ANALYTIC TABLEAUX FOR NON-MONOTONIC REASONING

Nicola Olivetti
Camilla Schwind

Outline

- Introduction

- Motivation for Nonmonotonic reasoning
- Tableaux for Nonmonotonic reasoning

- Circumscription

- Nonmonotonic Modal Logics

- Default Logic
- Theory Revision

Introduction

- Classical Logic is *monotonic*:

If $A \vdash p$ and $A \subset B$ then $B \vdash p$

- Common sense Reasoning is *not* monotonic.
- Nonmonotonic Reasoning applies to:

- databases (closed world assumption)
- logic programming (negation as failure)
- reasoning about actions (frame problem)
- inheritance reasoning
- diagnosis

Types of Common Sense Reasoning

- Normally, students are young.

==> Circumscription

- Typically, birds fly.

==> Default Logic

- If penguins would not fly, we would know about it

==> Nonmonotonic modal logic
(Autoepistemic Logic)

Circumscription (John MacCarthy 1980)

Motivation

Example:

$Bird(Tweety) \wedge Snake(Kaa)$

Difficulty to deduce that

Kaa is not a bird!!

Impossible in classical Logic

Solution: Minimize the predicate Bird

Notation

P, Q predicates

$P \leq Q$: $\forall x(P(x) \rightarrow Q(x))$

Tuples of predicates

$P = P_1, P_2, ...P_n$

$Q = Q_1, Q_2, ...Q_n$

$P \leq Q$

$P_1 \leq Q_1 \wedge P_2 \leq Q_2 \wedge ...P_n \leq Q_n.$

$P = Q$ iff $P \leq Q \wedge Q \leq P$

$P < Q \, : \, P \leq Q \wedge \neg(Q \leq P)$

Circumscription Schema

P predicate constants, $A(P)$ formula

Circumscription of P in $A(P)$ is

$A(P) \wedge \neg\exists\Phi(A(\Phi) \wedge \Phi < P)$

We note $Circ(A; P)$

$Circ(Bird(Tweety) \wedge Snake(Kaa); Bird) =$
$Bird(Tweety) \wedge Snake(Kaa) \wedge$
$\neg\exists\Phi(\Phi(Tweety) \wedge Snake(Kaa) \wedge \Phi < Bird)$

Instantiation of Φ by $x = Tweety$ yields

$Circ(Bird(Tweety) \wedge Snake(Kaa); Bird)$
$\vdash \neg Bird(Kaa)$

(using the unique names assumption)
$Tweety \neq Kaa$

Minimizing Exceptions

Tweety is a bird.
Penguins are birds which do not fly.
A bird flies unless it is an exceptional bird.

We want to deduce that Tweety flies.

$$Bird(Tweety)$$

$$\forall x((Penguin(x) \rightarrow Bird(x) \wedge \neg fly(x))$$

$$\forall x((Bird(x) \wedge \neg exception(x)) \rightarrow fly(x))$$

Minimize exception BUT Predicate circumscription does not allow to derive $fly(Tweety)$

Dependency between $exception$ and fly

Formula Circumscription

Circumscription of a formula
with a set of predicate and function constants
allowed to vary

P tuple of predicate constants
Z tuple of function/predicate constants
disjoint from P
occurring within a formula $A(P, Z)$

Circumscription of P in $A(P, Z)$ with
variables Z allowed to vary
$A(P, Z) \wedge \neg \exists \Phi \exists \phi (A(\Phi, \phi) \wedge \Phi < P)$
Note $Circ(A; P; Z)$

Circumscribing *exception* with fly varying
permits to derive $fly(Tweety)$.

Minimal models

Partial Preorder between Models

$\mid I \mid_P$ extension of predicate P in interpretation I

I and J Interpretations $I <_{P,Z} J$

1. $\mid I \mid_P \subset \mid J \mid_P$
2. $\mid I \mid_Q = \mid J \mid_Q$
   for predicates Q not in P and in Z

$<_{P,Z}$ is a preorder:

$I <_{P,Z} J$ and $J <_{P,Z} I$ without $I = J$

Interpretation $I \in S$ is minimal in S
with respect to $<_{P,Z}$ iff
there is no interpretation $I' \in S$
such that $I' <_{P,Z} I$

Propositional Circumscription

Let be $Prop$ the set of all propositional variables

$P \in Prop$ set of propositional variables (circumscribed)
$Z \in Prop$ set of propositional variables (varying)
disjoint from P

I propositional interpretation
Denote $I_P = I \cap P$

Le be I and J propositional interpretations:

$I \leq_{P,Z} J$ iff

1. $I_P \subset J_P$ and
2. $I_Q = J_Q$ for $Q = Prop \setminus (P \cup Z)$

Correctness of Circumscription

A First-order theory

P Predicate symbols

Z Predicate and Function symbols

Theorem (McCarthy)

Every $<_{P,Z}$-minimal model for A satisfies every instance of $Circ(A; P; Z)$

Completeness of Circumscription

Circumscription is not complete
over the class of universal finitary theories

Example:

$$I(0)$$

$$\forall x \forall y I(x) \wedge (S(x,y) \vee S(y,z)) \rightarrow I(y)$$

$$\forall x \neg S(x,0)$$

$$\forall x \forall y \forall z S(x,z) \wedge S(y,z)) \rightarrow x = y$$

$$\forall x \forall y \forall z S(x,y) \wedge S(x,z)) \rightarrow z = y$$

Completeness of Circumscription

Circumscription is complete
over the class of theories
with a domain closure axiom

$$\forall x(x = c_1 \vee x = c_2 \vee x = c_3 \ldots \vee x = c_n$$

for domain D, with $D = \{c_1, c_2, \ldots c_n\}$

Completeness of Circumscription

Circumscription is complete
over the class of well-founded theories
such that P is disjunctively definable
with parameters in $C, (P)$

for domain D, with $D = \{c_1, c_2, \ldots c_n\}$

Default Logic (Raymond Reiter 1980)

$\Delta = (W, D)$

W set of closed first order sentences "world"

D a set of default rules

Default: $\dfrac{\alpha : \{\beta_1, \beta_2, \ldots, \beta_n\}}{\gamma}$

α, β_i and γ are closed first order sentences

α prerequisite

$\beta_1, \beta_2, \ldots, \beta_i, \ldots \beta_n$ justifications

γ consequent

Extensions

R- Extension of a default theory (W, D):
Let S be a set of formulae without free variables
Let $\Gamma(S)$ be the smallest set satisfying

D1 $W \subset \Gamma(S)$

D2 $\Gamma(S) = Th(\Gamma(S))$

D3 if $\dfrac{\alpha : \{\beta_1, \beta_2, \ldots, \beta_n\}}{\gamma} \in D$ and $\alpha \in \Gamma(S)$
and
$\neg\beta_i \notin S$ for $1 \leq i \leq n$
then $\gamma \in \Gamma(S)$

E is an extension iff $E = \Gamma(E)$

$Th(\Gamma) = \{\phi : \Gamma \vdash \phi\}$

Non-Existence of Extension

$\Delta = (W, D)$, with $W = \emptyset$ and $D = \dfrac{: \neg A}{A}$

normal defaults always admit an extension

Normal default: $\dfrac{\alpha : \beta}{\beta}$

One prerequisite and identical to the consequent.

