coin \Rightarrow drink) \rightarrow regularservice)$
(3) $x : seton \Rightarrow coin \Rightarrow rightamount$[15]

The above knowledge base could be read as representing the following facts: (1)*: Having set the machine on and then having put some coin, if the amount is right, the customer will get his drink.* (2)*: having set the machine on, it operates correctly (regular service) whenever it gives a drink after inserting coin.* (3)*: Having set the machine on and then having put some coin, it results that the amount is right.*

We show that the goal "Having set the machine on, it gives a regular service?"

$$x : seton \Rightarrow regularservice$$

derives from Γ. *The derivation is presented in Figure 9.*

7. CONDLEAN: A THEOREM PROVER FOR CONDITIONAL LOGICS

In this section we present CondLean, a theorem prover implementing the sequent calculi SeqS; it is a SICStus Prolog program inspired by leanT^AP [Beckert and

[15]This formula is an abbreviation for $x : seton \Rightarrow coin \Rightarrow (\top \rightarrow rightamount)$.

Posegga 1995]. The program comprises a set of clauses, each one of them represents a sequent rule or axiom. The proof search is provided for free by the mere depth-first search mechanism of Prolog, without any additional ad hoc mechanism. CondLean is available for free download at `http://www.di.unito.it/~pozzato/condlean3.1`.

We represent each component of a sequent (antecedent and consequent) by a *list* of labelled formulas, partitioned into three sub-lists: atomic formulas, transitions and complex formulas. Atomic and complex formulas (i.e. the world formulas) are represented by a list like `[x,a]`, where `x` is a Prolog constant and `a` is a formula. A transition $x \xrightarrow{A} y$ is represented by `[x,a,y]`. For example, the sequent $x : A \Rightarrow B, x : A \Rightarrow C, x \xrightarrow{A} y \vdash y : B, x : C, x : A \rightarrow B$ is represented by the following lists (the upper one represents the antecedent, the lower one represents the consequent):

```
[[],[[x,a,y]],[[x,a=>b],[x,a=>c]]]
[[y,b],[x,c]],[],[[x,a->b]]
```

Prolog constants are used to represent SeqS's labels. The sequent calculi are implemented by the predicate

prove(Sigma, Delta, Labels).

This predicate succeeds if and only if $\Sigma \vdash \Delta$ is derivable in SeqS, where `Sigma` and `Delta` are the lists representing the multisets Σ and Δ, respectively and `Labels` is the list of labels introduced in that branch. For instance, to prove

$$x\text{: } A \Rightarrow (B \wedge C)^{16} \vdash x\text{: } A \Rightarrow B,\ x\text{: } C$$

in CK, one queries CondLean with the goal:

```
prove([[],[],[[x, a=>(b and c)]]], [[[x,c],[],[[x, a=>b]]], [x]).
```

Each clause of `prove` implements one axiom or rule of SeqS; for example, the clause implementing ($\Rightarrow$ L) is:

```
prove([LitSigma,TransSigma,ComplexSigma],[LitDelta,TransDelta,
    ComplexDelta], Labels):-
  member([X,A=>B],ComplexSigma), member(Y,Labels),
  put([Y,B],LitSigma,ComplexSigma,NewLitSigma,NewComplexSigma),
  prove([LitSigma,TransSigma,ComplexSigma],
    [LitDelta,[[X,A,Y]|TransDelta],ComplexDelta],Labels),
  prove([NewLitSigma,TransSigma,NewComplexSigma],
    [LitDelta,TransDelta,ComplexDelta],Labels).
```

The predicate `put` is used to put `[Y,B]` in the proper sub-list of the antecedent. To search a derivation of a sequent $\Sigma \vdash \Delta$, CondLean proceeds as follows. First of all, if $\Sigma \vdash \Delta$ is an axiom, the goal will succeed immediately by using the clauses for the axioms. If it is not, then the first applicable rule will be chosen, e.g. if `ComplexSigma` contains a formula `[X,A and B]`, then the clause for ($\wedge$ L) rule will be used, invoking `prove` on the unique premise of ($\wedge$ L). CondLean proceeds in a

[16] CondLean extends the sequent calculi to formulas containing also $\neg$, $\wedge$, $\vee$ and $\top$.

similar way for the other rules. The ordering of the clauses is such that the application of the branching rules is postponed as much as possible. In order to control the application of rules ($\Rightarrow$ L), (ID), (MP), (CEM), and (CS), i.e. rules in which premises have a higher complexity than the conclusion, additional parameters are added to the predicate `prove`, implementing the restricted applications described in section 4.

