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Graphs and Combinatorial Optimization

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Abstract/Résumé

The integer programming models known as set packing and set covering have a wide range of applications, many of which arise in the context of graph theory. Sometimes, because of the special structure of the constraint matrix, the natural linear programming relaxation yields an optimal solution that is integral, thus solving the problem. Sometimes, both the linear programming relaxation and its dual have integer optimal solutions. Under which conditions do such integrality properties hold? This question is of both theoretical and practical interest. Min-max theorems, polyhedral combinatorics and graph theory all come together in this rich area of discrete mathematics. In addition to min-max and polyhedral results, some of the deepest results in this area come in two flavors: "excluded structure" characterizations and "decomposition" results. These notes provide an introduction to this area. In particular, they survey the celebrated Strong Perfect Graph Conjecture and its recent solution by Chudnovsky, Robertson, Seymour and Thomas, and Lehman's characterization of ideal clutters. The monograph "Combinatorial Optimization: Packing and Covering" by Cornuéjols, CBMS 74, SIAM (2001) provides background material.

Keywords: combinatorial optimization, packing, covering, ideal clutter, perfect graph.

Les problèmes de recouvrement et d'empaquetage ont de nombreuses applications, en particulier en théorie des graphes. Parfois, en raison de la structure particulière de la matrice des contraintes, la relaxation naturelle de ces problèmes sous forme d'un programme linéaire produit une solution optimale *entière*, ce qui résoud donc le problème initial. Sous quelle conditions a-t-on cette propriété d'intégralité? Cette question a un intérêt à la fois théorique et pratique. Les théorèmes Min-Max, la combinatoire polyèdrale et la théorie des graphes se retrouvent dans ce domaine riche des mathématiques discrètes. A côté des théorèmes polyèdraux et Min-Max qui sont souvent très élégants, les résultats les plus profonds dans ce domaine ont tendance à se présenter sous deux formes: caractérisation par "structures exclues" et théorèmes de "décomposition". Ces notes présentent une introduction à ce domaine. En particulier elles survolent la célèbre conjecture des graphes parfaits et sa solution récente par Chudnovsky, Robertson, Seymour and Thomas, et la caractérisation de Lehman des hypergraphes idéaux.

Mots-clés : optimisation combinatoire, empaquetage, recouvrement, hypergraphe, idéal, graphe parfait.

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1. Clutters

A *clutter* $\mathcal{C}$ is a family $E(\mathcal{C})$ of subsets of a finite ground set $V(\mathcal{C})$ with the property that $S_1 \not\subseteq S_2$ for all distinct $S_1, S_2 \in E(\mathcal{C})$. $V(\mathcal{C})$ is called the set of *vertices* and $E(\mathcal{C})$ the set of *edges* of $\mathcal{C}$. Clutters are also called *Sperner families* in the literature. A clutter is *trivial* if it has no edge or if it has the empty set as the only edge.

For example, a simple graph (no multiple edges or loops) is a clutter where every edge has cardinality two. We refer the reader to West [**56**] for a basic reference in graph theory. In these notes, all graphs are assumed to be finite and simple.

In a clutter, a *matching* is a set of pairwise disjoint edges. A *transversal* is a set of vertices that intersects all the edges. A clutter is said to *pack* if the maximum cardinality of a matching equals the minimum cardinality of a transversal. This terminology is due to Seymour [**54**]. Many min-max theorems in graph theory can be rephrased by saying that a clutter packs. We give three examples. The first is König's theorem (for a proof, see Theorem 3.1.11 in West [**56**]).

Theorem 1.1. (König [**37**]) *In a bipartite graph, the maximum cardinality of a matching equals the minimum cardinality of a transversal.*

As a second example, consider the edge version of Menger's theorem (for a proof, see Theorem 4.2.18 in West [**56**]).

Theorem 1.2. (Menger [**44**]) *Let s and t be distinct nodes of a graph G. The maximum number of pairwise edge-disjoint st-paths in G equals the minimum number of edges in an st-cut.*

Let $\mathcal{C}_1$ be the clutter whose vertices are the edges of G and whose edges are the st-paths of G (Following West's terminology [**56**], paths and cycles have no repeated nodes). We call $\mathcal{C}_1$ the *clutter of st-paths.* Its transversals are the st-cuts. Thus, rephrased in terms of clutters, Menger's theorem states that the clutter of st-paths packs.

Interestingly, some famous results and difficult conjectures can be rephrased by saying that certain clutters pack. In a graph G, a set of edges is called a *postman set* if it induces an acyclic graph whose odd degree nodes coincide with the odd degree nodes of G.

Conjecture 1.3. (Conforti and Johnson [**19**]) *The clutter of postman sets packs in planar graphs.*

It turns out that, if true, this conjecture implies the four color theorem [**1**] stating that every planar graph is 4-colorable, i.e. the nodes of a planar graph can be colored with four colors so that no edge has its endnodes colored with the same color (see [**20**] for details). This indicates that a full understanding of the clutters that pack must be extremely difficult. More restricted notions are amenable to beautiful theories, while still containing rich classes of examples. We introduce several such concepts in the following section.

1.1. The Max Flow Min Cut Property and Idealness.

Given a nontrivial clutter $\mathcal{C}$, we define $M(\mathcal{C})$ to be a 0,1 matrix whose columns are indexed by $V(\mathcal{C})$, whose rows are indexed by $E(\mathcal{C})$ and where $m_{ij} = 1$ if and only if the vertex corresponding to column j belongs to the edge corresponding to row i. In other words, the rows of $M(\mathcal{C})$ are the characteristic vectors of the sets in $E(\mathcal{C})$. Note that the definition of $M(\mathcal{C})$ is unique up to permutation of rows and permutation of columns. Furthermore, $M(\mathcal{C})$ contains no dominating row, since $\mathcal{C}$ is a clutter (A vector $r \in F$ is said to be *dominating* if there exists $v \in F$ distinct from r such that $r \geq v$). A 0,1 matrix containing no dominating rows is called a *clutter* *matrix*. Given any 0,1 clutter matrix M, let $\mathcal{C}(M)$ denote the clutter such that $M(\mathcal{C}(M)) = M$.

Let $M \neq 0$ be a 0,1 clutter matrix and consider the following pair of dual linear programs, where x and y denote the vectors of unknowns.

$$\min\{wx : x \geq 0,\ Mx \geq \mathbf{1}\} \tag{1.1}$$

$$= \max\{y\mathbf{1} : y \geq 0,\ yM \leq w\} \tag{1.2}$$

Here x and $\mathbf{1}$ are column vectors while w and y are row vectors. $\mathbf{1}$ denotes a vector all of whose components are equal to 1. Equality holds between (1.1) and (1.2) as a consequence of the duality theorem of linear programming (see for example Section 7.4 in Schrijver [**52**]).

We say that a vector is *integral* if all its components are integers. In these notes, our interest will focus on integral solutions to (1.1) or (1.2). First, we reformulate the definition of a clutter that packs.

Definition 1.4. Clutter $\mathcal{C}(M)$ *packs* if both (1.1) and (1.2) have optimal solution vectors x and y that are integral when $w = \mathbf{1}$.

This corresponds to the earlier definition because, when $w = \mathbf{1}$, if (1.1) and (1.2) have integral optimal solutions, they also have 0,1 optimal solutions x and y. Furthermore, x is the characteristic vector of a transversal of $\mathcal{C}(M)$ and y is the characteristic vector of a matching.

Definition 1.5. Clutter $\mathcal{C}(M)$ has the *packing property* if both (1.1) and (1.2) have optimal solution vectors x and y that are integral for all vectors w with components equal to $0, 1$ or $+\infty$. (Note: When $w_j = 0$, it is optimal to set $x_j = 1$ and when $w_j = +\infty$, it is optimal to set $x_j = 0$. By convention, we consider that $w_j x_j = 0$ in this case.)

Definition 1.6. Clutter $\mathcal{C}(M)$ has the *Max Flow Min Cut* property (or MFMC property) if both (1.1) and (1.2) have optimal solution vectors x and y that are integral for all nonnegative integral vectors w.

