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## A note on the Roberts characterization of proper and unit interval graphs

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### Abstract/Résumé

In this note, a constructive proof is given that the classes of proper interval graphs and unit interval graphs coincide, a result originally established by F.S. Roberts.

**Keywords:** proper interval graphs, unit interval graphs, characterizations, constructive proof.

Dans cette note, nous donnons une preuve constructive d'un résultat de F.S. Roberts qui établit que les classe des graphes d'intervalles propres et des graphes d'intervalles unitaires coïncident.

**Mots-clés :** graphes d'intervalles propres, graphes d'intervalles unitaires, caractérisations, preuve constructive.

**Relecteurs/Reviewers:** Victor Chepoi, Michel Van Canegehem, Jean-François Maurras.

A graph $G = (V, E)$ is an *interval graph* if to each vertex $v \in V$ a closed (resp. open) interval $I_v = [l_v, r_v]$ on the real line can be associated, such that any pairwise distinct vertices $u, v \in V$ are adjacent if and only if $I_u \cap I_v \neq \emptyset$. The family $\{I_v\}_{v \in V}$ is an *interval representation* of $G$. The complement $\overline{G} = (V, F)$ of an interval graph $G$ can be transitively oriented by setting $(u, v) \in \vec{F}$ if $r_u < l_v$. The orientation $\vec{F}$ of the edges of $\overline{G}$ induces a partial order called *interval order* (we shall write $I_u \prec I_v$ if $r_u < l_v$). $G$ is a *proper interval graph* if there is an interval representation of $G$ in which no interval properly contains another. In the same way, $G$ is an *unit interval graph* if there is an interval representation of $G$ in which all the intervals have the same length. For more details about the world of interval graphs, the reader can consult [2].

In 1969, F.S. Roberts proved that the classes of proper interval graphs and unit interval graphs coincide [4]. He showed notably that $K_{1,3}$-free interval graphs are unit interval graphs by using the Scott-Suppes characterization of semiorders [5]. Then, the trivial implications "unit $\Rightarrow$ proper $\Rightarrow$ $K_{1,3}$-free" for interval graphs enabled him to establish the whole result. In this note, a consructive proof which does not rely on some characterizations by forbidden subgraphs is presented. The main idea is to build directly an unit interval representation from the maximal cliques of the proper interval graph. For this, we make use of a strong structural property of these maximal cliques (assertion (3) of Theorem) which is first simply established thanks to a characterization inspired from [3] (assertion (2) of Theorem).

The reader shall note that another constructive proof was recently given by K.P. Bogart and D.B. West [1], where proper intervals are converted into unit intervals by means of successive contractions, dilatations and translations. For additional characterizations of proper and unit interval graphs, the reader is referred to [3, 6]. All graph-theoretical terms not defined here can be found in [2].

**Theorem** *For an undirected graph $G = (V, E)$, the following conditions are equivalent:*

*(1) $G$ is a proper interval graph,*

*(2) there exists a linear order $v_1 < \cdots < v_n$ on $V$ such that for all $v_i < v_j$, $v_i v_j \in E$ implies that $\{v_i, \ldots, v_j\}$ is a clique in $G$,*

*(3) the clique matrix of $G$ (maximal cliques-versus-vertices incidence matrix) has the consecutive 1's property for rows and columns,*

*(4) $G$ is an unit interval graph,*

*(5) $G$ is a $K_{1,3}$-free interval graph.*

**Proof.** (1) $\Rightarrow$ (2). Let $\{I_v\}_{v \in V}$ be a proper interval representation of $G$. For $a, b \in V$, set $a < b$ if $l_a < l_b$ or $l_a = l_b$ and $r_a \leq r_b$. Trivially, $<$ is a linear order on $V$. Let $a, b, c \in V$ with $a < b < c$ and $ac \in E$. Since $I_a, I_b, I_c$ are proper, we have $l_a \leq l_b \leq l_c < r_a \leq r_b \leq r_c$ and also $ab, bc \in E$. Now, let $v_i < \cdots < v_j \in V$ with $v_i v_j \in E$. By applying the previous assertion first to each triplet $\{v_i, v_k, v_j\}$ and then to all remaining triplets of $\{v_i, \ldots, v_j\}$, we conclude that this set induces a clique in $G$.

(2) $\Rightarrow$ (3). A $(0,1)$-matrix has the consecutive 1's property for columns (resp. rows) if its rows (resp. columns) can be permuted in such a way that the 1's in each column (resp. row) occur consecutively. Let $\{v_1, \ldots, v_n\}$ be the set $V$ of vertices ordered according to $<$ and $M$ be the clique matrix of $G$ such that a column $j$ corresponds to the vertex $v_j$. Suppose that there is a row $i$ of $M$ which has not the consecutive 1's property, that is there is in $G$ a maximal clique $C_i$ containing two vertices $v_a < v_b$ with $a$ strictly lower than $b$. Now $v_a v_b \in E$ implies that $\{v_a, \ldots, v_b\}$ is a clique, which is in contradiction with the maximality of $C_i$. Consequently, $M$ has the consecutive 1's property for rows. By ordering the rows of $M$ according to the 1's most to the left, the consecutive 1's property for rows yields the consecutive 1's property for columns.

