On the Validity of the Two Raster Sequences Distance Transform Algorithm

Édouard THIEL

Laboratoire Informatique et Systèmes (LIS)
Université d'Aix-Marseille (AMU)

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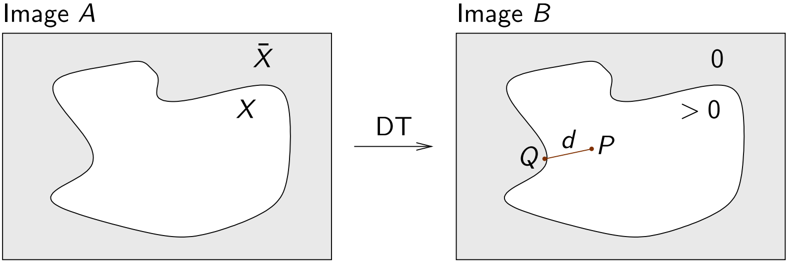
The slides and Python sources are available here:
https://pageperso.lis-lab.fr/~edouard.thiel/DGMM2022/
Short link: https://bit.ly/et-dgmm22

Summary

1 - Introduction



X = set of shape points, $\bar{X}$ = background points
$d : E \longrightarrow F$ a distance, e.g., $E = \mathbb{Z}^n$, $F = \mathbb{Z}_+$

Distance Transform:

$$\mathrm{DT} \left(\begin{array}{ccc} E & \longrightarrow & F \\ P & \longmapsto & d(P, \bar{X}) = \inf\{\, d(P, Q) \,:\, Q \in \bar{X} \,\} \end{array} \right.$$

Founding paper

> Rosenfeld, A., Pfaltz, J.: Sequential operations in digital picture processing. Journal of ACM **13**(4), 471–494 (1966) [pdf]

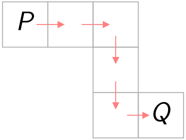
Several important contributions:

- Notion of distance transform;
- Definition of path-based distances d_4 and d_8;
- A raster sequences DT algorithm (RSDT) in two scans for these distances;
- A constructive proof that, for any given local transformation on an image, the sequential and parallel approaches are mathematically equivalent.
- ...

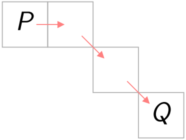
Path-based distances d_4 and d_8

(introduced as d and d^* in the original paper)

The distance is the length of a shortest path using 4 or 8 neighbours:



$d_4(P, Q) = 5$



$d_8(P, Q) = 3$

They correspond to the Minkowski distances:

$$d_4(P, Q) = d_1(P, Q) = |x_Q - x_P| + |y_Q - y_P|$$

$$d_8(P, Q) = d_\infty(P, Q) = \max\left(|x_Q - x_P|,\ |y_Q - y_P|\right)$$

Naive parallel DT for d_4

Compute min's on 4-neighbourhood, $+1$:

– At step 0, let B^0 be a copy of A, where the foreground points are set to the special value $\mu = \infty$.
– For each step $k > 0$, compute the image $B^k = (b^k_{i,j})$, where

$$b^k_{i,j} = \min \left\{ b^{k-1}_{i+1,j} + 1,\ b^{k-1}_{i-1,j} + 1,\ b^{k-1}_{i,j+1} + 1,\ b^{k-1}_{i,j-1} + 1 \right\}$$

The process is repeated until no point value changes.

Same algorithm for d_8 using the 8-neighbourhood.

Example of parallel DT for d_4

a

1	1	1	1	1
1	1	0	1	1
1	1	1	1	1

b^0

μ	μ	μ	μ	μ
μ	μ	0	μ	μ
μ	μ	μ	μ	μ

b^1

μ	μ	1	μ	μ
μ	1	0	1	μ
μ	μ	1	μ	μ

b^2

μ	2	1	2	μ
2	1	0	1	2
μ	2	1	2	μ

b^3

3	2	1	2	3
2	1	0	1	2
3	2	1	2	3

b^4

3	2	1	2	3
2	1	0	1	2
3	2	1	2	3

Image a: $0 =$ background, $1 =$ shape
Images b^k: $\mu = \infty$

Remark: one extra step is needed to detect no changes.

The number of iterations is bounded by the largest distance
$\rightarrow$ inefficient on a sequential machine.