Clearly, the MFMC property for a clutter implies the packing property which itself implies that the clutter packs. Conforti and Cornuéjols [**15**] conjectured that, in fact, the MFMC property and the packing property are identical. This conjecture is still open.

Conjecture 1.7. *A clutter has the MFMC property if and only if it has the packing property.*

Definition 1.8. Clutter $\mathcal{C}(M)$ is *ideal* if (1.1) has an optimal solution vector x that is integral for all $w \geq 0$.

It is easy to show that the MFMC property implies idealness. Indeed, if (1.1) has an optimal solution vector x for all nonnegative integral vectors w, then (1.1) has an optimal solution x for all nonnegative rational vectors w. In fact, the packing property implies idealness.

Theorem 1.9. *If a clutter has the packing property, then it is ideal.*

This follows from a result of Lehman [**39**] (see Theorem 3.19 presented later).

Figure 1 shows inclusion relationships between the classes of clutters introduced above. The following clutter matrices show that some of these inclusions are strict:

$$Q_6 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},\ C_3^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},\ C_4^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix},\ Q_6^+$$

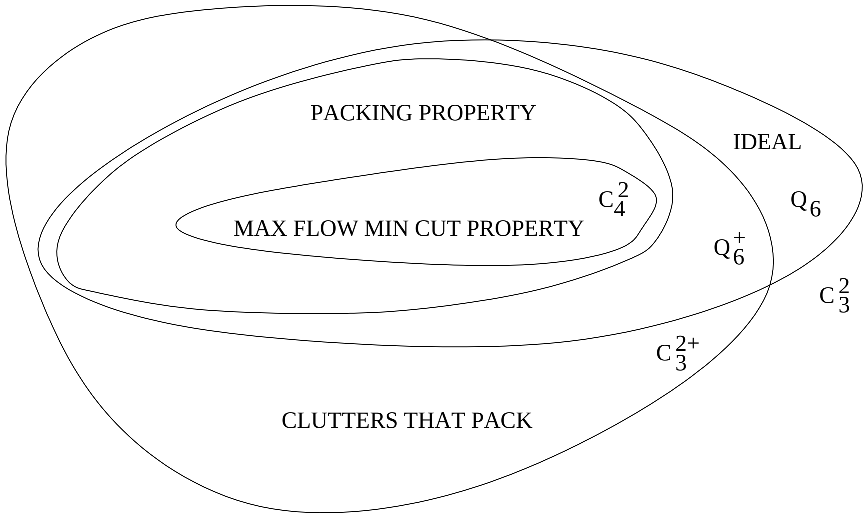

FIGURE 1. Classes of clutters.

and C_3^{2+}: For an $m \times n$ 0,1 matrix M, we use M^+ to denote the $m \times (n+1)$ matrix obtained from M by appending the column vector $\mathbf{1}$. The reader is encouraged to verify that these five clutters satisfy the properties suggested in Figure 1.

A subset P of $\mathbb{R}^n$ is called a *polyhedron* if it is the intersection of finitely many affine half-spaces, i.e. $P = \{x : Ax \leq b\}$ for some matrix A and vector b. Observe that $\mathcal{C}(M)$ is ideal if and only if the polyhedron $P = \{x \geq 0 : Mx \geq \mathbf{1}\}$ is an *integral polyhedron*, that is, all the extreme points of P have only integer coordinates (an *extreme point* of P is a point x that cannot be written as a convex combination of $x^1, x^2 \in P$ where $x^1 \neq x$ and $x^2 \neq x$). Equivalently, $\mathcal{C}$ is ideal if and only if

$$\begin{aligned} x(S) &\geq 1 \quad \text{ for all } \; S \in E(\mathcal{C}) \\ x &\geq 0 \end{aligned}$$

is an integral polyhedron, where $x(S)$ denotes $\sum_{i \in S} x_i$.

A linear system $Ax \geq b$ is *Totally Dual Integral* (TDI) if the linear program $\min\{wx : \; Ax \geq b\}$ has an integral optimal dual solution y for every integral w for which this linear program has a finite optimum. Edmonds and Giles [**27**] showed that, if $Ax \geq b$ is TDI and b is integral, then $P = \{x : Ax \geq b\}$ is an integral polyhedron. The interested reader can find the proof of the Edmonds-Giles theorem in Schrijver [**52**] pages 310–311, or Nemhauser and Wolsey [**45**] pages 536–537. It follows that $\mathcal{C}(M)$ has the MFMC property if and only if (1.2) has an optimal integral solution y for all nonnegative integral vectors w.

For convenience, we say that trivial clutters have all the above properties: MFMC, idealness, etc.

1.2. Perfection. The min-max equation (1.1)=(1.2) has a close max-min relative

$$\max\{wx : x \geq 0, \; Mx \leq \mathbf{1}\} \tag{1.3}$$
$$= \min\{y\mathbf{1} : y \geq 0, \; yM \geq w\} \tag{1.4}$$

The clutter $\mathcal{C}(M)$ *covers* if both (1.3) and (1.4) have optimal solution vectors x and y that are integral when $w = \mathbf{1}$. This is the analog of a clutter that packs. Like for clutters that pack, a full understanding of the clutters that cover appears to be extremely difficult. Thus like for the min-max equation (1.1)=(1.2) where we introduced the notions of packing property, MFMC property and idealness, it makes sense to introduce three analogous definitions for the max-min equation (1.3)=(1.4). Surprisingly, these three notions turn out to be identical.

DEFINITION 1.10. A *perfect matrix* M is a $0,1$ matrix with no column of 0's such that $P = \{x \geq 0 :\ Mx \leq \mathbf{1}\}$ is an integral polyhedron.

When M is a perfect matrix, the linear program $\max\{wx :\ x \geq 0,\ Mx \leq \mathbf{1}\}$ has an integral optimal solution x for all w, and therefore the *set packing problem* $\max\{wx :\ Mx \leq \mathbf{1},\ x \in \{0,1\}^n\}$ can be solved in polynomial time as a linear program. By contrast, for a general 0,1 matrix M, the set packing problem is NP-hard [**32**].

As we mentioned already, Edmonds and Giles [**27**] observed that, when a linear system $Ax \leq b,\ x \geq 0$ is TDI and b is integral, the polyhedron $\{x :\ Ax \leq b,\ x \geq 0\}$ is integral. The converse is not true in general (recall Q_6 in Figure 1 for the linear system $-Q_6 x \leq -\mathbf{1},\ x \geq 0$). But it is true when A is a 0,1 matrix and $b = \mathbf{1}$.

THEOREM 1.11. (Lovász [**40**], Fulkerson [**31**], Chvátal [**10**]) *For a 0,1 matrix M with no column of 0's, the following statements are equivalent:*

(i) *the linear system $Mx \leq \mathbf{1},\ x \geq 0$ is TDI,*
(ii) *the matrix M is perfect,*
(iii) *max $\{wx :\ Mx \leq \mathbf{1},\ x \geq 0\}$ has an integral optimal solution x for all $w \in \{0,1\}^n$.*

Clearly (i) implies (ii) implies (iii), where the first implication is the Edmonds-Giles property and the second follows from the definition of perfection. What is surprising is that (iii) implies (i) and, in fact, that (ii) implies (i). We will prove this theorem in the next section. It is convenient to present this material in terms of graphs.

The concept of perfect graph (to be defined in the next section) was introduced by Berge [**2**] at a workshop in 1960. In a graph, a *clique* is a set of pairwise adjacent nodes. The *clique-node matrix* of a graph is a 0,1 matrix M with columns indexed by the nodes of G where the rows of M are the characteristic vectors of the maximal cliques of G. Thus entry m_{ij} of M is 1 if and only if node j belongs to maximal clique i. Chvátal [**10**] established the following connection between perfect graphs and perfect matrices: A 0,1 clutter matrix with no column of 0's is perfect if and only if it is the clique-node matrix of a perfect graph. Therefore we may reason in terms of graphs rather than 0,1 matrices.