Examples

$\Delta_1 = (W_1, D_1)$, with $W_1 = \emptyset$ and
$D_1 = \{\frac{:a}{\neg b}, \frac{:b}{\neg a}\}$

has two extensions

$E_1 = Th(\{\neg b\})$
$E_2 = Th(\{\neg a\})$

$\Delta_2 = (W_2, D_2)$, with $W_2 = \emptyset$ and
$D_2 = \{\frac{:a}{\neg b}, \frac{:b}{\neg c}, \frac{:c}{\neg a}\}$

has no extension

$\Delta_3 = (W_3, D_3)$, with $W_3 = \{a, \neg b\}$ and
$D_3 = \{\frac{a:\neg c}{b}\}$

has no extension.

But: a : Karl is German

b : Karl drinks beer

c: to be temperate

Justified Default Logic (Lukasiewicz)

Let $\Delta = (W, D)$ be a default theory.
Let D' be any subset of D.

$E = Th(W \cup CONS(D')$ is a j-extension of Δ with respect to $F = Th(W \cup JUST(D') \cup CONS(D'))$ iff D' is a maximal grounded subset of D, such that for all $\beta \in JUST(D')$, $\neg\beta \notin E$.

Theorem 1 *A default theory has always a j-extension.*

Constrained Default Logic (Schaub, Brewka)

Let $\Delta = (W, D)$ be a default theory.
Let D' be any subset of D.

$E = Th(W \cup CONS(D')$ is a c-extension of Δ
with respect to
$C = Th(W \cup JUST(D') \cup CONS(D'))$ iff
D' is a maximal grounded subset of D',
such that $E \cup JUST(D')$ is consistent.

Theorem 2 *A default theory has always a c-extension.*

Example

$$W = \{FRENCH(Dupond), FRENCH(Durand), CANADIAN(Mike), \neg DR\text{-}WINE(Durand)\},$$

$$\hat{D} = \{\frac{FRENCH(x) : \neg TEMPERATE(x)}{DR\text{-}WINE(x)}, \frac{\top : TEMPERATE(x)}{DR\text{-}WATER(x)}\}.$$

One j-extension:

$$E = Th(W \cup \{DR\text{-}WINE(Dupond), DR\text{-}WATER(Dupond), DR\text{-}WATER(Durand), DR\text{-}WATER(Mike)\}).$$

No R-extension

Two c-extensions:

$$E_1 = Th(W \cup \{DR\text{-}WINE(Dupond), DR\text{-}WATER(Durand), DR\text{-}WATER(Mike)\})$$

$$E_2 = Th(W \cup \{DR\text{-}WATER(Dupond), DR\text{-}WATER(Durand), DR\text{-}WATER(Mike)\}).$$

Nonmonotonic Modal Logics

(McDermott and Doyle 1980)

• Built on a given modal logic S:
propositional logic + L

• "Students are typically young":
$s \wedge \neg L \neg y \rightarrow y$

• Possible sets of nonmonotonic conclusions from a set of premises Σ: S-expansions of Σ.

• A set of formulas Δ is an S**-expansion** of Σ iff

$$\Delta = \{\phi \mid \Sigma \cup \neg L\overline{\Delta} \vdash_S \phi\}$$

where $\vdash_S$ is the derivability relation of the logic S
and $\neg L\overline{\Delta} = \{\neg L\phi \mid \phi \in For_L - \Delta\}$.

Autoepistemic Logic (Moore 1980)

- Sets of autoepistemic conclusions from a set of premises Σ
==> **stable expansions** of Σ

- A set of formulas Δ is **stable expansion** of Σ iff

$$\Delta = \{\phi \mid \Sigma \cup L\Delta \cup \neg L\overline{\Delta} \vdash \phi\}$$

where $\vdash$ is the propositional derivability relation

$L\Delta = \{L\phi \mid \phi \in \Delta\}$ and

$\neg L\overline{\Delta} = \{\neg L\phi \mid \phi \in For_L - \Delta\}$.

- No modal axioms applied, but a necessitation rule
for positive introspection:
if ϕ is believed, so is $L\phi$

Examples

$\{Lp \to p\}$ has two expansions

$$E_1 = Th(\{p, Lp, \neg L \neg p, \ldots\})$$

$$E_2 = Th(\{\neg p, \neg Lp, L\neg p, L\neg Lp, \ldots\})$$

$\{Lp\}$

has no expansion

Nonmonotonic Reasoning Tasks

• Extensions/expansions give possible sets of "conclusions"
• Premises yielding multiple extensions/expansions

==> Cautious (skeptical) and brave (credulous) approach
• ϕ is a **cautious** conclusion from a set of premises if it belongs to every extension/expasnsion of the premises.

• ϕ is a **brave** conclusion from a set of premises if it belongs to some extension/expansion of the premises.

Decidability and Complexity

- First-order case not even semi-decidable
- Propositional case decidable
- Nonmonotonic reasoning (circumscription, DL, AEL) is complete w.r.t. the second level of the polynomial time hierarchy (PH) in the polynomial case. Cautious reasoning is Π_2^p complete
Brave reasoning is Σ_2^p complete

- Nonmonotonic reasoning is strictly harder than classical reasoning (unless $P = NP$).

Use of Tableaux for NMR

Tableaux are likely the most flexible and promising proof method for nonmonotonic reasoning.

- For *fixpoint based nonmonotonic* logics: tableaux can be used to check both *provability* and *consistency* in a natural way.
- For *preference-based nonmonotonic logics*: one can define a preference criterium among open branches which mirrors the intended semantic preference and make use of it to eliminate unwanted models.

Tableaux, (as any other standard proof-method) are monotonic:
if T results to be closed by the rules of tableau expansion, then whatever we add to (any branch of) T, T remains closed.

How can a tableau methodology be used to represent nonmonotonic mechanisms?

Making tableaux nonmonotonic

Some general principles:

- *Operation on Branches*: addition and removal of formulas from completed branches, according to some conditions.
- *Relative closure*: the closure of a tableau may depend on another related tableau being *open*, in this way we can incorporate unprovability checks.
- *Negative closure conditions*: some branches in a completed tableau are forced to be "closed" because they do not contain some formulas.
- *Selection*: open branches of a tableau are ruled out because of an external selection criteria.

Background Notions and Notation

We identify a tableau with *a set* T *of sets* B *of signed formulas*:

$$\mathbf{T}\phi, \mathbf{F}\phi,$$

each set B is called a *branch*.

Signed formulas are divided in two types: α-type and β-type.

Rules of tableau expansion

replace a branch containing a signed formula ϕ by one or more branches as follows:

- if a branch B contain ϕ of type α, whose α-subformulas are ϕ_1, ϕ_2, then replace B by

$$B' = (B - \{\phi\}) \cup \{\phi_1, \phi_2\}$$

- if a branch B contain ψ of type β, whose β-subformulas are ψ_1, ψ_2, then replace B by

$$B_1 = (B - \{\psi\}) \cup \{\psi_1\}$$
$$B_2 = (B - \{\psi\}) \cup \{\psi_2\}$$

α	α_1	α_2
$\mathbf{T}(A \wedge B)$	$\mathbf{T}A$	$\mathbf{T}B$
$\mathbf{T}\neg(A \vee B)$	$\mathbf{T}\neg A$	$\mathbf{T}\neg B$
$\mathbf{T}\neg(A \rightarrow B)$	$\mathbf{T}A$	$\mathbf{T}B$
$\mathbf{T}\neg\neg A$	$\mathbf{T}A$	
$\mathbf{F}(A \vee B)$	$\mathbf{F}A$	$\mathbf{F}B$
$\mathbf{F}\neg(A \wedge B)$	$\mathbf{F}\neg A$	$\mathbf{F}\neg B$
$\mathbf{F}(A \rightarrow B)$	$\mathbf{F}\neg A$	$\mathbf{F}B$
$\mathbf{F}\neg\neg A$	$\mathbf{F}A$	

β	β_1	β_2
$\mathbf{T}(A \vee B)$	$\mathbf{T}A$	$\mathbf{T}B$
$\mathbf{T}\neg(A \wedge B)$	$\mathbf{T}\neg A$	$\mathbf{T}\neg B$
$\mathbf{T}(A \rightarrow B)$	$\mathbf{T}\neg A$	$\mathbf{T}B$
$\mathbf{F}(A \wedge B)$	$\mathbf{F}A$	$\mathbf{F}B$
$\mathbf{F}\neg(A \wedge B)$	$\mathbf{F}\neg A$	$\mathbf{F}\neg B$
$\mathbf{F}\neg(A \rightarrow B)$	$\mathbf{F}A$	$\mathbf{F}\neg B$

Given a set of formulas Σ and a formula ϕ, we say that a tableau T is **for** (Σ, ϕ)

if T contains as initial data $\mathbf{T}\psi$ for every $\psi \in \Sigma$ and $\mathbf{F}\phi$.