Considering systems SeqCK and SeqID, CondLean implements the refinements described in section 5, i.e. ($\Rightarrow$ L), (ID), and (MP) are not invertible (see Figures 4 and 7). In particular, we have shown that ($\Rightarrow$ L) needs to be applied *only once* in each derivation branch (Lemma 5.10). Therefore, when the clause implementing ($\Rightarrow$ L) is used, a backtracking point is introduced by the choice of the label `Y` occurring in the premises of the rule. Choosing, sooner or later, the right label to apply ($\Rightarrow$ L) may strongly affect the theorem prover's efficiency: if there are n labels to choose for an application of ($\Rightarrow$ L) the computation might succeed only after n-1 backtracking steps, with a significant loss of efficiency. Therefore, for these systems we present a second implementation, called *free-variables*, which makes use of *Prolog variables* to represent all the labels that can be used in a single application of the ($\Rightarrow$ L) rule. This version represents labels by integers starting from 1; by using integers we can easily express constraints on the range of the variable-labels. To this regard the library `clpfd` is used to manage free-variable domains.

We have also developed a third version for systems SeqCK and SeqID, called *heuristic version*, that performs a "two-phase" computation: in "Phase 1" an *incomplete* theorem prover searches a derivation exploring a *reduced search space*; in case of failure, the free-variables version is called ("Phase 2"). Intuitively, the reduction of the search space in Phase 1 is obtained by committing the choice of the label to instantiate a free variable, whereby blocking the backtracking.

More details on free-variables and heuristic versions are given in [Olivetti and Pozzato 2003].

The performances of the theorem prover are promising even on a small machine. To test our program we used samples generated by modifying the samples from [Beckert and Goré 1997] and from [Viganò 2000].

We have tested CondLean, SeqCK system, obtaining the following results[17]:

(1) the *standard* version, called *constant labels* version, succeeds in 79 tests over 90 in less than 2 seconds (78 in less than one second);
(2) the free-variables version succeeds in 73 tests over 90 in less than 2 seconds (but 67 in less than 10 mseconds);
(3) the heuristic version succeeds in 78 tests over 90 in less than 2 seconds (70 in less than 500 mseconds).

Considering the sequent-degree (defined as the maximum level of nesting of the $\Rightarrow$ operator) as a parameter, we have the following results, obtained by testing the SeqCK free-variables version:

Sequent degree	2	6	9	11	15
Time to succeed (mseconds)	5	500	650	1000	2000

[17]These results are obtained running SICStus Prolog 3.10.0 on an Intel Pentium 166 MMX, 96 MB RAM machine.

It is worth noticing that our theorem prover is only inspired by the lean methodology, but it does not fit its style in a rigorous manner. For instance, we use some auxiliary predicates, such as `put`, `select`, and `member`, whereas leanT^AP only relies on Prolog's clause indexing scheme and backtracking. We use these auxiliary predicates in order to control the derivation process. In particular, by partitioning formulas of each sequent into three sub-lists (atoms, transitions, complex formulas) and by using the `select` (resp. the `member`) predicate, we can have a better control of proof search. For instance we can postpone the application of branching rules (including the critical rule of left conditional) since we can choose the rule to apply by selecting the next complex formula to process, instead of processing always the leftmost one. On the other hand, the first argument of leanT^AP is the next formula to be processed, which is always the leftmost formula in a single-sided sequent, and this allows it to use the first-argument indexing refinements available in SICStus Prolog; our code does not present this characteristic, so it cannot take advantage of these refinements.