2 - Raster Sequences DT in two scans for d_4 and d_8

[Rosenfeld and Pfaltz, 1966]

Idea: on a sequential machine, increase the convergence rate by a clever choice of the order:

– the forward scan processes the image row by row in the raster sequence $a_{1,1}, \ldots, a_{1,n}$, $a_{2,1}, \ldots, a_{2,n}$, …, $a_{m,1}, \ldots, a_{m,n}$;
– the backward scan processes the points in the reverse order.

The proposed algorithm converges in only two scans, independently of the thickness of the shapes in the image.

Raster Sequences DT in two scans for d_4

Forward scan: f_1 is applied on A to obtain B

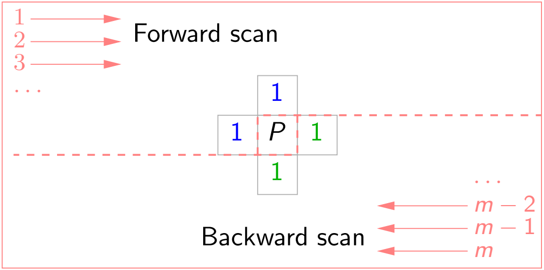
$$
\begin{aligned}
f_1 :\; b_{i,j} &= 0 && \text{if } a_{i,j} = 0 \\
&= \mu && \text{if } a_{i,j} = 1 \text{ and } (i,j) = (1,1) \\
&= \min\left(b_{i-1,j} + 1\,,\, b_{i,j-1} + 1\right) && \text{if } a_{i,j} = 1 \text{ and } (i,j) \neq (1,1)
\end{aligned}
$$

Backward scan: f_2 is applied on B to obtain C

$$
f_2 :\; c_{i,j} = \min\left(b_{i,j},\; c_{i+1,j} + 1\,,\, c_{i,j+1} + 1\right)
$$

Illustration of RSDT for d_4

$$
\begin{aligned}
f_1 :\ b_{i,j} &= 0 && \text{if } a_{i,j} = 0\\
&= \mu && \text{if } a_{i,j} = 1 \text{ and } (i,j) = (1,1)\\
&= \min\left(b_{i-1,j} + 1\,,\, b_{i,j-1} + 1\right) && \text{if } a_{i,j} = 1 \text{ and } (i,j) \neq (1,1)\\
f_2 :\ c_{i,j} &= \min\left(b_{i,j},\ c_{i+1,j} + 1\,,\, c_{i,j+1} + 1\right)
\end{aligned}
$$



Example:

1	1	1	1	1
1	1	0	1	1
1	1	1	1	1

$\xrightarrow{f_1}$

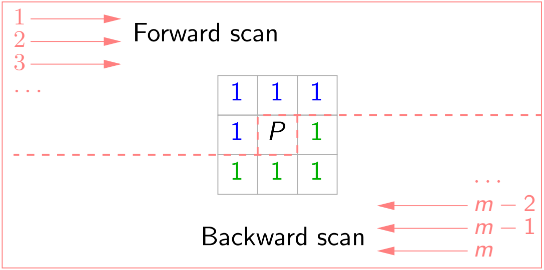
μ	μ	μ	μ	μ
μ	μ	0	1	2
μ	μ	1	2	3

$\xrightarrow{f_2}$

3	2	1	2	3
2	1	0	1	2
3	2	1	2	3

RSDT for d_8

Same algorithm for d_8: add indirect neighbours in the min's.



Example:

1	1	1	1	1
1	1	0	1	1
1	1	1	1	1

$\xrightarrow{f_1}$

μ	μ	μ	μ	μ
μ	μ	0	1	2
μ	1	1	1	2

$\xrightarrow{f_2}$

2	1	1	1	2
2	1	0	1	2
2	1	1	1	2

3 - Raster Sequences DT for weighted distances

Families of distances defined by "masks" of displacements and weights:

Montanari distances [Montanari, 1968]:
masks M_k of size $(2k+1)(2k+1)$, using visible points as displacements, and Euclidean lengths as weights;

Chamfer distances [Borgefors 1984], [Borgefors 1986]:
masks using integer weights.

$\sqrt{2}$	1	$\sqrt{2}$
1	O	1
$\sqrt{2}$	1	$\sqrt{2}$

	$\sqrt{5}$		$\sqrt{5}$	
$\sqrt{5}$	$\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{5}$
	1	O	1	
$\sqrt{5}$	$\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{5}$
	$\sqrt{5}$		$\sqrt{5}$	

4	3	4
3	O	3
4	3	4

	11		11	
11	7	5	7	11
	5	O	5	
11	7	5	7	11
	11		11	

Definitions: weighting and Chamfer Mask

Weighting $(\vec{v}, w)$:
a displacement $\vec{v} \neq \vec{0}$, associated to a weight $w > 0$.