2. Perfect Graphs

The node set of graph G is denoted by $V(G)$ and its edge set by $E(G)$. A *stable set* is a set of nodes no two of which are adjacent. A *clique* is a set of nodes every pair of which are adjacent. The cardinality of a largest clique in graph G is denoted by $\omega(G)$. The cardinality of a largest stable set is denoted by $\alpha(G)$. A *k-coloring* is a partition of the nodes into k stable sets (these stable sets are called *color classes*). The *chromatic number* $\chi(G)$ is the smallest value of k for which there exists a k-coloring. Obviously, $\omega(G) \leq \chi(G)$ since the nodes of a clique must be in distinct color classes of a k-coloring. An *induced subgraph* of G is a graph with node set $S \subseteq V(G)$ and edge set comprising all the edges of G with both ends in S. It is denoted by $G(S)$. The graph $G(V(G) - S)$ is denoted by $G \setminus S$. A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. A graph is *minimally imperfect* if it is not perfect but all its proper induced subgraphs are.

A *hole* is a graph induced by a cycle of length at least 4. A hole is *odd* if it contains an odd number of nodes. Odd holes are not perfect since their chromatic number is 3 whereas the size of their largest clique is 2. It is easy to check that odd holes are minimally imperfect. The complement of a graph G is the graph $\bar{G}$ with the same node set as G, and uv is an edge of $\bar{G}$ if and only if it is not an edge of G. It is easy to check that complements of odd holes are also minimally imperfect. In the early sixties Berge [**2**] proposed the *Strong Perfect Graph Conjecture*: The odd holes and their complements are the only minimally imperfect graphs. This conjecture attracted much attention over the next forty years. It was proved in May 2002 by Chudnovsky, Robertson, Seymour and Thomas [**7**] in a very impressive paper. Claude Berge passed away in June 2002 knowing that his famous conjecture is true.

THEOREM 2.1. **(Strong Perfect Graph Theorem)** (Chudnovsky, Robertson, Seymour and Thomas [**7**]) *The only minimally imperfect graphs are the odd holes and their complements.*

In this section, we survey key aspects of the proof of the Strong Perfect Graph Theorem. A *Berge graph* is a graph that does not contain an odd hole or its complement as an induced subgraph. Clearly, every perfect graph is a Berge graph. The Strong Perfect Graph Theorem states that the converse is also true: Every Berge graph is perfect. The idea of the proof is to show that every Berge graph either falls into one of four basic classes of perfect graphs, or that it has a kind of separation that cannot occur in a minimally imperfect graph. This implies that there exists no minimally imperfect Berge graph, and therefore every Berge graph is perfect.

In [**2**], Berge also made a weaker conjecture, which states that a graph G is perfect if and only if its complement $\bar{G}$ is perfect. This conjecture was proved by Lovász [**40**] in 1972. We give a short elegant proof due to Gasparyan [**33**] by proving first the following stronger result.

THEOREM 2.2. (Lovász [**41**]) *A graph G is perfect if and only if, for every induced subgraph H, the number of nodes of H is at most $\alpha(H)\omega(H)$.*

PROOF. First assume that G is perfect. Then, for every induced subgraph H, $\omega(H) = \chi(H)$. Since the number of nodes of H is at most $\alpha(H)\chi(H)$, the inequality follows.

We give a proof of the converse due to Gasparyan [**33**]. Assume that G is not perfect. Let H be a minimally imperfect subgraph of G and let n be the number of nodes of H. Let $\alpha = \alpha(H)$ and $\omega = \omega(H)$. Then H satisfies

$$\omega = \chi(H\backslash v) \text{ for every node } v \in V(H)$$
$$\text{and } \omega = \omega(H\backslash S) \text{ for every stable set } S \subseteq V(H).$$

Let A_0 be a stable set of size α in H. Fix an ω-coloring of each of the α graphs $H\backslash s$ for $s \in A_0$, let $A_1, \ldots, A_{\alpha\omega}$ be the stable sets occuring as a color class in one of these colorings and let $\mathcal{A} := \{A_0, A_1, \ldots, A_{\alpha\omega}\}$. Let $\mathbf{A}$ be the corresponding stable set versus node incidence matrix. Define $\mathcal{B} := \{B_0, B_1, \ldots, B_{\alpha\omega}\}$ where B_i is a clique of size ω in $H\backslash A_i$. Let $\mathbf{B}$ be the corresponding clique versus node incidence matrix.

Claim: Every clique of size ω in H intersects all but one of the stables sets in $\mathcal{A}$.

Proof: Let $S_1, \ldots, S_\omega$ be any ω-coloring of $H \setminus v$. Since any clique C of size ω in H has at most one node in each S_i, C intersects all S_i's if $v \notin C$ and all but one if $v \in C$. Since C has at most one node in A_0, the claim follows.

In particular, it follows that $\mathbf{A}\mathbf{B}^T = J - I$ where J is the matrix filled with ones and I the identity. Since $J - I$ is nonsingular, $\mathbf{A}$ and $\mathbf{B}$ have at least as many columns as rows, that is $n \geq \alpha\omega + 1$. $\square$

COROLLARY 2.3. **(Perfect Graph Theorem)** (Lovász [**40**]) *Graph G is perfect if and only if graph $\bar{G}$ is perfect.*

PROOF. Since, for any graph H, $\alpha(H) = \omega(\bar{H})$ and $\omega(H) = \alpha(\bar{H})$, Theorem 2.2 implies Corollary 2.3.

□

2.1. Four Basic Classes of Perfect Graphs. Bipartite graphs are perfect since, for any induced subgraph H, the bipartition implies that $\chi(H) \leq 2$ and therefore $\omega(H) = \chi(H)$.

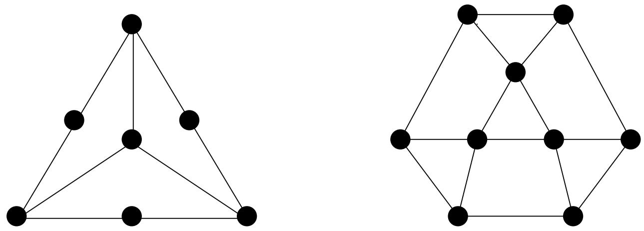

FIGURE 2. A bipartite graph and its line graph.

A graph L is the *line graph* of a graph G if $V(L) = E(G)$ and two nodes of L are adjacent if and only if the corresponding edges of G share a vertex. See Figure 2 for an example.

PROPOSITION 2.4. *Line graphs of bipartite graphs are perfect.*

PROOF. If G is bipartite, $\chi'(G) = \Delta(G)$ by a theorem of König [**37**], where χ' denotes the edge-chromatic number and Δ the largest node degree.

If L is the line graph of a bipartite graph G, then $\chi(L) = \chi'(G)$ and $\omega(L) = \Delta(G)$. Therefore $\chi(L) = \omega(L)$. Since induced subgraphs of L are also line graphs of bipartite graphs, the result follows. □

Since bipartite graphs and line graphs of bipartite graphs are perfect, it follows from Lovász's perfect graph theorem (Corollary 2.3) that the complements of bipartite graphs and of line graphs of bipartite graphs are perfect. This can also be verified directly, without using the perfect graph theorem. To summarize, in this section we have introduced four classes of perfect graphs:

- bipartite graphs and their complements, and
- line graphs of bipartite graphs and their complements.

These graphs are called *basic.*

2.2. 2-Join, Homogeneous Pair and Skew Partition. The proof of the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour and Thomas is based on a decomposition theorem for Berge graphs: Every Berge graph is basic as defined above, or it can be decomposed using a 2-join, a homogeneous pair or a skew partition. We define these three decompositions in this section.

2-Join

A graph G has a *2-join* if its nodes can be partitioned into sets V_1 and V_2, each of cardinality at least three, with nonempty disjoint subsets $A_1, B_1 \subseteq V_1$ and $A_2, B_2 \subseteq V_2$, such that all the nodes of A_1 are adjacent to all the nodes of A_2, all the nodes of B_1 are adjacent to all the nodes of B_2 and these are the only adjacencies between V_1 and V_2. See Figure 3. 2-joins were introduced by Cornuéjols and Cunningham [**21**] in 1985. They gave an $O(|V(G)|^2|E(G)|^2)$ algorithm to find whether a graph G has a 2-join.