(3) $\Rightarrow$ (4). One can suppose without loss of generality that $G$ is connected. Let $M = (a_{ij})_{mn}$ be the $m \times n$ clique matrix of $G$ such that the 1's of each row and each column appear consecutively (with the rows ordered according to the 1's most to the left). For each row $i$, set $a(i) = \min\{j \mid a_{ij} = 1\}$ and $b(i) = \max\{j \mid a_{ij} = 1\}$. Denote by $v_j$ the vertex of $G$ corresponding to the column $j$: each row $i$ defines a maximal clique $C_i = \{v_{a(i)}, \ldots, v_{b(i)}\}$. Then, the consecutive 1's property for columns implies that $C_1, \ldots, C_m$ are linearly ordered such that, for every vertex $v_j \in V$, the maximal cliques containing $v_j$ occur consecutively. Now an inductive process is proposed to construct an unit interval representation $\{I_j\}_{v_j \in V}$ of $G$; to each vertex $v_j \in V$ shall be associated an interval $I_j$ of unit length $U$ on the real line. The inductive step is the following. Having an unit interval representation of the subgraph induced by $\bigcup_{j=1}^{i} C_j$ such that $I_1$ is the interval most to the left and no two intervals share a common endpoint, one can obtain an unit interval representation of the subgraph induced by $\bigcup_{j=1}^{i+1} C_j$ having the same properties. Recalling that $C_i = \{v_{a(i)}, \ldots, v_{b(i)}\}$ and $C_{i+1} = \{v_{a(i+1)}, \ldots, v_{b(i+1)}\}$, the maximality of $C_i$ and the connectivity of $G$ yield $a(i) < a(i+1) < b(i) < b(i+1)$. Thus, we only have to define the endpoints of the set of unit intervals $\mathcal{I}_r = \{I_{b(i)+1}, \ldots, I_{b(i+1)}\}$ in such way that these ones intersect all the unit intervals of the set $\mathcal{I}_m = \{I_{a(i+1)}, \ldots, I_{b(i)}\}$ but none of the set $\mathcal{I}_l = \{I_{a(i)}, \ldots, I_{a(i+1)-1}\}$ (see Fig. 1). To realize this, we propose the following construction. Define $L = r_{a(i+1)} - r_{a(i+1)-1}$ to be the portion of $I_{a(i+1)} \in \mathcal{I}_m$ which shall contain the left endpoints of the intervals of $\mathcal{I}_r$. Clearly, this portion is intersected by every interval of $\mathcal{I}_m$ but none of $\mathcal{I}_l$. Then, set $\epsilon = \frac{L}{b(i+1)-b(i)+1}$ and for $j \in \{b(i)+1, \ldots, b(i+1)\}$, define $I_j = [l_j, l_j + U]$ with $l_j = r_{a(i+1)-1} + \epsilon(j - b(i))$. The value $\epsilon$ represents the shift between two consecutive intervals; this one is calculated so that the left endpoints of the $b(i+1) - b(i)$ intervals of $\mathcal{I}_r$ belong to the portion $L$ of $I_{a(i+1)}$. Consequently, this construction gives a correct unit interval representation of the subgraph induced by $\bigcup_{j=1}^{i+1} C_j$. Finally, we conclude by giving the initial step of the induction. An unit interval representation of $C_1$ having the desired properties is easily obtained by setting $\epsilon = \frac{U}{b(1)}$ and, for $j \in \{a(1), \ldots, b(1)\}$, defining $I_j = [l_j, l_j + U]$ with $l_j = \epsilon(j-1)$ (here $a(1) = 1$ and the cardinality of $C_1$ equals $b(1)$). Then, the application of the initial and inductive steps leads to a complete unit interval representation of $G$ (an example of such a contruction is given in Appendix).