Given a set of signed formulas Q we denote by $Tab(Q)$ any *completed* tableau with input formulas Q.

Composition of tableaux (see [Schwind90])
Given two tableaux $T1$ and $T2$ we let

$$T_1 \otimes T2 = \{X \cup Y : X \in T1 \ \wedge \ Y \in T_2\}.$$

Tableaux for Propositional Circumscription

We first consider the case when all atoms are minimized.

We adopt the formulation of the rules of tableau expansion without sign-switching rules for negation.

We say that a branch B is $\mathbf{T}$-completed ($\mathbf{F}$-completed) if all its $\mathbf{T}$-signed ($\mathbf{F}$-signed formulas) are literals. A branch B is completed if it is both $\mathbf{T}$- completed and $\mathbf{F}$-completed.

Closure conditions

A branch B is **ordinary closed** if

- it contains a $\mathbf{T}\phi$ and $\mathbf{T}\neg\phi$, ($\mathbf{T}$-closed), or
- it contains $\mathbf{F}\phi$ and $\mathbf{F}\neg\phi$) ($\mathbf{F}$-closed), or,
- it contains a formula $\mathbf{T}\phi$ and $\mathbf{F}\phi$.

A branch B is **m-closed** if

- it is $\mathbf{T}$-completed;
- for some literal $\neg p$, we have $\mathbf{F}\neg p \in B$, but $\mathbf{T}p \notin B$.

B is **closed** if it is either ordinary closed or m-closed.

Ignoring Branches

Given a branch B we let:

$$At(B) = \{p : \mathbf{T}p \in B \text{ and } p \text{ is an atom}\}.$$

Given a branch B in a tableau T, we say that B is **ignorable** if there is another branch $B' \in T$, such that:

(a) B' is $\mathbf{T}$-completed,

(b) B' is not $\mathbf{T}$-closed, and

(c) $At(B') \subset At(B)$.

We say that a tableau T is **closed** iff all branches of T are either closed or ignorable.

Example

$p \vee q \models_m \neg p \vee \neg q$.

Initial tableau:

$$\{\mathbf{T}p \vee q, \mathbf{F}\neg p \vee \neg q\},$$

$$\{\mathbf{T}p, \mathbf{F}\neg p, \mathbf{F}\neg q\}, \quad \{\mathbf{T}q, \mathbf{F}\neg p, \mathbf{F}\neg q\}.$$

$p \to q \models_m \neg q$.

Initial tableaux:

$$\{\mathbf{T}p \to q, \mathbf{F}\neg q\},$$

$$\{\mathbf{T}\neg p, \mathbf{F}\neg q\}, \quad \{\mathbf{T}q, \mathbf{F}\neg q\},$$

the first branch is m-closed the second one is ignorable, thus the tableau is closed.

Examples (continued)

$$p \vee q, p \rightarrow r \not\models_m \neg r.$$

Initial tableau:

$$\{\mathbf{T}p \vee q, \mathbf{T}p \rightarrow r, \mathbf{F}\neg r\},$$

$$\{\mathbf{T}p, \mathbf{T}r, \mathbf{F}\neg r\} \quad \{\mathbf{T}q, \mathbf{T}\neg p, \mathbf{F}\neg r\} \quad \{\mathbf{T}q, \mathbf{T}r, \mathbf{F}\neg r\}.$$

B_2 is m-closed

branch B_3 is ignorable because of B_2,

but B_1 is a counterexample branch.

Let V, with $V(p) = V(r) = 1$, is a minimal model of $(p \vee q) \wedge (p \rightarrow r)$, which falsifies $\neg r$.

Theorem (Olivetti 1992)

For any finite set of formulas A and formulas ϕ, it holds that $A \models_m \phi$ if and only if any tableau for (A, ϕ) is closed.

Remark

The method can be made more efficient if formulas are in *clausal form*; in particular for the *Horn case*, it can be developed a variant which requires no branch split and no ignorability test (similar to bottom-up evaluation).

Dealing with variable atoms

The method can be easily adapted to minimal entailment with variable atoms (that one we denoted by $\models_{\mathbf{P},\mathbf{Z}}$,
(no fixed atoms, i.e. $\mathbf{P} \cup \mathbf{Z} = PVar$)

- **M-closure**: B is $\mathbf{P}$-m-closed if it is $\mathbf{T}$-completed, and for some literal $\neg p$, such that $p \in \mathbf{P}$, we have $\mathbf{F}\neg p \in B$, but $\mathbf{T}p \notin B$.

- **Ignorability condition**: let

$$\mathbf{P}(B) = \{p \in \mathbf{P} \mid \mathbf{T}p \in B\}$$

  A branch B in a tableau T is $\mathbf{P}$-*ignorable* if there is another branch $B' \in T$, such that

  – (a) B' is $\mathbf{T}$-completed,

  – (b) it is not $\mathbf{T}$-closed,

  – (c) $\mathbf{P}(B') \subset \mathbf{P}(B)$.

Example by Brewka Let A be the conjunction of the following formulas:

$$german(tom) \wedge \neg abnormal(tom) \rightarrow drink\_beer(tom),$$
$$german(tom),$$
$$eats\_cake(tom) \rightarrow \neg drink\_beer(tom).$$

$$\mathbf{P} = \{abnormal(tom), eats\_cake(tom)\}.$$

Let T be a tabelau for $(A, drink\_beer(tom))$. The following branches are generated (we only list those one which are not $\mathbf{T}$-closed):

$$\begin{aligned}
B_1 &= \{\mathbf{T}abn(tom), \mathbf{T}ger(tom), \mathbf{T}\neg eats_c(tom), \\
&\quad \mathbf{F}drink_b(tom)\}, \\
B_2 &= \{\mathbf{T}abn(tom), \mathbf{T}ger(tom), \mathbf{T}\neg drink_b(tom \\
&\quad \mathbf{F}drink_b(tom)\} \\
B_3 &= \{\mathbf{T}drink_b(tom), \mathbf{T}ger(tom), \mathbf{T}\neg eats_c(to\ n), \\
&\quad \mathbf{F}drink_b(tom)\}
\end{aligned}$$

Example (continued)

We have that

- B_3 is closed;
- B_1 and B_2 are **P**-ignorable because of B_3.

T proves that $A \models_{\mathbf{P}} \mathit{drink\_beer}(\mathit{tom})$.

General case with fixed atoms

(we recall that fixed-atoms can be always eliminated):

[Kuhna93] tableaux generation of $\leq_{\mathbf{P},\mathbf{Z}}$ minimal models, with fixed atoms

$$\mathbf{Q} = PVar - (\mathbf{P} \cup \mathbf{Z})$$

Characterization of minimal models in terms of tableaux branches for the general case.