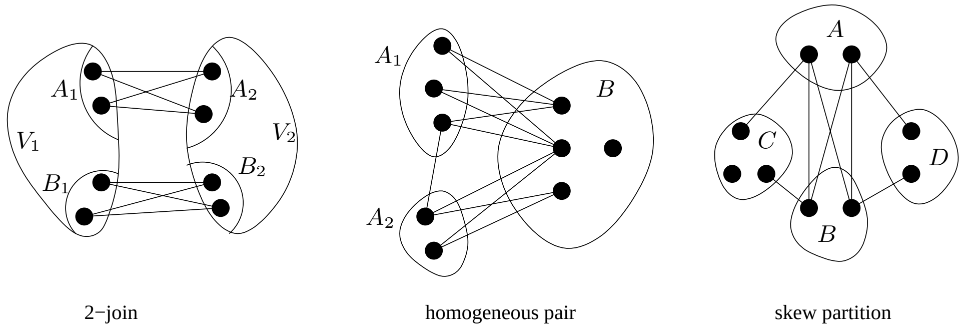

FIGURE 3. Decompositions.

When G contains a 2-join, we can decompose G into two blocks G_1 and G_2 defined as follows.

DEFINITION 2.5. If A_2 and B_2 are in different connected components of $G(V_2)$, define *block* G_1 to be $G(V_1 \cup \{p_1, q_1\})$, where $p_1 \in A_2$ and $q_1 \in B_2$. Otherwise, let P_1 be a shortest path from A_2 to B_2 in $G(V_2)$ and define *block* G_1 to be $G(V_1 \cup V(P_1))$. Block G_2 is defined similarly.

THEOREM 2.6. **(2-Join Decomposition Theorem)** (Cornuéjols and Cunningham [**21**], see also [**20**]) *Graph G is perfect if and only if its blocks G_1 and G_2 are perfect.*

COROLLARY 2.7. *If a minimally imperfect graph G has a 2-join, then G is an odd hole.*

PROOF. Since G is not perfect, Theorem 2.6 implies that block G_1 or G_2 is not perfect, say G_1. Since G_1 is an induced subgraph of G and G is minimally imperfect, it follows that $G = G_1$. Thus, since $|V_2| \geq 3$, V_2 induces a chordless path P_1. Therefore G is a minimally imperfect graph with a node of degree 2. It is well known that such a graph G is an odd hole [**47**]. $\square$

Homogeneous Pair

The notion of homogeneous pair was introduced by Chvátal and Sbihi [**13**]. A graph G has a *homogeneous pair* if $V(G)$ can be partitioned into subsets A_1, A_2 and B, such that:

- $|A_1| + |A_2| \geq 3$ and $|B| \geq 2$.
- If a node of B is adjacent to a node of A_i then it is adjacent to all the nodes of A_i, for $i \in \{1, 2\}$.

THEOREM 2.8. **(Homogeneous Pair Theorem)** (Chvátal and Sbihi [**13**]) *No minimally imperfect graph has a homogeneous pair.*

Skew Partition

A graph G has a *skew partition* if its nodes can be partitioned into four nonempty sets A, B, C, D such that there are all the possible edges between A and B and no edges from C to D. Chvátal [**11**] introduced skew partitions in 1985 and he conjectured that no minimally imperfect graph has a skew partition. He observed that the conjecture holds for a *star cutset*, defined to be a skew partition where $|A| = 1$.

LEMMA 2.9. **(Star Cutset Lemma)** (Chvátal [**11**]) *No minimally imperfect graph has a star cutset.*

Proof. Assume that G is minimally imperfect and has a star cutset. Let G_1 be the graph induced by $A \cup B \cup C$ and G_2 the graph induced by $A \cup B \cup D$. The graphs G_1 and G_2 are perfect. Let S_i be the color class of an $\omega(G)$-coloring of G_i that contains the unique node of A, for $i \in \{1, 2\}$. Then S_i meets all the cliques of size $\omega(G)$ in G_i, i.e. $\omega(G \setminus (S_1 \cup S_2)) < \omega(G)$. It follows that $G \setminus (S_1 \cup S_2)$ can be colored with fewer than $\omega(G)$ colors, since it is perfect. Since $S_1 \cup S_2$ is a stable set, G can be colored with $\omega(G)$ colors, a contradiction. □

Noteworthy contributions towards the skew partition conjecture were made by Hoàng [**36**] and Roussel and Rubio [**51**]. The conjecture was settled by Chudnovsky, Robertson, Seymour and Thomas [**7**]. They obtained it as a consequence of the Strong Perfect Graph Theorem.

Theorem 2.10. **(Skew Partition Theorem)** (Chudnovsky, Robertson, Seymour and Thomas [**7**]) *No minimally imperfect graph has a skew partition.*

In order to prove the Strong Perfect Graph Theorem, Chudnovsky, Robertson, Seymour and Thomas first proved the following weaker result.

A skew partition is *balanced* if
(i) every induced path of length at least 2 in G with ends in $A \cup B$ and interior in $C \cup D$ is even, and
(ii) every induced path of length at least 2 in $\bar{G}$ with ends in $C \cup D$ and interior in $A \cup B$ is even.

Theorem 2.11. (Chudnovsky, Robertson, Seymour and Thomas [**6**]) *A minimally imperfect Berge graph with smallest number of nodes cannot have a balanced skew partition.*

We give the proof of Theorem 2.11. It uses Lovász's Replication Lemma [**40**] which we discuss next. Incidentally, the Replication Lemma was the step that Fulkerson missed in his attempt to prove the Perfect Graph Theorem. Because Fulkerson had convinced himself that the Replication Lemma was likely to be false, he had not tried very hard to prove it. Fulkerson [**31**] says: "In the Spring of 1971, I received a postcard from Berge saying that he had just heard that Lovász had a proof of the perfect graph conjecture. This immediately rekindled my interest, naturally, and so I sat down at my desk and thought again about the replication lemma. Some four or five hours later, I saw a simple proof of it."

Lemma 2.12. **(Replication Lemma)** (Lovász [**40**]) *Let G be a perfect graph and $v \in V(G)$. Create a new node v' and join it to v and to all the neighbors of v. Then the resulting graph G' is perfect.*

Proof. It suffices to show $\chi(G') = \omega(G')$ since, for induced subgraphs, the proof follows similarly. We distinguish two cases.

- **Case 1:** Node v is contained in some clique of size $\omega(G)$ in G. Then $\omega(G') = \omega(G) + 1$. This implies $\chi(G') \leq \omega(G')$, since at most one new color is needed in G'. Clearly $\chi(G') = \omega(G')$ follows.
- **Case 2:** Node v is not contained in any clique of size $\omega(G)$ in G. Consider any $\omega(G)$-coloring of G and let S be the color class containing v. Then $\omega(G \setminus (S - \{v\})) = \omega(G) - 1$, since every clique of size $\omega(G)$ in G meets $S - \{v\}$. By the perfection of G, the graph $G \setminus (S - \{v\})$ can be colored with $\omega(G) - 1$ colors. Using one additional color for the nodes $(S - \{v\}) \cup \{v'\}$, we obtain a $\omega(G)$-coloring of G'.