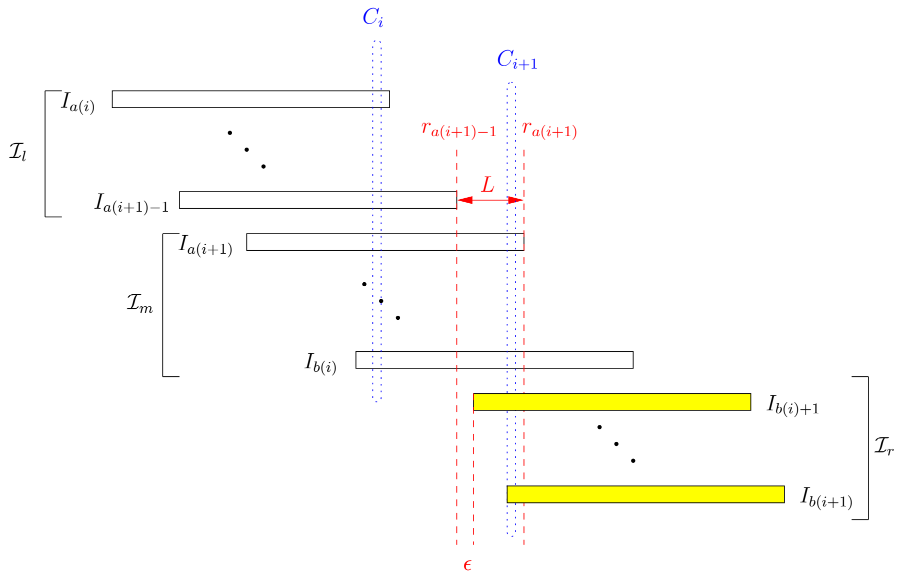


**Fig. 1.** The construction of $C_{i+1}$ from $C_i$.

(4) $\Rightarrow$ (5). Let $\{I_v\}_{v\in V}$ be an unit interval representation of $G$ and suppose the existence of four intervals $I_a, I_b, I_c, I_d$ inducing a copy of $K_{1,3}$ with $I_a \prec I_b \prec I_c$. This imposes on $I_b$ to be strictly smaller than $I_d$, which is a contradiction.

(5) $\Rightarrow$ (1). Let $\{I_v\}_{v\in V}$ be an interval representation of $G$. Suppose that two intervals $I_b, I_d$ are such that $I_b \subset I_d$. Since $G$ is $K_{1,3}$-free, there cannot exist $I_a, I_c$ both intersecting $I_d$ with $I_a \prec I_b \prec I_c$. Consequently, if there is an interval $I_a$ (resp. $I_c$) intersecting $I_d$ such that $I_a \prec I_b$ (resp. $I_b \prec I_c$), then one can always extend the right endpoint (resp. the left endpoint) of $I_b$ until $r_d + \epsilon$ (resp. $l_d - \epsilon$), $\epsilon > 0$, without modifying the graph $G$. By repeating this operation while there exists an interval containing another, we obtain a proper interval representation of $G$. $\square$

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# Appendix

Here is an application of the method previously described to build an unit interval representation from the clique matrix of a proper interval graph. The set of considered intervals is $\mathcal{I} = \{1, 2, 3, 4, 5, 6, 7\}$ defined below on Fig. 2. One can easily verify that these intervals *induce* a proper interval graph.

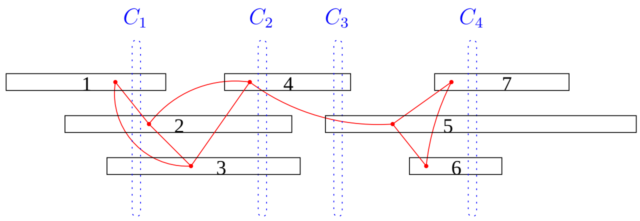


**Fig. 2.** The seven intervals of the set $\mathcal{I}$.

The corresponding clique matrix is:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $C_1$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $C_2$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| $C_3$ | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $C_4$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 |

The result of the construction appears below on Fig. 3.

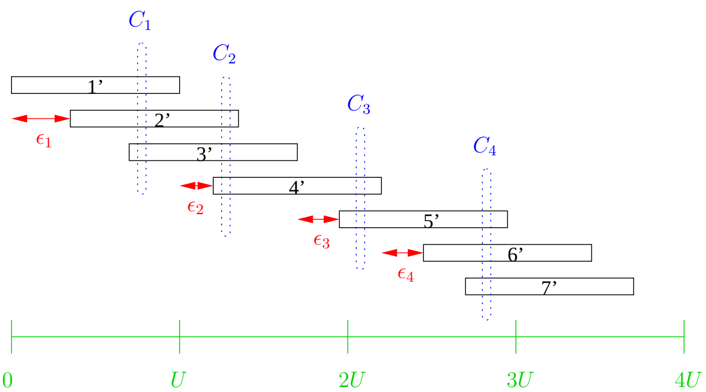


**Fig. 3.** The result of the construction.

Sucessively, we set $\epsilon_1 = \frac{U}{3}$, $\epsilon_2 = \frac{U}{6}$, $\epsilon_3 = \frac{U}{4}$ and $\epsilon_4 = \frac{U}{4}$ in order to obtain the set of unit intervals $1' = [0, U]$, $2' = [\frac{U}{3}, \frac{4U}{3}]$, $3' = [\frac{2U}{3}, \frac{5U}{3}]$, $4' = [\frac{7U}{6}, \frac{13U}{6}]$, $5' = [\frac{23U}{12}, \frac{35U}{12}]$, $6' = [\frac{29U}{12}, \frac{41U}{12}]$ and $7' = [\frac{8U}{3}, \frac{11U}{3}]$.