Let $\mathbf{Q}(B) = \{r \in \mathbf{Q} \mid \mathbf{T}p \in B\}$

Given a completed open branch $B \in Tab(\mathbf{T}A)$, consider the following property:

> Let $B_1, \ldots, B_k \in Tab(\mathbf{T}A)$ all open branches such that $\mathbf{P}(B_i) \subset \mathbf{P}(B)$. If $k > 0$, then for $i = 1, \ldots, k$, there are literals $l_i \in Fixed(B_i)$, such that $B \cup \{\mathbf{T}l_1^c, \ldots, \mathbf{T}l_k^c\}$ is open.

Fact

- if B is open and satisfies the above property, for some literals $l_1, \ldots, l_k$, then every maximal consistent extension of

$$B \cup \{\neg p : \; p \notin \mathbf{P}(B)\} \cup \{\mathbf{T}l_1^c, \ldots, \mathbf{T}l_k^c\}$$

  is a $\leq_{\mathbf{P},\mathbf{Z}}$ minimal model of A.

- If M is $\leq_{\mathbf{P},\mathbf{Z}}$ minimal model of A then there is a corresponding open branch B which satisfies the above property.

Example

$$T_1 = \{\mathbf{T}S(a), \mathbf{T}S(b)\}$$
$$T_2 = \{\mathbf{T}P(a), \mathbf{T}S(a)\}$$

We have $\mathbf{P}(T_1 \subset \mathbf{P}(T_2)$

Let $\mathbf{Q} = \{\mathbf{T}S(a), \mathbf{T}S(b)\}$

Suppose we want select the models minimal for $\mathbf{P}, \mathbf{Z}$

$T_2 <_{\mathbf{P},\mathbf{Z}} T_1$?

No!

Expanding T_2 to

$$T_2' = \{\mathbf{T}P(a), \mathbf{T}S(a), \mathbf{T}S(b)\} \text{ and}$$
$$T_2'' = \{\mathbf{T}P(a), \mathbf{T}S(a), \mathbf{T}\neg S(b)\}$$

yields an "equivalent" tableau
with comparable branches

A method based on hyperresolution

[Niemela96]

Clauses:

$$K: \; a_1 \wedge \ldots \wedge a_m \rightarrow b_1 \vee \ldots \vee b_n$$

(we allow $m, n = 0$.)

$$Head(K) = \{b_1, \ldots, b_n\}$$
$$Body(K) = \{a_1, \ldots, a_m\}$$

Rules of Branch Expansion

$$\frac{\begin{array}{c} a_1 \wedge \ldots \wedge a_m \rightarrow b_1 \vee \ldots \vee b_n \\ a_1, \ldots, a_m \\ \neg b_1, \ldots, \neg b_{j-1}, \neg b_{j+1}, \ldots, \neg b_n \end{array}}{b_j}$$

$$\frac{\begin{array}{c} a_1 \wedge \ldots \wedge a_m \rightarrow b_1 \vee \ldots \vee b_n \\ a_1, \ldots, a_m \end{array}}{\neg b_j \mid b_j}$$

A branch B is *finished* if, whenever $a_1 \wedge \ldots \wedge a_m \rightarrow b_1 \vee \ldots \vee b_n \in B$, and $a_1, \ldots a_m \in B$, then some $b_j \in B$.

Property of minimal models

Let A be a set of clauses, M be an interpretation and

$$N_A(M) = \{\neg a \mid \exists K \in A, a \in Head(K), M \not\models a\}.$$

> M is a minimal model of A iff $M \models A$ and for every atom a, $M \models a$ implies $A \cup N_A(M) \models a$.

We add a closure condition on branches B

$$N_A(B) = \{\neg a \mid \exists K \in A, a \in Head(K), a \notin B\}.$$

A branches B is **ungrounded wrt. A** iff there is an atom $a \in B$ such that $A \cup N_A(B) \not\models a$.

B is **MM-closed** wrt. A iff it is (ordinary) closed or ungrounded wrt. A.

A tableau T for (A, ϕ) is **MM-closed** iff every branch if MM-closed wrt. A

Theorem

$A \models_m \phi$ iff every finished tableaux for (A, ϕ) is MM-closed.

Remarks

- No comparison of branches is required (polynomial space complexity); every minimal model is generated only once.
- Call to classical (un)provability checks for testing ungroundness.
- Conjecture: the method can be adapted to deal with more general forms of minimal entailment.
- There are other uses of hyperresolution for minimal models generation: see [Bry and Yahya96]

Calculus MLK for minimal entailment [Olivetti92]

(for the case of minimization of every atom.)

Sequents have the form

$$\Gamma \vdash \Delta,$$

where Γ and Δ are sets of formulas.

- **Logical rules** (an upside version of tableaux rules: if a formula A occurs in the left (right) part, then it is treated as $\mathbf{T}A$ ($\mathbf{F}A$).

$$\frac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A_1, A_2}{\Gamma \vdash \Delta, A}$$

if A is of type α and A_1, A_2 are the α-subformulas of A.

$$\frac{\Gamma, B_1, \vdash \Delta \quad \Gamma, B_2, \vdash \Delta}{\Gamma, B \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B_1 \quad \Gamma \vdash \Delta, B_2}{\Gamma \vdash \Delta, B}$$

if B is of type β and B_1, B_2 are the β-subformulas of A.

- **Structural rules**

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

$$\frac{\Gamma \vdash A, \Delta_1 \quad \Gamma, A \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}$$

- **Initial sequents**

$$\Gamma \vdash \Delta,$$

provided either

(a) $\Gamma \cap \Delta \neq \emptyset$, or for some A, both $A, \neg A \in \Gamma$, or $A, \neg A \in \Delta$, or

(b) for some literal $\neg p \in \Delta$, $p \notin \Gamma$.

We show that $p \vee q \models_m \neg p \vee \neg q$, by a derivation in MLK:

$$\dfrac{\dfrac{p \vdash \neg p, \neg q}{p \vdash \neg p \vee \neg q} \quad \dfrac{q \vdash \neg p, \neg q}{q \vdash \neg p \vee \neg q}}{p \vee q \vdash \neg p \vee \neg q}$$

We show that $p \rightarrow q \models_m \neg q$, by the following derivation:

$$\dfrac{\dfrac{\vdash \neg p, q}{\vdash p \rightarrow q} \quad \vdash \neg q}{p \rightarrow q \vdash \neg q}$$

The last step of the derivation above is by CM rule.

Theorem Let Δ be a set of formulas, and A a formula we have that

$$\Delta \models_m A \;\Leftrightarrow\; \Delta \vdash A \text{ is derivable in MLK.}$$

Remark In MLK, cut is not eliminatable.

Proposition The following are equivalent:

1. No tableaux proof of $\Gamma \models_m A$ requires branch comparison.
2. There is a MLK derivation of $\Gamma \vdash A$, which does not make use neither of CM rule, nor of CUT rule.

Tableaux for Autoepistemic Logic
[Niemela88]

In order to check if ψ is in some (all) stable expansions of a given set of premises A, we start a tableau for (A, ψ).

In the tableau construction formulas of the kind $L\phi$ may occur.

To check $L\phi$ we start a new tableau for (A, ϕ). If there are other formulas of the form $L\chi$, we iterate the process and we generate several tableaux.

Basic idea:

- $L\phi$ must be put in a stable expansion E of A, if ϕ is provable from E, and
- $\neg L\phi$ must be put in E if ϕ is not provable from E.