□

Proof of Theorem 2.11: Let G be a minimally imperfect Berge graph with smallest number of nodes. Suppose that G has a balanced skew partition A, B, C, D. Note that each of C, D has cardinality at least two since, otherwise, the minimally imperfect graph $\bar{G}$ has a star cutset, contradicting Lemma 2.9. Let G' be the graph obtained from G by adding a node v adjacent to all the nodes of A and to no other node of G. If G' contains an odd hole, then G has an odd path contradicting (i) in the definition of a balanced skew partition. Similarly, if $\bar{G}'$ contains an odd hole, (ii) is contradicted. Therefore G' is a Berge graph. Now consider $G_1 = G' \setminus D$ and $G_2 = G' \setminus C$. For $i \in \{1, 2\}$, the graph G_i is perfect since it is Berge and has fewer nodes than G. Replicate node v in G_i so that v belongs to a clique of size $\omega(G)$. By the Replication Lemma 2.12, the resulting graph R_i is perfect. Consider $\omega(G)$-colorings of R_1 and R_2 respectively. Both colorings have the same number of colors in A and assume w.l.o.g. that these colors are $1, 2, \ldots, k$. Let K be the subgraph of G induced by the nodes with colors $1, 2, \ldots, k$ and let H be the subgraph of G induced by the nodes with other colors. Since every clique of size $\omega(G)$ in G is in $G \setminus D$ or $G \setminus C$, the largest clique in K has size k and the largest clique in H has size $\omega(G) - k$. The graphs H and K are perfect since they are proper subgraphs of G. Color K with k colors and H with $\omega(G) - k$ colors. Now G is colored with $\omega(G)$ colors, a contradiction to the assumption that G is minimally imperfect. $\square$

Theorem 2.11 was presented in September 2001 at a workshop in Princeton. As the next step towards Theorem 2.10, Chudnovsky and Seymour obtained the following theorem in January 2002. Its proof is much more difficult than that of Theorem 2.11.

Theorem 2.13. (Chudnovsky and Seymour [**8**]) *A minimally imperfect Berge graph with smallest number of nodes cannot have a skew partition.*

2.3. Decomposition of Berge Graphs. Conforti, Cornuéjols and Vušković proposed the following approach to solving the Strong Perfect Graph Conjecture.

Conjecture 2.14. (Conforti, Cornuéjols and Vušković (2001)) **(Decomposition Conjecture)** *Every Berge graph G is basic or has a skew partition, or one of G or $\bar{G}$ has a 2-join.*

Chudnovsky, Robertson, Seymour and Thomas proved the following variation of this conjecture.

Theorem 2.15. (Chudnovsky, Robertson, Seymour and Thomas [**7**]) **(Decomposition Theorem)** *Every Berge graph G is basic or has a skew partition or a homogeneous pair, or one of G or $\bar{G}$ has a 2-join.*

This theorem implies the Strong Perfect Graph Theorem. Indeed, suppose that the Decomposition Theorem holds and that there exists a minimally imperfect graph G distinct from an odd hole or its complement. Choose G with the smallest number of nodes. G cannot have a skew partition by Theorem 2.13. G cannot have a homogeneous pair by Theorem 2.8. Neither G nor $\bar{G}$ can have a 2-join by Corollary 2.7. Since G is a Berge graph, G must be basic by the Decomposition Theorem. Therefore G is perfect, a contradiction.

Theorem 2.15 was already known to hold in several special cases. For example, in 1984 Burlet and Fonlupt [**4**] worked out the case when G is a Meyniel graph (A graph is *Meyniel* if all its odd cycles have at least two chords). For a given graph X, a graph G is *X-free* if G does not contain X as an induced subgraph. Theorem 2.15 was known to hold for several classes of X-free graphs (see Figure 4). For example, it was known when G is claw-free (Chvatal and Sbihi [**14**] in 1988 and Maffray and

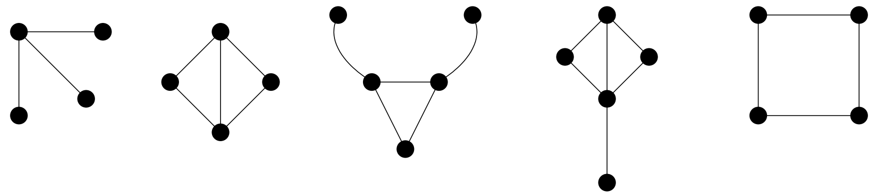

FIGURE 4. Small graphs.

Reed [**43**] in 1999), diamond-free (Fonlupt and Zemirline [**29**] in 1987), bull-free (Chvátal and Sbihi [**13**] in 1987), or dart-free (Chvátal, Fonlupt, Sun and Zemirline [**12**] in 2000). All these results involve special types of skew partitions (such as star cutsets) and, in some cases, homogeneous pairs [**13**]. A special case of 2-join called augmentation of a flat edge appears in [**43**]. In 1999, Conforti and Cornuéjols [**16**] used more general 2-joins to prove Conjecture 2.14 for WP-free Berge graphs, a class of graphs that contains all bipartite graphs and all line graphs of bipartite graphs. [**16**] was the precursor of a sequence of decomposition results involving 2-joins. The following result was obtained in February 2001. A *square* is a hole of length four.

THEOREM 2.16. (Conforti, Cornuéjols and Vušković [**17**]) *A square-free Berge graph is bipartite, the line graph of a bipartite graph, or has a 2-join or a star cutset.*

A breakthrough occured in September 2001 when Chudnovsky, Robertson, Seymour and Thomas announced that they could prove the Decomposition Conjecture in the following important special case.

THEOREM 2.17. (Chudnovsky, Robertson, Seymour and Thomas [**6**]) *If G is a Berge graph that contains the line graph of a bipartite subdivision of a 3-connected graph, then G has a balanced skew partition, or one of G or $\bar{G}$ has a 2-join or is the line graph of a bipartite graph.*

Eight months later, Chudnovsky, Robertson, Seymour and Thomas completed the proof of Theorem 2.15 and therefore of the Strong Perfect Graph Theorem. These results are contained in a monumental paper [**7**].

Conforti, Cornuéjols and Vušković [**18**] proved a weaker version of the Decomposition Conjecture where "skew partition" is replaced by "double star cutset". A *double star* is a node set S that contains two adjacent nodes u, v and a subset of the neighbors of u or v. Clearly, if G has a skew partition, then G has a double star cutset: Take $S = A \cup B$, $u \in A$ and $v \in B$. Although the decomposition result in [**18**] is weaker than Conjecture 2.14 for Berge graphs, it holds for a larger class of graphs than Berge graphs: By changing the decomposition from "skew partition" to "double star cutset", the result can be obtained for all odd-hole-free graphs instead of just Berge graphs.

THEOREM 2.18. (Conforti, Cornuéjols and Vušković [**18**]) *If G is an odd-hole-free graph, then G is a bipartite graph or the line graph of a bipartite graph or the complement of the line graph of a bipartite graph, or G has a double star cutset or a 2-join.*

Decomposition Theorem 2.18 was used by Cornuéjols, Liu and Vušković [**22**] to construct a polynomial time recognition algorithm for perfect graphs. Independently, Chudnovsky and Seymour [**9**] found an algorithm for perfect graph recognition which does not use decomposition. Both algorithms [**9**], [**22**] build on the companion paper [**5**] which performs a certain "cleaning" step in polynomial time. A key step in the cleaning procedure [**5**] is due to Chudnovsky and Seymour.

2.4. Proof of Theorem 1.11. It suffices to prove that (iii) implies (i). Assume that M is a 0,1 matrix with no column of 0's such that

(iii) max $\{wx : \ Mx \leq \mathbf{1}, \ x \geq 0\}$ has an integral optimal solution x for all $w \in \{0,1\}^n$.

We can assume w.l.o.g. that M is a clutter matrix. Let $G(M)$ be the graph having a node for each column of M and an edge between nodes j and k if M has a row i with $m_{ij} = m_{ik} = 1$.

Claim 1: M is the clique-node matrix of graph $G(M)$.

Proof: Suppose not. Then there exists a clique K of size at least 3 in $G(M)$ such that, for every row i, $K \not\subseteq N_i := \{j : m_{ij} = 1\}$. Choose K to be minimal with this property. Then, for each $j \in K$ there exists a distinct row i_j such that $K \cap N_{i_j} = K \setminus \{j\}$.

Let $w_j = 1$ for $j \in K$ and 0 otherwise. Consider the solution $x_j = \frac{1}{|K|-1}$ for $j \in K$ and 0 otherwise. It satisfies the constraints $Mx \leq \mathbf{1}, \ x \geq 0$ and it has objective value $wx = \frac{|K|}{|K|-1} > 1$, contradicting the assumption that max $\{wx : \ Mx \leq \mathbf{1}, \ x \geq 0\}$ has an integral optimal solution (an integral solution has objective value at most 1 since $x_j = 1$ for at most one node of K). This proves Claim 1.