Mask $\mathcal{M}$: a non-empty set of weightings.

Chamfer mask $\mathcal{M}$:

- reachability: the set of displacements contains at least a basis of the image points;
- central-symmetry: $\forall(\vec{v}, w) \in \mathcal{M}$, $(-\vec{v}, w) \in \mathcal{M}$.

Definitions: weighted distance

Points P and Q are $\mathcal{M}$-neighbours:
there exists $(\vec{v}, w) \in \mathcal{M}$ such that $\vec{PQ} = \vec{v}$.

$\mathcal{M}$-path $\mathcal{P}$ between points P and Q:
a sequence of distinct points $P_0 = P$, P_1, $\ldots$, $P_k = Q$
with P_i a $\mathcal{M}$-neighbour of P_{i-1}, $1 \leq i \leq k$.

Cost of the $\mathcal{M}$-path $\mathcal{P}$:
the sum of the weights of the displacements in $\mathcal{P}$.

Weighted distance $d_{\mathcal{M}}(P, Q)$:
cost of a path having minimal cost:

$$d_{\mathcal{M}}(P, Q) = \min \left\{ \sum \lambda_i w_i \ : \ \sum \lambda_i \vec{v}_i = \vec{PQ},\ (\vec{v}_i, w_i) \in \mathcal{M},\ \lambda_i \in \mathbb{Z}_+ \right\}$$

Metric property

Let $\mathcal{M}$ be a chamfer mask, then $d_{\mathcal{M}}$ is a metric:

– defined, positive, symmetric: from hypotheses of a chamfer mask;

– triangular inequality: because minimal paths can be concatenated.

Adaptation of the RSDT for weighted distances

$\mathcal{M}$ is split in two half-masks:

$\sqrt{2}$	1	$\sqrt{2}$
1	O	1
$\sqrt{2}$	1	$\sqrt{2}$

	$\sqrt{5}$		$\sqrt{5}$	
$\sqrt{5}$	$\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{5}$
	1	O	1	
$\sqrt{5}$	$\sqrt{2}$	1	$\sqrt{2}$	$\sqrt{5}$
	$\sqrt{5}$		$\sqrt{5}$	

4	3	4
3	O	3
4	3	4

	11		11	
11	7	5	7	11
	5	O	5	
11	7	5	7	11
	11		11	

Forward scan:

$$B_P = \min \{ B_{P+\vec{v}} + w \; : \; (\vec{v}, w) \in \mathcal{M}_{\text{forward}} \}$$

Backward scan:

$$B_P = \min \{ B_P \, ; \, B_{P+\vec{v}} + w \; : \; (\vec{v}, w) \in \mathcal{M}_{\text{backward}} \}$$

In the min's, value of external image points is $\mu = \infty$.

Convergence of the RSDT in two scans

In [Rosenfeld and Pfaltz, 1966], original proof of convergence of the RSDT in two scans, given for d_4 in $\mathbb{Z}^2$, by induction on the distances values $\leqslant k$ in a 4-neighbourhood:

- detailed in our paper;
- can be extended to d_1 and d_∞ in $\mathbb{Z}^n$;
- doesn't work for weights $\neq 1$, nor larger neighbourhoods.

In [Montanari 1968]: direct distance formulas using Farey series; proof of convergence in two iterations.

In [Borgefors 1984, 1986]: DT in $\mathbb{Z}^n$; but no proof of convergence.

Question: does the RSDT converge in two scans for any chamfer mask?

4 - Search of counter-examples

We have developed a small tool in Python language, available in the annex of our paper (licence CC-BY):

- chamfer2D.py: image class, weighted mask and distance transforms in 2D;
- showWDT.py: show DTs passes for a weighted mask;
- checkWDT.py: systematic comparisons of parallel vs raster sequences DTs with weighted masks

Short link: `https://bit.ly/et-dgmm22`

RSDT algorithms in $\mathbb{Z}^2$

```
def compute_sequential_DT_in_two_scans (img, half_mask) :
    compute_one_DT_scan (img, half_mask, 1)
    compute_one_DT_scan (img, half_mask, 2)

def compute_sequential_DT_multi_scans (img, half_mask) :
    scan_num = 1
    while True :
        if compute_one_DT_scan (img, half_mask, scan_num) :
            scan_num += 1
        else : break  # no change during this scan
    return scan_num
```

To check if the RSDT converges in two scan, we can compare the results with the parallel DT, or check if the multi scans version returns `scan_num` $\leqslant 3$.