The *(un-)provability* of one formula may depend on the *(un-)provability* of other formulas.

$$\Downarrow$$

We must make **arbitrary but consistent** decisions about *provability* of every formula ϕ such that $L\phi$ (or $\neg L\phi$) occurs in a given tableau:

$$\Downarrow$$

Having declared ϕ "provable" ("non-provable"), the addition of $L\phi$ (respectively $\neg L\phi$) to the tableau does not change the provability status of the goal formula of any tableau involved in the construction.

Each **consistent decision** about provability exactly determines one **stable expansion**.

Algorithm for checking $A \vdash_{ske} \phi$ and $A \vdash_{cre} \phi$

We build up the following tableaux-structure

$$(A, \phi, T_0, X),$$

where T_0 is a tableau for (A, ϕ), and X is the least set satisfying the following conditions:

- $T_0 \in X$;
- if $\mathbf{T}L\psi$ or $\mathbf{F}L\psi$ occurs in an *open* branch of some tableau in X, then X contains a tableau for (A, ψ).

A **labelling** l for (A, ϕ, T_0, X) is a function which assigns a label $OPEN$ or $CLOSED$ to each tableau in X.

1. Build up the structure (A, ϕ, T_0, X), first completing T_0, and then completing every tableau in X.

2. For each labeling l and tableau $T \in X$, we let the tableau $upd(l, T, X)$ be obtained from T by adding

   - (a) $\mathbf{T}L\psi$ to every branch of T if there is a tableau $T' \in X$ for (A, ψ) such that $l(T') = CLOSED$, and
   - (b) $\mathbf{F}L\psi$ to every branch of T if there is a tableau $T' \in X$ for (A, ψ) such that $l(T') = OPEN$.

3. for each labeling l we check whether l is **admissible**. We say that l is admissible for (A, ϕ, T_0, X) whenever, for every $T \in X$, it holds that:

   - if $l(T) = OPEN$ then $upd(l, T, X)$ contains an open branch;
   - if $l(T) = CLOSED$ then all branches of $upd(l, T, X)$ are closed.

4. • **Output** $A \vdash_{ske} \phi$ iff in all admissible labellings for (A, ϕ, T_0, X), T_0 is labelled $CLOSED$;

   • **Output** $A \vdash_{cre} \phi$ if there is an admissible labeling for (A, ϕ, T_0, X) in which T_0 is labeled $CLOSED$.

In particular, $A \vdash_{ske} \phi$ if there are no admissible labellings.

Soundness and Completeness

- If S is a stable expansion of A, then there is an admissible labeling l of (A, ϕ, T_0, X), such that

$$\phi \in S \quad \text{iff} \quad l(T_0) = CLOSED.$$

- Let l be an admissible labeling of (A, ϕ, T_0, X), then there is a stable expansion S of A such that for every tableau $T \in X$ for (A, ψ),

$$\psi \in S \quad \text{iff} \quad l(T) = CLOSED.$$

Example Let $A = \neg Lp \rightarrow \neg p$.

We show that $A \vdash_{ske} \neg p$.

Let T_0 be a tableau for $(A, \neg p)$.

$$T_0 = \quad \{\{\mathbf{T}Lp, \mathbf{T}p\},\ \{\mathbf{F}p, \mathbf{T}p\}\}.$$

Let T_1 be a tableau for (A, p).

$$T_1 = \quad \{\{\mathbf{T}Lp, \mathbf{F}p\},\ \{\mathbf{F}p\}\}.$$

$X = \{T_0, T_1\}$.

Let $l_1(T_1) = OPEN$, we add $\mathbf{F}Lp$,

- T_1 remains **open**,

- T_0 becames **closed**

if $l_1(T_0) = CLOSED$, then l_1 is **admissible**, otherwise it is not.

Let $l_2(T_1) = CLOSED$, we add $\mathbf{T}Lp$

But T_1 remains **open**, thus l_2 is **not admissible**.

The only admissible labelling is l_1,

$l_1(T_1) = OPEN$ and $l_1(T_0) = CLOSED$. and T_0 is **closed** in l_1.

Example Let $A = \{\neg Lp \to q, \neg Lq \to p\}$. We show that $A \vdash_{ske} p \vee q$.

Let T_0 be the tableau for $(A, p \vee q)$. T_0:

$$
\begin{aligned}
B_{0,1} &= \{\mathbf{F}p, \mathbf{F}q, \mathbf{T}Lp, \mathbf{T}Lq\}, \\
B_{0,2} &= \{\mathbf{F}p, \mathbf{F}q, \mathbf{T}Lp, \mathbf{T}p\}, \\
B_{0,3} &= \{\mathbf{F}p, \mathbf{F}q, \mathbf{T}q, \mathbf{T}Lq\}, \\
B_{0,4} &= \{\mathbf{F}p, \mathbf{F}q, \mathbf{T}q, \mathbf{T}p\}.
\end{aligned}
$$

$B_{0,2}$, $B_{0,3}$, and $B_{0,4}$ are **closed**. T_0 is **open**.

Let T_1 be a tableau for (A, p):

$$
\begin{aligned}
B_{1,1} &= \{\mathbf{F}p, \mathbf{T}Lp, \mathbf{T}Lq\}, \\
B_{1,2} &= \{\mathbf{F}p, \mathbf{T}Lp, \mathbf{T}p\}, \\
B_{1,3} &= \{\mathbf{F}p, \mathbf{T}q, \mathbf{T}Lq\}, \\
B_{1,4} &= \{\mathbf{F}p, \mathbf{T}q, \mathbf{T}p\},
\end{aligned}
$$

where $B_{1,2}$ and $B_{1,4}$ are **closed**. T_1 is **open**.

T_2 be a tableau for (A, q):

$$
\begin{aligned}
B_{2,1} &= \{\mathbf{F}q, \mathbf{T}Lp, \mathbf{T}Lq\}, \\
B_{2,2} &= \{\mathbf{F}q, \mathbf{T}Lp, \mathbf{T}p\}, \\
B_{2,3} &= \{\mathbf{F}q, \mathbf{T}q, \mathbf{T}Lq\}, \\
B_{2,4} &= \{\mathbf{F}q, \mathbf{T}q, \mathbf{T}p\},
\end{aligned}
$$

$B_{2,3}$ and $B_{2,4}$ are **closed**. T_2 is **open**.

The only admissible labelings are l_1 and l_2 with

- $l_i(T_0) = CLOSED$, for $i = 1, 2$,
- $l_1(T_1) = CLOSED,\ l_1(T_2) = OPEN$,
- $L_2(T_1) = OPEN, l_2(T_2) = CLOSED$.

Since T_0 is $CLOSED$ under both of them, we can conclude that $A \vdash_{ske} p \vee q$.

Example

Let $A = \neg Lp \to p$.

Let T_0 be a tableau for A:

$$T_0 = \{\{\mathbf{T}Lp\}, \{\mathbf{T}p\}\},$$

Let T_1 be a tableau for (A, p):

$$T_1 = \{\{\mathbf{T}Lp, \mathbf{F}p\}, \{\mathbf{T}p, \mathbf{F}p\}\}.$$

- Suppose that $l(T_1) = CLOSED$, then we add $\mathbf{T}Lp$ and T_1 remains open.
- Suppose that $l(T_1) = OPEN$, then we add $\mathbf{F}Lp$ and T_1 becomes closed.

Thus, there are not admissible labellings.