Claim 2: $G(M)$ is a perfect graph.

Proof: In the 70's, this claim was proved using Lovász's characterization of minimally imperfect graphs [**41**] or Padberg's theorem [**47**]. Since we now have the Strong Perfect Graph Theorem at our disposal, let us use it. Thus, supposing that $G(M)$ is not perfect, $G(M)$ has a node set H that induces an odd hole or the complement of an odd hole. Let $w_j = 1$ for $j \in H$ and 0 otherwise.

If H induces an odd hole, consider the solution $x_j = \frac{1}{2}$ for $j \in H$ and 0 otherwise. It satisfies the constraints $Mx \leq \mathbf{1}, \ x \geq 0$ and it has objective value $wx = \frac{|H|}{2}$, contradicting the assumption that max $\{wx : \ Mx \leq \mathbf{1}, \ x \geq 0\}$ has an integral optimal solution (an integral solution is the characteristic vector of a stable set and thus it has objective value at most $\frac{|H|-1}{2}$ since a stable set intersects H in at most $\frac{|H|-1}{2}$ nodes).

If H induces the complement of an odd hole, consider the solution $x_j = \frac{2}{|H|-1}$ for $j \in H$ and 0 otherwise. It satisfies the constraints $Mx \leq \mathbf{1}, \ x \geq 0$ and it has objective value $wx = \frac{2|H|}{|H|-1} > 2$, contradicting the assumption that max $\{wx : \ Mx \leq \mathbf{1}, \ x \geq 0\}$ has an integral optimal solution (a stable set intersects H in at most 2 nodes). This proves Claim 2.

Since $G(M)$ is perfect, the complement graph $\bar{G}(M)$ is also perfect by the Perfect Graph Theorem. Let w be a nonnegative integral vector. If $w_i = 0$, delete node i from $\bar{G}(M)$ (and remove all the edges incident with node i). If $w_i \geq 2$, replicate node i of $\bar{G}(M)$ so that the resulting graph has w_i copies of node i (including node i itself), pairwise connected. By the Replication Lemma, the resulting graph $\bar{W}(M)$ is perfect. Thus the maximum cardinality of a clique in $\bar{W}(M)$ equals the chromatic number of $\bar{W}(M)$, i.e. a minimum cover of the nodes of $\bar{W}(M)$ by stable sets. In terms of the matrix M, this max-min equation is

$$\begin{array}{llcll} \max & wx & = & \min & \sum_i y_i \\ & Mx \leq \mathbf{1} & & & yM \geq w \\ & x \in \{0,1\}^n & & & y \in Z_+^m. \end{array}$$

Therefore the linear program max $\{wx : \; Mx \le \mathbf{1}, \; x \ge 0\}$ has an integral optimal dual solution y. In other words,

(i) the linear system $Mx \le \mathbf{1}, \; x \ge 0$ is TDI. $\square$

3. Ideal Clutters

3.1. Blockers. A *transversal* of a clutter $\mathcal{C}$ is a set of vertices that intersects all the edges. The *blocker* $b(\mathcal{C})$ of a clutter $\mathcal{C}$ is the clutter with $V(\mathcal{C})$ as vertex set and the minimal transversals of $\mathcal{C}$ as edge set. That is, $E\big(b(\mathcal{C})\big)$ consists of the minimal members of $\{B \subseteq V(\mathcal{C}) : |B \cap A| \ge 1 \text{ for all } A \in E(\mathcal{C})\}$. In other words, the rows of $M\big(b(\mathcal{C})\big)$ are the minimal 0,1 vectors x^T such that x belongs to the polyhedron $P(\mathcal{C}) = \{x \ge 0 : \; M(\mathcal{C})x \ge \mathbf{1}\}$.

EXAMPLE 3.1. Let G be a graph and s, t be distinct nodes of G. If $\mathcal{C}$ is the clutter of st-paths, then $b(\mathcal{C})$ is the clutter of minimal st-cuts.

Edmonds and Fulkerson [**26**] observed that $b(b(\mathcal{C})) = \mathcal{C}$. Before proving this property, we make the following remark.

REMARK 3.2. Let $\mathcal{H}$ and $\mathcal{K}$ be two clutters defined on the same vertex set. If
(i) every edge of $\mathcal{H}$ contains an edge of $\mathcal{K}$ and
(ii) every edge of $\mathcal{K}$ contains an edge of $\mathcal{H}$,
then $\mathcal{H} = \mathcal{K}$.

THEOREM 3.3. *If $\mathcal{C}$ is a clutter, then $b(b(\mathcal{C})) = \mathcal{C}$.*

PROOF. Let A be an edge of $\mathcal{C}$. The definition of $b(\mathcal{C})$ implies that $|A \cap B| \ge 1$, for every edge B of $b(\mathcal{C})$. So A is a transversal of $b(\mathcal{C})$, i.e. A contains an edge of $b(b(\mathcal{C}))$.

Now let A be an edge of $b(b(\mathcal{C}))$. We claim that A contains an edge of $\mathcal{C}$. Suppose otherwise. Then $V(\mathcal{C}) - A$ is a transversal of $\mathcal{C}$ and therefore it contains an edge B of $b(\mathcal{C})$. But then $A \cap B = \emptyset$ contradicts the fact that A is an edge of $b(b(\mathcal{C}))$. So the claim holds.

Now the theorem follows from Remark 3.2. $\square$

Two 0,1 matrices of the form $M(\mathcal{C})$ and $M(b(\mathcal{C}))$ are said to form a *blocking pair*. The next theorem is an important result due to Lehman [**38**]. It states that, for a blocking pair A, B of 0,1 matrices, the polyhedron P defined by

$$Ax \ge \mathbf{1} \tag{3.1}$$

$$x \ge 0 \tag{3.2}$$

is integral if and only if the polyhedron Q defined by

$$Bx \ge \mathbf{1} \tag{3.3}$$

$$x \ge 0 \tag{3.4}$$

is integral. The proof of this result uses the following remark.

REMARK 3.4.
(i) The rows of B are exactly the 0,1 extreme points of P.
(ii) If an extreme point x of P satisfies $x^T \ge \lambda^T B$ where $\lambda_i \ge 0$ and $\sum \lambda_i = 1$, then x is a 0,1 extreme point of P.

PROOF. (i) follows from the fact that the rows of B are the minimal 0,1 vectors in P.

To prove (ii), note that x is an extreme point of $P_I = \{\chi : \; \chi^T \ge \lambda^T B \text{ where } \lambda_i \ge 0 \text{ and } \sum \lambda_i = 1\}$ for otherwise x would be a convex combination of distinct $x^1, x^2 \in P_I$ and, since $P_I \subseteq P$, this would contradict the assumption that x is an extreme point of P. Now (ii) follows by observing that the extreme points of P_I are exactly the rows of B. $\square$

THEOREM 3.5. (Lehman [**38**]) *A clutter is ideal if and only if its blocker is.*

PROOF. By Theorem 3.3, it suffices to show that if P defined by (3.1)-(3.2) is integral, then Q defined by (3.3)-(3.4) is also integral.

Let a be an arbitrary extreme point of Q. By (3.3), $Ba \geq \mathbf{1}$, i.e. $a^T x \geq 1$ is satisfied by every x such that x^T is a row of B. Since P is an integral polyhedron, it follows from Remark 3.4(i) that $a^T x \geq 1$ is satisfied by all the extreme points of P. By (3.4), $a \geq 0$. Therefore $a^T x \geq 1$ is satisfied by all points in P. Furthermore, $a^T x = 1$ for some $x \in P$. Now, by linear programming duality, we have

$$1 \;=\; \min\{a^T x : \, x \in P\} \;=\; \max\{\lambda^T \mathbf{1} : \, \lambda^T A \leq a^T, \, \lambda \geq 0\}.$$

Therefore, by Remark 3.4(ii), a is a 0,1 extreme point of Q. $\square$

3.2. Deletion, Contraction and Minors. Let $\mathcal{C}$ be a clutter. For $j \in V(\mathcal{C})$, the *contraction* $\mathcal{C}/j$ and *deletion* $\mathcal{C} \setminus j$ are clutters defined as follows: Both have $V(\mathcal{C}) - \{j\}$ as vertex set, $E(\mathcal{C}/j)$ is the set of minimal members in $\{S - \{j\} : S \in E(\mathcal{C})\}$ and $E(\mathcal{C} \setminus j) = \{S : j \notin S \in E(\mathcal{C})\}$.