Computation of one RSDT scan in $\mathbb{Z}^2$ (1/2)

```
def compute_one_DT_scan (img, half_mask, scan_num) :
    forward = scan_num % 2 == 1
    if forward :
        i_start = 0 ; i_end = img.m                     # 0 to m-1
        j_start = 0 ; j_end = img.n ; step = 1          # 0 to n-1
    else :
        i_start = img.m-1 ; i_end = -1                  # m-1 to 0
        j_start = img.n-1 ; j_end = -1 ; step = -1      # n-1 to 0
    changed = False
    for i in range (i_start, i_end, step) :
        for j in range (j_start, j_end, step) :
...
                img.mat[i][j] = min_w
    return changed
```

Computation of one RSDT scan in $\mathbb{Z}^2$ (2/2)

```
...
    changed = False
    for i in range (i_start, i_end, step) :
        for j in range (j_start, j_end, step) :
            if img.mat[i][j] == 0 : continue
            min_w = -1 if scan_num == 1 else img.mat[i][j]
            for p_i, p_j, p_w in half_mask.weightings :
                q_i = i - p_i*step ; q_j = j - p_j*step
                if not img.is_inside (q_i, q_j) : continue
                if img.mat[q_i][q_j] == -1 : continue
                q_w = img.mat[q_i][q_j] + p_w
                if min_w == -1 or q_w < min_w : min_w = q_w
            if img.mat[i][j] != min_w : changed = True
            img.mat[i][j] = min_w   # can be -1
    return changed
```

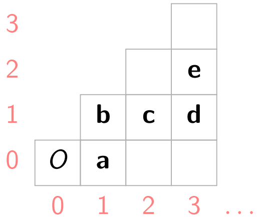
We use $\mu = -1$ in place of ∞ for non-propagated distances.

Search and mask notation

The search is limited in $\mathbb{Z}^2$ to grid-symmetrical masks.

The weightings are chosen in the first octant ($0 \leqslant i \leqslant j$), then the grid symmetries are performed to populate the mask.

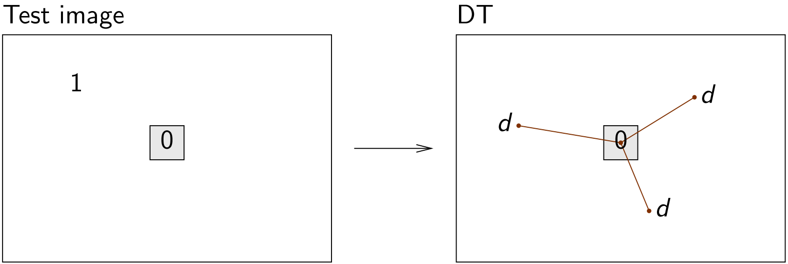
The displacements are chosen among the visible points (s.t. $\gcd(i, j) = 1$), named in the column-row order:



A grid-symmetrical mask constituted by a set of weightings $(\mathbf{v}, w)$ where $\mathbf{v}$ is a visible point is denoted by $\langle(\mathbf{v}, w), \ldots\rangle$:

Mask for d_4: $\langle(\mathbf{a}, 1)\rangle$
Mask for d_8: $\langle(\mathbf{a}, 1), (\mathbf{b}, 1)\rangle$
Mask for chamfer distance 5,7,11: $\langle(\mathbf{a}, 5), (\mathbf{b}, 7), (\mathbf{c}, 11)\rangle$

Test images



Images where all points have value 1 (shape points), except one point which has value 0 (background) in the centre.