Finitary characterization of Expansions

Notion of **Full Set**

Let Σ be a finite set of formulas,

$Sf^L(\Sigma) = \{L\phi \in Sub(\Sigma)\}$

$\neg Sf^L(\Sigma) = \{\neg L\phi \mid L\phi \in Sf^L(\Sigma)\}$

$B \subseteq Sf^L(\Sigma) \cup \neg Sf^L(\Sigma)$ is a **full set** for Σ, if for every $L\phi \in Sf^L(\Sigma)$

- $L\phi \in B \;\Leftrightarrow\; \Sigma \cup B \vdash_{AE} \phi$;
- $\neg L\phi \in B \;\Leftrightarrow\; \Sigma \cup B \not\vdash_{AE} \not\phi$;

$\vdash_{AE}$ is the extension of propositional calculus in which L-formulas are treated as propositional atoms)

Full Sets and Stable expansions

Define (recursively)

$$\Sigma \vdash_{AEL} \phi \ \Leftrightarrow \Sigma \cup SB_{\Sigma}(\phi) \vdash_{AE} \phi$$

where

$$\begin{array}{rcl} SB_{\Sigma}(\phi) & = & \{L\chi \in S^{IL}(\phi) \mid \Sigma \vdash_{AEL} \chi\} \\ & \cup & \{\neg L\chi \mid \chi \in S^{IL}(\phi) \ \wedge \ \Sigma \not\vdash_{AEL} \chi\} \end{array}$$

where $S^{IL}(\phi)$ is the set of immediate L-subformulas of ϕ.

$\Sigma \vdash_{AEL} \psi$ is decidable (for instance, by using tableaux)

Proposition There is a bijection between *full sets* and *stable expansions*:

$$E_{\Sigma}(B) = \{\psi \mid \Sigma \cup B \vdash_{AEL} \psi\}$$

B is a full set, $E_{\Sigma}(B)$ is a stable expansion.

Connection with tableaux

Let l be an admissible labeling of (Σ, ϕ, T_0, X), where T_0 if for $(\Sigma, true)$,

for each $\psi \in Sf^L(\Sigma)$ there is a tableau $T_\psi \in X$

Define

$B^+ = \{L\psi \mid l(T_\psi) = CLOSED\}$

$B^- = \{\neg L\phi \mid l(T_\phi) = OPEN\}$

$B = B^+ \cup B^-$,

B is a *full set* and every full set can be obtain in this way.

A new decision method[Niemela95]

Based on a more efficient way of checking **full set** existence.

We say that B *covers* Σ iff for every$\psi \in Sf^L(\Sigma)$, either $L\psi \in B$ or $\neg L\psi \in B$.

In the procedure $FULL\_FIND(\Sigma, B, F)$

- B represent the temptative full set being created;
- F represent the set of $L\psi$ yet "uncovered" by B

The initial call is $FULL\_FIND(\Sigma, \emptyset, \emptyset)$ and it returns either a full set B of Σ or **false** if it does not exist.

$EXTEND(\Sigma, B, F)$

repeat $B' := B$

for all $\phi \in Sf^L(\Sigma)$

$B := B \cup \{L\phi\}$ if $\Sigma \cup B \cup F \vdash_{AE} \phi$

until $B = B'$

return B

$FULL\_FIND(\Sigma, B, F)$

$B := EXTEND(\Sigma, B, F)$

if B covers Σ **then** return **true**,

else

if $Sf^L(\Sigma) \subseteq Sf^L(B) \cup F$

then return **false**;

else let $L\chi$ such that $L\chi \notin B$, nor $\neg L\chi \notin B$

if $FULL\_FIND(\Sigma, B' \cup \{\neg L\chi\}, F) =$
false

then
return $FULL\_FIND(\Sigma, B', F \cup \{L\chi\})$

else return **true**

The algortihm can be adapted to obtain a decision method for both skeptical and credulous consequence:

Instead of returning **true**, we return the outcome of a derivability test with respect to the full sets:

- $\Sigma \vdash_{cre} \phi \;\Leftrightarrow\; \exists \text{B full set} : \; \Sigma \cup B \vdash_{AEL} \phi$;
- $\Sigma \vdash_{ske} \phi \;\Leftrightarrow\; \neg\exists \text{B full set} : \; \Sigma \cup B \not\vdash_{AEL} \phi$;

Alternative Approach

[Donini, Massacci et al.96]

A uniform tableaux method for nomonotonic modal logics based on their modal preferential semantics.

- It is parametric wrt. the underlying modal logic;
- It is parametric wrt. the preference criteria on Kripke structures

It can capture Autoepistemic logic, the Logic of Minimal Knowledge [Halpern and Moses 85], and default logic, via its modal translation.

Proof Systems for Default Logic

- Normal defaults (Reiter 80)

- Normal defaults without prerequisites (Besnard et al. 83)

- TMS Based Approach (Junker Brewka 91, Levy 91, Junker Konolige 90)

- Tableaux Based Approach for Normal Defaults (Schwind 90)

- Tableaux Based Approach "global" (Risch and Schwind 94)

- Tableaux Rule Approach "incremental" Approach (Amati et al 96)

Definitions

Generating defaults of an extension E of a default theory without free variables $\Delta = (W, D)$

$GD(E, D) =$
$\{d : d \in D, d = \dfrac{\alpha : \{\beta_1, \beta_2, \ldots, \beta_n\}}{\gamma}, \alpha \in E$ and $\neg\beta \notin E\}$

Consequents of a set of defaults D

$$CONS(D) = \{\gamma : \frac{\alpha : \{\beta_1, \beta_2, \ldots, \beta_n\}}{\gamma} \in D\}$$

Justifications of a default
$d = \dfrac{\alpha : \{\beta_1, \beta_2, \ldots, \beta_n\}}{\gamma}$

$JUST(d) = \{\beta_1, \beta_2, ...\beta_n\}$
and the set of justifications of a set of defaults is the union of all the justification sets.

For a set of formulae, F $\neg F = \{\neg f : f \in F\}$

Default proofs for Normal Theories

Default proof for F from $\Delta = (W, D)$
Finite sequence D_0, D_1, $\ldots D_k$ of subsets of D such that

1. $W \cup CONS(D_0) \vdash F$
2. For $1 \leq i \leq k, W \cup CONS(D_i) \vdash PREREQ(D_{i-1})$
3. $D_k = \emptyset$
4. $W \cup \bigcup_{i=0}^{k} CONS(D_i)$

Extension membership problem is not semi-decidable

"incremental" approach

Characterizing Extensions

Theorem (Reiter80):

If E is an extension for a default theory without free variables $\Delta = (W, D)$, then

$$E \;=\; Th(W \;\cup\; CONS(GD(E,\;\Delta)))$$

? inverse?

Groundedness

A set of defaults without free variables is grounded in W iff for all $d \in D$ there are d_0, d_1,...$d_n \in D$

$$d_i = \frac{\alpha_i : \beta_1^i \cdots \beta_{n_i}^i}{\gamma_i} \text{ such that}$$

$$\alpha_0 \in Th(W)$$

$\alpha_{i+1} \in Th(W \cup CONS(\{d_0, d_1, ...d_i\}))$ for $0 \leq i \leq n-1$ and
$d_n = d$.

Complete Characterization of Extensions

R-Extensions

A default theory $\Delta = (W, D)$
has an extension, E
iff there exists $D' \subseteq D$, D' grounded in W and
$E = Th(W \cup CONS(D'))$ and $\forall\, d = \dfrac{\alpha : \beta}{\gamma}$

(i) if $d \in D'$ then $\alpha \in Th(W \cup CONS(D'))$ and $\neg\beta_i \notin Th(W \cup CONS(D'))$, for all i such that $1 \leq i \leq n$;

(ii) if $d \notin D'$ then $\alpha \notin Th(W \cup CONS(D'))$ or $\neg\beta_i \in Th(W \cup CONS(D'))$ for some i such that $1 \leq i \leq n$.