EXAMPLE 3.6. Let $\mathcal{C}$ be the clutter of st-paths in a graph G where s, t are distinct nodes of G, and let j be an edge of G. Then $\mathcal{C}/j$ is the clutter of st-paths in the graph obtained from G by contracting edge j in the graph theoretical sense (West [**56**] page 65) and $\mathcal{C} \setminus j$ is the clutter of st-paths in the graph obtained from G by deleting the edge j in the graphical sense. See Figure 5 for an example.

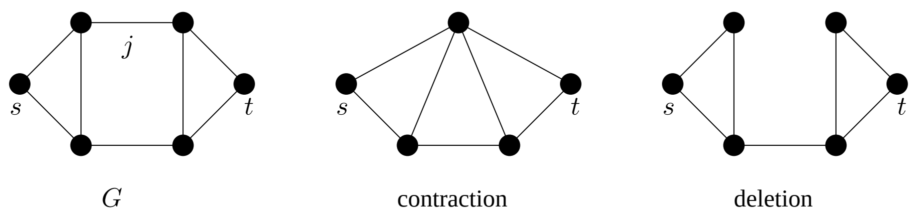

FIGURE 5. Examples of contraction and deletion in a graph.

Let $\mathcal{C}$ be a clutter and let M be the corresponding clutter matrix. Contracting $j \in V(\mathcal{C})$ corresponds to removing column j from matrix M as well as the resulting dominating rows. In the set covering constraints $Mx \geq \mathbf{1}$ of (1.1), this corresponds to setting $x_j = 0$. Deleting j corresponds to removing column j from M as well as all the rows with a 1 in column j. This corresponds to setting $x_j = 1$ in $Mx \geq \mathbf{1}$.

EXAMPLE 3.7. For the graphs of Figure 5, the clutters of st-paths have the following clutter matrices, where j corresponds to the last column of matrix M.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix},$$

$$M/j = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad M \setminus j = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Contractions and deletions of distinct vertices can be performed sequentially, and it is easy to show that the result does not depend on the order.

PROPOSITION 3.8. *For a clutter $\mathcal{C}$ and distinct vertices j_1, j_2,*

(i) $(\mathcal{C}\backslash j_1)\backslash j_2 = (\mathcal{C}\backslash j_2)\backslash j_1$
(ii) $(\mathcal{C}/j_1)/j_2 = (\mathcal{C}/j_2)/j_1$
(iii) $(\mathcal{C}\backslash j_1)/j_2 = (\mathcal{C}/j_2)\backslash j_1$

PROOF. Use the definitions of contraction and deletion. □

DEFINITION 3.9. A clutter $\mathcal{D}$ obtained from $\mathcal{C}$ by a sequence of deletions and contractions is a *minor* of $\mathcal{C}$.

If V_1 and V_2 are disjoint subsets of $V(\mathcal{C})$, we let $\mathcal{C}/V_1\backslash V_2$ be the minor obtained from $\mathcal{C}$ by contracting all vertices of V_1 and deleting all vertices of V_2. If $V_1 \neq \emptyset$ or $V_2 \neq \emptyset$, the minor is *proper*.

PROPOSITION 3.10. *For a clutter $\mathcal{C}$ and $U \subset V(\mathcal{C})$,*

(i) $b(\mathcal{C}\backslash U) = b(\mathcal{C})/U$
(ii) $b(\mathcal{C}/U) = b(\mathcal{C})\backslash U$

PROOF. Use the definitions of contraction, deletion and blocker. □

We leave it as an exercise to prove the following result.

PROPOSITION 3.11. *If a clutter is ideal, then so are all its minors.*

COROLLARY 3.12. *Let M be a 0,1 matrix. The following are equivalent.*

(i) *The polyhedron $\{x \geq \mathbf{0},\ Mx \geq \mathbf{1}\}$ is integral.*
(ii) *The bounded polyhedron $\{0 \leq x \leq \mathbf{1},\ Mx \geq \mathbf{1}\}$ is integral.*

3.3. st-cuts and st-paths. Consider a graph G and distinct nodes $s, t \in V(G)$. Let $\mathcal{C}$ be the clutter where $V(\mathcal{C}) = E(G)$ and $E(\mathcal{C})$ is the family of st-paths of G.

THEOREM 3.13. (Ford and Fulkerson [**30**]) *The clutter $\mathcal{C}$ of st-paths has the MFMC property.*

This theorem is a restatement of the famous Max Flow Min Cut theorem of Ford-Fulkerson: For any vector $w \in \mathbb{Z}_+^{E(G)}$ of capacities, the minimum capacity of an st-cut equals the maximum number of st-paths such that every edge $e \in E(G)$ belongs to at most w_e of the paths. Indeed, the Ford-Fulkerson theorem states that both (1.1) and (1.2) have optimal solutions that are integral whenever the capacity vector w is integral.

Theorem 3.13 implies that $\mathcal{C}$ is ideal and therefore

$$\{x \in \mathbb{R}_+^{E(G)} : x(P) \geq 1 \text{ for all } st\text{-paths } P\}$$

is an integral polyhedron. Its extreme points are the minimal st-cuts. It will be convenient to refer to minimal st-cuts simply as st-cuts.

As a consequence of Lehman's theorem (Theorem 3.5), the clutter of st-cuts is also ideal. So

$$\{x \in \mathbb{R}_+^{E(G)} : x(C) \geq 1 \text{ for all } st\text{-cuts } C\}$$

is an integral polyhedron. In fact, it is easy to show that the clutter of st-cuts has the MFMC property.

3.4. T-cuts and T-joins. Consider a connected graph G with nonnegative edge weights w_e, for $e \in E(G)$. The Chinese Postman Problem consists in finding a minimum weight closed walk going through each edge at least once (the edges of the graph represent roads where mail must be delivered and w_e is the length of the road). Equivalently, the postman must find a minimum weight set of edges $J \subseteq E(G)$ such that $J \cup E(G)$ induces an Eulerian graph, i.e. J induces a graph whose odd degree nodes coincide with the odd degree nodes of G. Since $w \geq 0$, we can assume w.l.o.g. that J is acyclic. Such an edge set J is called a *postman set*.

The problem is generalized as follows. Let G be a graph and T a node set of G of even cardinality. An edge set J of G is called a *T-join* if it induces an acyclic graph the odd degree nodes of which coincide with T. For disjoint node sets S_1, S_2, let (S_1, S_2) denote the set of edges with one endnode in S_1 and the other in S_2. A *T-cut* is a minimal edge set of the form $(S, V(G) - S)$ where S is a set of nodes with $|T \cap S|$ odd. Clearly every T-cut intersects every T-join. Note that when $|T| = 2$, say $T = \{s, t\}$, T-joins and T-cuts are nothing but st-paths and st-cuts. Therefore the theory of T-joins and T-cuts is a generalization of Ford and Fulkerson's network flow theory. In this section, we will show that the clutters of T-joins and T-cuts are ideal.

Edmonds and Johnson [**28**] considered the problem of finding a minimum weight T-join. One way to solve this problem is to reduce it to the perfect matching problem in a complete graph K_p, where $p = |T|$. Namely, compute the lengths of shortest paths in G between all pairs of nodes in T, use these values as edge weights in K_p and find a minimum weight perfect matching in K_p. The union of the corresponding paths in G is a minimum weight T-join. Edmonds and Johnson developed a direct primal-dual algorithm for the minimum weight T-join problem and, as a by-product, obtained that the clutter of T-cuts is ideal.