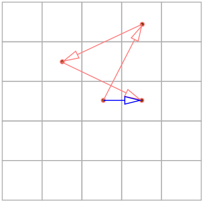
checkWDT.py usage

Given a set of displacements, e.g. $\langle \mathbf{a}, \mathbf{c} \rangle$, the program loops on several weights, and computes the DTs on test images of several sizes:

```
$ python3 checkWDT.py -i 3 5 -j 4 5 -w 1 4 -v ac
Parameters : m=[3..5[, n=[4..5[, w=[1..4[, names=ac,
  multi=False, show=False
Parallel vs sequential DT in two passes ...
Image 3 x 4  mask < a=1, c=1 >  equal: True  homog: True
Image 3 x 4  mask < a=1, c=2 >  equal: True  homog: True
Image 3 x 4  mask < a=1, c=3 >  equal: True  homog: True
Image 3 x 4  mask < a=2, c=1 >  equal: False homog: False
...
Image 3 x 4  mask < a=3, c=3 >  equal: True  homog: True
Image 4 x 4  mask < a=1, c=1 >  equal: True  homog: False
...
Image 4 x 4  mask < a=3, c=3 >  equal: True  homog: False
```


Interesting counter-example : $\langle(\mathbf{c}, 1)\rangle$

·	1	·	1	·
1	·	·	·	1
·	·	O	·	·
1	·	·	·	1
·	1	·	1	·



Reachability: $\langle(\mathbf{c}, 1)\rangle$ is a chamfer mask because the vector $(1, 0)$ can be obtained using the symmetrical displacements of $\mathbf{c}$.

Distance known as the Knight distance [Das and Chatterji, 1988].

The RSDT does not always converge in two scans for this mask.

Parallel DT for $\langle(\mathbf{c},1)\rangle$

(a)

1	1	1	1
1	1	0	1
1	1	1	1

(b)

μ	μ	μ	μ
μ	μ	0	μ
μ	μ	μ	μ

(c)

1	μ	μ	μ
μ	μ	0	μ
1	μ	μ	μ

(d)

1	2	μ	μ
μ	μ	0	μ
1	2	μ	μ

(e)

1	2	3	μ
μ	μ	0	3
1	2	3	μ

(f)

1	2	3	4
4	μ	0	3
1	2	3	4

(g)

1	2	3	4
4	5	0	3
1	2	3	4

(h)

1	2	3	4
4	5	0	3
1	2	3	4

(a) original image, (b) initialization, (c–h) passes 1–6.

RSDT for $\langle(\mathbf{c}, 1)\rangle$

(a)

1	1	1	1
1	1	0	1
1	1	1	1

(b)

μ	μ	μ	μ
μ	μ	0	μ
1	μ	μ	μ

(c)

1	2	μ	μ
μ	μ	0	μ
1	μ	μ	μ

(d)

1	2	μ	μ
μ	μ	0	3
1	2	3	μ

(e)

1	2	3	4
4	μ	0	3
1	2	3	μ

(f)

1	2	3	4
4	5	0	3
1	2	3	4

(g)

1	2	3	4
4	5	0	3
1	2	3	4

(a) original image, (b–g) passes 1–6, (b,d,f) forward passes, (c,e,g) backward passes.

5 - Validity holds for chamfer norms

A metric d in $\mathbb{Z}^n$ induces a discrete norm g defined by $g(q-p) = d(p,q)$ if g satisfies the property of homogeneity over $\mathbb{Z}$:

$$\forall \vec{x} \in \mathbb{Z}^n, \forall \lambda \in \mathbb{Z},\; g(\lambda \vec{x}) = |\lambda|\, g(\vec{x})\,.$$

A chamfer norm is a discrete norm induced by a chamfer mask.

Examples:

- Norms: $\langle (\mathbf{a},1) \rangle$ for d_4, $\langle (\mathbf{a},1),\, (\mathbf{b},1) \rangle$ for d_8, $\langle (\mathbf{a},3),\, (\mathbf{b},4) \rangle$, $\langle (\mathbf{a},5),\, (\mathbf{b},7),\, (\mathbf{c},11) \rangle$.
- Non-norm: $\langle (\mathbf{c},1) \rangle$ (no homogeneity: let $P = (0,1)$, then $g(\vec{OP}) = 3$ and $g(2.\vec{OP}) = 2 \neq 2.g(\vec{OP})$).