Justified Default Logic

j-Extensions

Let $\Delta = (W, D)$ be a default theory.
Let D' be any subset of D.

$E = Th(W \cup CONS(D')$ is a j-extension of Δ with respect to $F = Th(W \cup JUST(D') \cup CONS(D'))$ iff D' is a maximal grounded subset of D, such that for all $\beta \in JUST(D')$, $\neg\beta \notin E$.

Constrained Default Logic

c-Extensions

Let $\Delta = (W, D)$ be a default theory.
Let D' be any subset of D.

$E = Th(W \cup CONS(D')$ is a c-extension of Δ
with respect to
$C = Th(W \cup JUST(D') \cup CONS(D'))$ iff
D' is a maximal grounded subset of D',
such that $E \cup JUST(D')$ is consistent.

Computing extensions

Idea:

Extension E of a default theory $\Delta = (W, D)$
Consequences of $W \cup CONS(D')$
where for D' (i) and (ii) hold

Maximal set Find all subsets D' of D
corresponding to extensions

Algorithm

Starting with $W \cup CONS(D)$

1. Maximal consistent subsets of $W \cup CONS(D)$ containing W

2. Justification conditions

3. Grounding $Th(W \cup CONS(D_i^j))$

4. Minimization

The first step is a special case of the second step.

Algorithm

sets D_1, ...,D_n of D, such that

$W \cup CONS(D_i)$ is maximal consistent testing candidate $Th(W \cup CONS(D_i))$

Maximal subsets D_i^j of D_i such that

$Th(W \cup CONS(D_i^j)) \wedge \neg JUST(D_i^j)) = \emptyset$.

Eliminate within D_i^j defaults which are not grounded by verifying for every default

$d = \dfrac{\alpha : \{\beta_1, \beta_2, \ldots, \beta_n\}}{\gamma}$ of D_i^j such that

$\alpha \in Th(W \cup CONS(D_i^j \setminus \{d\}))$

if $\alpha \in Th(W)$, then $\{d\}$ is grounded.

If $\alpha \in CONS(D_i^j \setminus \{d\})$ then $D_i^j \setminus \{d\}$ has to be grounded.

Tableaux System

TP as a mapping producing sets of literals from sets of formulas as follows:

$$
\begin{aligned}
TP(F) = \;& \{F\} \text{ if } F \text{ is a set of literals} \\
TP(F) = \;& TP(F' \cup \{f\} \\
& \quad \text{if } \neg\neg f \in F \text{ and } F' = F \setminus \{\neg\neg f\} \\
TP(F) = \;& TP(F' \cup \{\alpha_1\} \cup \{\alpha_2\}) \\
& \quad \text{if } \alpha \in F \text{ and } F' = F \setminus \{\alpha\} \\
TP(F) = \;& TP(F' \cup \{\beta_1\}) \cup TP(F' \cup \{\beta_2\}) \\
& \quad \text{if } \beta \in F \text{ and } F' = F \setminus \{\beta\}
\end{aligned}
$$

Fundamental Property

Set of literals is *closed* if it contains ℓ and $\neg\ell$
Set of literals is *open* otherwise
Set of sets of literals is closed if each of its elements is closed
and open otherwise

Completeness

f is a theorem iff $TP(\{\neg f\})$ is closed

Properties and Operations

$TP(\{f \wedge g\}) =$
$\{X \cup Y : X \in TP(\{f\}) and Y \in TP(\{g\})\}$
==> Consider conjunction as an operation upon sets of sets of literals

Notation:

$f \otimes g = \{X \cup Y : X \in TP(\{f\})$ and $Y \in TP(\{g\})\}$

Thus $TP(\{f \wedge g\}) = TP(\{f\}) \otimes TP(\{g\})$

$TS(M) = TP(M) \setminus \{X : X \in TP(M)$ and X closed or $\exists Y \in TP(M)$ such that $Y \subseteq X$ and $X \neq Y\}$

Why Analytic Tableaux

Given a default $d = \frac{\alpha : \beta}{\gamma}$, using the fundamental property of TP one obtains for R-extensions:

(i) If $d \in D'$ then
$TP(\{\neg\alpha\}) \otimes TS(W) \otimes TP(CONS(D'))$ is closed
and $TP(\{\beta\} \otimes TS(W) \otimes TP(CONS(D'))$ is open

(ii) If $d \notin D'$ then
$TP(\{\neg\alpha\} \otimes TS(W) \otimes TP(CONS(D'))$ is open or
$TP(\{\beta\} \otimes TS(W) \otimes TP(CONS(D'))$ is closed

Groundedness testing

Choose $d \in D'$, $d = \dfrac{\alpha : \beta}{\gamma}$. If $TP(W \otimes CONS(D' \setminus \{d\})) \otimes \{\neg\alpha\})$ is closed, a finite sequence $d_0, \ldots, d_n$ of elements of D' has to be found, $d_i = \dfrac{\alpha_i : \beta_i}{\gamma_i}$, such that $TP(\{\neg\alpha_0\}) \otimes TP(W)$ is closed, $TP(\{\neg\alpha_{i+1}\}) \otimes TP(W) \otimes TP(CONS(\{d_0, \ldots, d_i\})$ is closed for $0 \leq i \leq n-1$ and $d_n = d$. However, if there exist k, $0 \leq k \leq n$, such that $TP(\{\neg\alpha_k\}) \otimes TP(W) \otimes TP(CONS(D' \setminus \{d_k, \ldots, d_n\}))$ is open i.e. $\alpha_k \notin Th(W \cup CONS(D' \setminus \{d_k, \ldots, d_n\}))$ the sequence $d_k, \ldots, d_n$ has to be withdrawn from D'. In this way all grounded extensions are obtained.

Example

$\Delta = (W, D)$ where $W = \{A, F\}$ and

$D = \{\frac{A : B \wedge \neg C}{B}, \frac{F : C}{C}\}$

$TS(W) = \{\{A, F\}\}$,

$TP(CONS(D)) = \{\{B_1, C_2\}\}$

(B_1 and B_2 are the consequents of the first and the second default respectively)

D possible candidate for an extension

BUT justification condition of *(i)* is not true for the first default

D has one extension:

$$E_2 = Th(W_1 \cup CONS(\{\frac{F : C}{C}\})).$$

Example

$\Delta = (W, D)$ where
$W = \{P, S\}, D = \{\frac{: Q \wedge S}{P}, \frac{: Q \rightarrow \neg S}{\neg R}\}$.
$TS(W) = \{\{P, S\}\}$,
$TP(CONS(D)) = \{\{P_1, \neg R_2\}\}$. First consider D as the possible unique candidate in order to get an extension. The justification condition of *(i)* holds for both of the defaults:

- $TS(W) \otimes TP(CONS(D) \otimes TP(\{Q_1 \wedge S_1\}) = \{\{P, S, P_1, \neg R_2, Q_1, S_1\}\}$ is open.
- $TS(W) \otimes TP(CONS(D) \otimes TP(\{Q_2 \rightarrow \neg S_2\})$
  $=$
  $\{\{P, S, P_1, \neg R_2, \neg Q_2\}, \{P, S, P_1, \neg R_2, \neg S_2\}\}$
  is open;

Since D is obviously grounded, Δ has a single extension

$$E = Th(W \cup CONS(\{\frac{: Q \wedge \neg S}{P}, \frac{: Q \rightarrow \neg S}{\neg R}\})).$$

Example

$\Delta = (\emptyset, D)$ where $D = \{\frac{A : B}{B}, \frac{B : A}{A}\}$. The justification condition holds for both defaults of D. However, D is not grounded, neither is $\{\frac{A : B}{B}\}$ or $\{\frac{B : A}{A}\}$. Hence D has only one extension $E = \emptyset$.