8. CONCLUSIONS

In this work we have provided a labelled calculus for minimal conditional logic CK, and its standard extensions with conditions ID, MP, CS and CEM. We have found cut-free and analytic systems for almost all studied systems, except for those presenting both MP and CEM. Basing on these calculi we have obtained a decision procedure for the respective logics. Moreover, we have been able to show that these logics are PSPACE. To the best of our knowledge, sequent calculi for these logics have not been previously studied and the complexity bound for them is new. Furthermore, we have presented a tighter space complexity bound for CK{+ID} which is based on the disjunction property of conditional formulas. We have also begun the investigation of a goal directed proof procedure for these conditional logics in the style of Miller's uniform proofs. Finally, we have implemented our calculi by a theorem prover, called CondLean, written in SICStus Prolog and which is inspired to the lean methodology.

Comparison with other works

We briefly remark on some related works. Most of the works have concentrated on extensions of CK: in the introduction we have made a distinction between conditional logics which admit the (more general) selection function semantics from conditional logics which admit the sphere semantics. The investigation of proof systems has been addressed to conditional logics of both types: Crocco, Fariñas, Artosi, Governatori, and Rotolo have studied proof systems for conditional logics based on the selection function semantics, whereas De Swart, Gent, Lamarre, Groeneboer, and Delgrande have studied proof systems for logics with sphere semantics.

Crocco and Fariñas [Crocco and del Cerro 1995] present sequent calculi for some conditional logics including CK, CEM, CO and others. Their calculi comprise two levels of sequents: principal sequents with $\vdash_P$ corresponds to the basic deduction relation, whereas auxiliary sequents with $\vdash_a$ corresponds to the conditional operator: thus the constituents of $\Gamma \vdash_P \Delta$ are sequents of the form $X \vdash_a Y$, where

X, Y are sets of formulas. The bridge between auxiliary sequents and conditional formulas is given by the *descending rule*:

$$\frac{X_1 \vdash_a B_1, \ldots X_{n-1} \vdash_a B_{n-1} \vdash_P X_n \vdash_a B_n}{A_1 \Rightarrow B_1, \ldots, A_{n-1} \Rightarrow B_{n-1} \vdash_P A_n \Rightarrow B_n}$$

where the $A_i = \bigwedge_i X_i$. These systems provide an interesting proof-theoretical interpretation of the conditional operator in terms of structural rules (eg. reduction and augmentation rules). It is not clear if these calculi can be used to obtain a decision procedure for the respective logics.

Artosi, Governatori, and Rotolo [Artosi et al. 2002] develop labelled tableau for the conditional logic CU that corresponds to cumulative non-monotonic logics. In their work they use labels similarly to ours. Formulas are labelled by path of worlds containing also variable worlds (see also our free-variable implementation). Their tableau method is defined primarily for a conditional language without nested conditionals; however, the authors discuss a possible extension to a non restricted conditional language. Differently from us, they do not use a specific rule to deal with equivalent antecedents of conditionals. They use instead a sophisticated unification procedure to propagate positive conditionals. The unification process itself checks the equivalence of antecedents. Their tableau system is based on KE and thus it contains a cut rule, called PB, whose application can be restricted in an analytic way. It is clear that we could incorporate an analytic cut rule in our system (as argued by Artosi, Governatori, and Rotolo): this would shorten the derivation size, although it can affect the proof search mechanism.

De Swart [de Swart 1983] and Gent [Gent 1992] give sequent/tableaux calculi for the strong conditional logics VC and VCS. Their proof systems are based on the entrenchment connective $\leq$, from which the conditional operator can be defined. Their systems are analytic and comprise an infinite set of rules $\leq F(n, m)$, with a uniform pattern, to decompose each sequent with m negative and n positive entrenchment formulas.

Lamarre [Lamarre 1993] presents tableaux systems for the conditional logics V, VN, VC, and VW. Lamarre's method is a consistency-checking procedure which tries to build a system of sphere falsifying the input formulas. The method makes use of a subroutine to compute the *core*, that is defined as the set of formulas characterizing the minimal sphere. The computation of the core needs in turn the consistency checking procedure. Thus there is a mutual recursive definition between the procedure for checking consistency and the procedure to compute the core.

Groeneboer and Delgrande [Delgrande and Groeneboer 1990] have developed a tableau method for the conditional logic VN which is based on the translation of this logic into the modal logic S4.3.