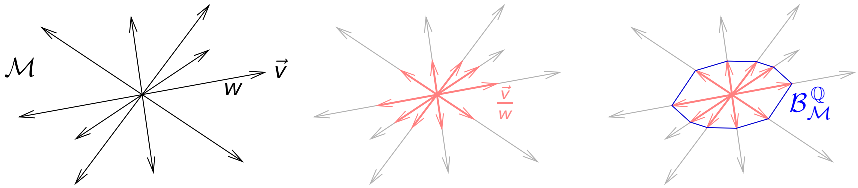
Chamfer norm condition

[Thiel 2001][Normand 2012]

The rational ball of a chamfer mask $\mathcal{M}$ is the set

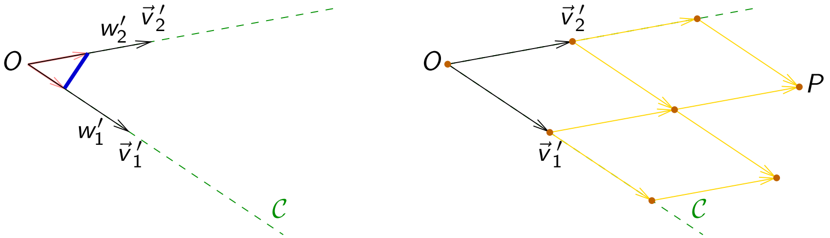
$$\mathcal{B}^{\mathbb{Q}}_{\mathcal{M}} = \text{conv}\left(\frac{\vec{v}}{w} \ : \ (\vec{v}, w) \in \mathcal{M} \right)$$

$\rightarrow$ convex polyhedron, whose geometry is the same as the distance balls up to a scale factor.



A chamfer mask $\mathcal{M}$ induces a discrete norm in $\mathbb{Z}^n$ if and only if it exists a triangulation $\mathcal{T}$ of $\mathcal{B}^{\mathbb{Q}}_{\mathcal{M}}$ in unimodular cones from O.

Minimal paths in unimodular cones



In this triangulation $\mathcal{T}$, each unimodular cone $\mathcal{C}$ is bounded by a subset $\mathcal{M}|_{\mathcal{C}} = \{\, (\vec{v}'_i, w'_i),\, 1 \leq i \leq n \,\}$ of n weightings of $\mathcal{M}$.

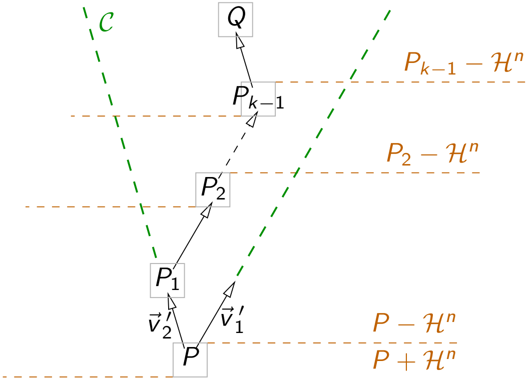
For each point P in $\mathcal{C}$, there is a minimal path from O to P which is a linear combination $\lambda_1 \vec{v}'_1 + \ldots + \lambda_n \vec{v}'_n$, $\lambda_i \in \mathbb{Z}_+$ and whose intermediate points are all included in $\mathcal{C}$.

Convergence of RSDT for chamfer norms

Proposition 1

Let $\mathcal{M}$ be a chamfer norm mask in $\mathbb{Z}^n$, then the two raster sequences DT algorithm provides the correct DT values for $d_{\mathcal{M}}$.

Idea of the proof



During a raster scan, each P_i is contained in the half-space $P_{i-1} - \mathcal{H}^n$, $1 \leq i \leq k$, so during the forward scan, each P_i is evaluated before P_{i-1}.

6 - Conclusion and future work

Contributions on the raster sequences DT:

- improvement of the original proof for d_4;
- hardened raster sequences DT algorithm;
- test programs in Python, available on-line;
- counter-example of the convergence in two scans;
- proof of convergence in two scans for chamfer norms in $\mathbb{Z}^n$.

Remark: the norm condition is sufficient but non necessary:
two scans are sufficients for the non-norm chamfer masks
$\langle(\mathbf{a},1),\ (\mathbf{b},1),\ (\mathbf{c},1)\rangle$, $\langle(\mathbf{a},1),\ (\mathbf{b},3),\ (\mathbf{c},2)\rangle$, $\langle(\mathbf{a},2),\ (\mathbf{b},3),\ (\mathbf{c},4)\rangle$, $\langle(\mathbf{a},1),\ (\mathbf{c},1)\rangle$, $\langle(\mathbf{a},2),\ (\mathbf{c},3)\rangle$, etc.

Future work: investigate if necessary conditions could be established on non-norm chamfer masks, or predict the number of passes for the raster sequences DT convergence.