Algorithm

COMPUTE-EXTENSIONS(W, D):

If W is not consistent

then PRINT(W is not consistent)

else For any maximal subset D' in D such that

$TP(W \otimes CONS(D'))$ is open do

$D_{just} := \emptyset$; $D_{grnd} := \emptyset$; $SD_{just} := \{D_{just}\}$;

JUSTIFICATIONS(D', D_{just}, SD_{just});

For any D_{just} in SD_{just} do

GROUNDING(D_{just}, $\emptyset$, D_{grnd});

EXTENSIONS(D_{just}, D_{grnd})

Justification Conditions

JUSTIFICATIONS(D', D_{just}, SD_{just}):

For any default $d = \dfrac{\alpha : \beta}{\gamma} \in D'$ do

If $TP(\{\beta\}) \otimes \Gamma(D')$ is open

then Add d to D_{just}

else For any maximal subset D'' in D' such that $TP(\{\beta\}) \otimes \Gamma(D'')$ is open, do JUSTIFICATIONS(D'', D_{just}, SD_{just});

If ($D_{just} \notin SD_{just}$) and (there is no $D'_{just} \in SD_{just}$ such that $D_{just} \subseteq D'_{just}$)

then Add D_{just} to SD_{just}

Grounding

GROUNDING(D', S, D_{grnd}):

For any default $d = \dfrac{\alpha : \beta}{\gamma} \in D' \setminus D_{grnd}$ do

$\Theta := Th(W \otimes CONS(D' \setminus (S \cup \{d\})));$

If $TP(\neg\alpha) \otimes \Theta$ is closed,

then If $\alpha \in Th(W)$ or comes from a grounded set of defaults

then Add d to D_{grnd}

else Let be D_Θ the set of defaults of $D' \setminus (S \cup \{d\})$

whose the consequent is used for deriving α, do

GROUNDING(D_Θ, $S \cup \{d\}$);

If D_Θ is grounded then Add d to D_{grnd};

else Remove the list of defaults $S \cup \{d\}$ from D'.

EXTENSIONS(D_{just}, D_{grnd}):

If D_{grnd} is a maximal grounded subset of D then

PRINT($Th(W \cup CONS(D_{grnd}))$ is a j-extension with respect to $JUST(D_{grnd})$)

Justified Default Logic

Let $\Delta = (W, D)$ be a default theory.
Let D' be any subset of D.

$E = Th(W \cup CONS(D')$ is a j-extension of Δ with respect to $F = Th(W \cup JUST(D') \cup CONS(D'))$ iff D' is a maximal grounded subset of D, such that for all $\beta \in JUST(D')$, $\neg\beta \notin E$.

Theorem 3 *A default theory has always a j-extension.*

Constrained Default Logic

Let $\Delta = (W, D)$ be a default theory.
Let D' be any subset of D.

$E = Th(W \cup CONS(D')$ is a c-extension of Δ
with respect to
$C = Th(W \cup JUST(D') \cup CONS(D'))$ iff
D' is a maximal grounded subset of D',
such that $E \cup JUST(D')$ is consistent.

Other approach (Amati et al.)

- incremental construction
- default rules as tableau rules

Default tableau restriction rule

For any default: $\dfrac{\alpha : \beta}{\gamma}$

there is a corresponding tableau rule:

$$\frac{S \vdash \alpha \quad S, \beta \nvdash \bot}{S \cup \{\gamma\}}$$

S is Δ-saturated if for all applicable default tableaux rules,

$$S \vdash \gamma$$

Sequence Construction

- (step 0) Let $T_0 = TP(\mathbf{T}W)$
- (step i+1). let $d = \dfrac{\alpha : \beta}{\gamma} \in D$, check whether

$T_i \otimes TP(\neg\alpha)$ is closed and
$T_i \otimes TP(\beta)$ is open.

and if so

$$T_{i+1} = T_i \otimes TP(\mathbf{T}\gamma)$$

if not try another $d \in D$

generate a finite sequence of tableaux
$T_0 \ldots T_n$

terminate with T_n, if
T_n is Δ-saturated, i.e.

for $d = \dfrac{\alpha : \beta}{\gamma}$, if d is applicable to T_n
then $T_n \otimes TP(\neg\gamma)$ is closed

Theorem

E is a consistent extension of $\Delta = (W, D)$ iff there is a finite sequence of tableaux $T_0 \ldots T_n$ such that

$T_0 = TP(W)$
T_n is Δ-saturated and satisfioes the

stability condition and

$E = Th(\{\phi \mid \phi \in T_n\} \cup \{\neg\psi \mid \neg\psi \in T_n\})$

Stability condition

For every $d = \frac{\alpha : \beta}{\gamma}$ used during the construction,

$TP(\beta) \otimes T_n$ is open

Theory Revision

Knowledge base K

Incorporate new information ϕ

$K + \phi = Th(K \cup \{\phi\})$

$K o \phi$ K revised by ϕ

Postulates for theory change

Alchurron, Gärdenfors, Makinson

Katsuno Mendelson

Theory Revision

$Mod(\phi 0 \psi) \subset Mod(\psi)$

Comparing distances

Tableaux allow to calculate and to compare distances
between models and model sets.

Model theoretic characterization of minimal change

set of all interpretations

To each propositional formula ϕ
assign a pre-order (partial order)
$\leq_\phi \subset$ I $\times$I

$I \leq_\phi J$ "I is bf closer to ϕ than J

The assignment is **faithful** if

1. if $I, I' \in Mod(\phi)$ then not $I \leq_\phi I'$
2. if $I, I' \in Mod(\phi)$ and $I' \notin Mod(\phi)$ then $I \leq_\phi I'$
3. if $\phi \leftrightarrow \psi$ then $\leq_\phi = \leq_\psi$

pre-order: Revision

partial order: Update

GENERAL FRAMEWORK

Specific revision and update operations have been defined by

Dala, Borgida, ⋮Winslett ⋮

Example for a specific distance notion:

M_1, M_2 sets of models

$M_1 - M_2 = \{m_1 \div m_2 \mid m_1 \in M_1, m_2 \in M_2$

Revision: $\circ$

$Mod(\phi \circ \psi)$
$= \{m \in Mod(\psi) \mid \exists m_0 \in Mod(\phi)$ and
$m_0 \div m \in Min\{M_1 - M_2, \subset\}\}$

Update: $\diamond$

$$Mod(\phi \diamond \psi) = \bigcup_{m \in Mod(\phi)} Update(m, \psi)$$

where $Update(m_0, \psi) = \{m \in Mod(\psi) \mid$
$m_0 \div m \in Min\{\{m_0\} - Mod(\psi)\}\}$

Revision with tableaux

$TP(\phi)$ $TP(\psi)$

$T_\phi - T_\psi = \{t - -t' \mid t \in TP(\phi)$ and $t' \in TH(\psi)\}$

where $t - -t' = \{l^+ \mid l \in t$ and $\neg l \in t'$ or $\neg l \in t$ and $l \in t'$

$t1 <_t t_2$ iff $t_1 - -t \subset t_2 - -t$

$Tab(\phi \circ \psi) = \{t \in TP(\psi) \mid \exists t_0 \in TP(\phi)$ such that $t_0 - -t \in Min\{T_\phi - T_\psi\}, \subset\}$