[Giordano et al. 2003] have defined a labelled tableaux calculus for the logic CE and some of its extensions. The flat fragment of CE corresponds to the nonmonotonic preferential logic P and admits a semantics in terms of preferential structures (possible worlds together with a family of preference relations). The tableau calculus makes use of pseudo-formulas, that are modalities in a hybrid language indexed on worlds. In that paper it is shown how to obtain a decision procedure for that logic by performing a kind of loop checking.

Finally, complexity results for some conditional logics have been obtained by Friedman and Halpern [Friedman and Halpern 1994]. Their results are based on a semantic analysis, by an argument about the size of possible countermodels. They do not give an explicit decision procedure for the logics studied. They consider conditional logics with the preferential semantics (which is related to the sphere semantics), therefore the systems they consider are either stronger or not comparable with ours[18]. Most of the logics they consider turn out to be PSPACE complete.

Future work

The proof theoretical and complexity analysis of the systems considered in this work presents some open issues. For the systems including CEM and MP we have not been able to prove the admissibility of cut although we conjecture that it holds and we hope to prove it in future research.

For CEM the complexity bound we have found is not optimal as it is known that this logic is co-NP complete. In this work we have given uniform and modular calculi for all the logics under consideration. It might be that one can derive from our calculi an optimized calculus for CEM matching the known complexity bound. Moreover, we can use our calculi to study other logical properties such as interpolation.

We would like to study labelled sequent calculi for other conditional logics based on the selection function semantics. Among the others, the following axioms/semantic conditions are well known in the literature:

(AC) $(A \Rightarrow B) \wedge (A \Rightarrow C) \rightarrow (A \wedge C \rightarrow B)$
If $f(w, [A]) \subseteq [B]$ then $f(w, [A \wedge B]) \subseteq f(w, [A])$
(CV) $(A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C) \rightarrow (A \wedge C \Rightarrow B)$
If $f(w, [A]) \subseteq [B]$ and $f(w, [A]) \cap [C] \neq \emptyset$ then $f(w, [A \wedge C]) \subseteq [B]$
(CA) $(A \Rightarrow C) \wedge (B \Rightarrow C) \rightarrow (A \vee B \Rightarrow C)$
$f(w, [A \vee B]) \subseteq f(w, [A]) \cup f(w, [B])$

These axioms/conditions are part of well-known conditional logics [Nute 1980]. Some of these conditions can be used to formalize non-monotonic inferences. For instance AC corresponds to the property of *cumulativity* and CV to the property of *rational monotony.* Preferential entailment corresponds to the first degree fragment (i.e. without nested conditionals) of CK+ID+AC+CUT+CA [Kraus et al. 1990].

We can think of extending our calculi to these logics. We would like to have a modular proof system in the form of a sequent calculus where each semantic condition/axioms corresponds to a well-defined group of rules[19]. To this regard, it is not difficult to devise rules capturing these semantic conditions. However, a straightforward encoding of the above semantic conditions results in a non-analytic

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[18]Among the others, they consider the semantic condition of centering, which is our MP, but they do not consider strong centering CS nor CEM.

[19]The systems of conditional logics with axioms AC, CA and CV enjoy an alternative semantics in terms of preferential models. If one adopts this alternative semantics, one can obtain analytic proof systems as shown in the mentioned [Giordano et al. 2003] and in [Giordano et al. 2005] (the latter presents calculi for KLM logics). However, these proof systems take advantage of the special nature of preferential models and do not fit uniformly into the family of calculi for conditional logics with selection function semantics presented in this work.

calculus where cut cannot be eliminated. The difficulty is that the selection function cannot be assumed to satisfy any *compositionality* principle: i.e. the value of $f(w, [A\#B])$ for any connective # is not a function of $f(w, [A])$ and $f(w, [B])$, at most f satisfies some constraints as the above ones. However, further research is needed to see how and whether we can capture the above semantic conditions and alike within the labelled calculus by analytic rules.

We also intend to develop goal-directed, or uniform proof, calculi for all conditional logics we have considered. This development could lead to define logic programming languages based on conditional logics.

Finally, we would like to improve our theorem prover CondLean by experimenting standard refinements and heuristics.

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