THEOREM 3.14. (Edmonds and Johnson [**28**]) *The polyhedron*

$$x(C) \geq 1 \quad \text{for all } T\text{-cuts } C \tag{3.5}$$

$$x_e \geq 0 \quad \text{for all } e \in E(G). \tag{3.6}$$

is an integral polyhedron.

In the next section, we give a non-algorithmic proof of this theorem suggested by Pulleyblank [**49**].

The Edmonds-Johnson theorem together with the fact that the blocker of an ideal clutter is ideal (Theorem 3.3 of Lehman) implies that the clutter of T-joins is also ideal. That is

$$\begin{aligned} x(J) &\geq 1 \quad \text{for all } T\text{-joins } J \\ x_e &\geq 0 \quad \text{for all } e \in E(G). \end{aligned}$$

is an integral polyhedron.

The clutter of T-cuts does not pack, nor does the clutter of T-joins. The reader is encouraged to find examples showing this.

Proof of the Edmonds-Johnson Theorem

First, we prove the following lemma. For $v \in V(G)$, let $\delta(v)$ denote the set of edges incident with v. A *star* is a tree where one node is adjacent to all the other nodes.

LEMMA 3.15. *Let $\tilde{x}$ be an extreme point of the polyhedron*

$$x(\delta(v)) \geq 1 \quad \textit{for all} \quad v \in T \tag{3.7}$$

$$x_e \geq 0 \quad \textit{for all} \quad e \in E(G). \tag{3.8}$$

The connected components of the graph $\tilde{G}$ induced by the edges such that $\tilde{x}_e > 0$ are either

(i) *odd cycles with nodes in T and edges $\tilde{x}_e = 1/2$, or*
(ii) *stars with nodes in T, except possibly the center, and edges $\tilde{x}_e = 1$.*

PROOF. Every connected component C of $\tilde{G}$ is either a tree or contains a unique cycle, since the number of edges in C is at most the number of inequalities (3.7) that hold with equality.

Assume first that C contains a unique cycle. Then (3.7) holds with equality for all nodes of C, which are therefore in T. Now C is a cycle since, otherwise, C has a pendant edge e with $\tilde{x}_e = 1$ and therefore C is disconnected, a contradiction. If C is an even cycle, then by alternately increasing and decreasing $\tilde{x}$ around the cycle by a small ϵ ($-\epsilon$ respectively), $\tilde{x}$ can be written as a convex combination of two points satisfying (3.7) and (3.8). So (i) must hold.

Assume now that C is a tree. Then (3.7) holds with equality for at least $|V(C)| - 1$ nodes of C. In particular, it holds with equality for at least one node of degree one. Since C is connected, this implies that C is a star and (ii) holds. □

Proof of Theorem 3.14: In order to prove the theorem, it suffices to show that every extreme point $\tilde{x}$ of the polyhedron (3.5)–(3.6) is the characteristic vector of a T-join. We proceed by induction on the number of nodes of G.

Suppose first that $\tilde{x}$ is an extreme point of the polyhedron (3.7)–(3.8). Consider a connected component of the graph $\tilde{G}$ induced by the edges such that $\tilde{x}_e > 0$ and let S be its node set. Since $\tilde{x}(S, V(G) - S) = 0$, it follows from (3.5) that S contains an even number of nodes of T. By Lemma 3.15, $\tilde{G}$ contains no odd cycle, showing that $\tilde{x}$ is an integral vector. Furthermore, $\tilde{x}$ is the characteristic vector of a T-join since, by Lemma 3.15 again, the component of $\tilde{G}$ induced by S is a star and $|S \cap T|$ even implies that the center is in T if and only if the star has an odd number of edges.

Assume now that $\tilde{x}$ is not an extreme point of the polyhedron (3.7)–(3.8). Then there is some T-cut $C = (V_1, V_2)$ with $|V_1| \geq 2$ and $|V_2| \geq 2$ such that

$$\tilde{x}(C) = 1.$$

Let $G_1 = (V_1 \cup \{v_2\}, E_1)$ be the graph obtained from G by contracting V_2 to a single node v_2. Similarly, $G_2 = (V_2 \cup \{v_1\}, E_2)$ is the graph obtained from G by contracting V_1 to a single node v_1. The new nodes v_1, v_2 belong to T. For $i = 1, 2$, let $\tilde{x}^i$ be the restriction of $\tilde{x}$ to E_i. Since every T-cut of G_i is also a T-cut of G, it follows by induction that $\tilde{x}^i$ is greater than or equal to a convex combination of characteristic vectors of T-joins of G_i. Let $\mathcal{T}_i$ be this set of T-joins. Each T-join in $\mathcal{T}_i$ has exactly one edge incident with v_i. Since $\tilde{x}^1$ and $\tilde{x}^2$ coincide on the edges of C, it follows that the T-joins of $\mathcal{T}_1$ can be combined with those of $\mathcal{T}_2$ to form T-joins of G and that $\tilde{x}$ is greater than or equal to a convex combination of characteristic vectors of T-joins of G. Since $\tilde{x}$ is an extreme point, it is the characteristic vector of a T-join. □

We have just proved that the clutter of T-cuts is ideal. It does not have the MFMC property in general graphs. However Seymour proved that it does in bipartite graphs. Seymour also showed that, in a general graph, if the edge weights

w_e are integral and their sum is even in every cycle, then the dual variables y in (1.2) can be chosen to be integral in an optimum solution.

3.5. Odd Cycles in Graphs. In this section, we consider the clutter $\mathcal{O}_G$ of odd cycles in a graph G, namely $V(\mathcal{O}_G) = E(G)$ and $E(\mathcal{O}_G)$ is the family of edge sets of the odd cycles in G.

EXAMPLE 3.16. For the complete graph K_4 on four nodes, the clutter of odd cycles has six vertices (the six edges of K_4) and four edges (the four triangles of K_4). The reader can check that its clutter matrix is Q_6. Recall that Q_6 is ideal but does not have the MFMC property.

For the complete graph K_5 on five nodes, the clutter of odd cycles has ten vertices (the ten edges of K_5) and its edges comprise all the triangles (there are ten) and all the pentagons (there are twelve). Note that $x_j = \frac{1}{3}$ for $j = 1, \ldots, 10$ is an extreme point of the polyhedron $\{x \in \mathbb{R}^{10}_+ : \ \mathcal{O}_{K_5} x \geq \mathbf{1}\}$ and it is obviously nonintegral. Therefore $\mathcal{O}_{K_5}$ is not ideal.

Seymour [**54**] characterized exactly the graphs G for which $\mathcal{O}_G$ has the MFMC property and Guenin [**35**] characterized exactly when $\mathcal{O}_G$ is ideal.

THEOREM 3.17. (Seymour [**54**]) *The clutter of odd cycles in a graph G has the MFMC property if and only if it contains no Q_6 minor.*

THEOREM 3.18. (Guenin [**35**]) *The clutter of odd cycles in a graph G is ideal if and only if it contains no $\mathcal{O}_{K_5}$ minor.*

A proof of these theorems can be found in Chapter 75 of Schrijver's book on *Combinatorial Optimization* [**53**].

3.6. Minimally Nonideal Matrices. Lehman (Theorem 3.5) showed that ideal 0,1 matrices always come in pairs (if M is ideal, so is its blocker $b(M)$). Another important result of Lehman about ideal 0,1 matrices is the following theorem, which is reminiscent of Theorem 1.11 on perfect matrices.

THEOREM 3.19. (Lehman [**39**]) *For a 0,1 matrix A, the following statements are equivalent:*

(i) *the matrix A is ideal,*
(ii) *min $\{cx : \ Ax \geq \mathbf{1}, \ x \geq \mathbf{0}\}$ has an integral optimal solution x for all $c \in \{0, 1, +\infty\}^n$.*

The fact that (i) implies (ii) is an immediate consequence of the definition of idealness. The difficult part of Lehman's theorem is that (ii) implies (i). The main purpose of this section is to prove this result. This is done by studying properties of minimally nonideal matrices.

Lehman's Characterization

A 0,1 matrix A is *minimally nonideal* (*mni*) if

(i) A is a clutter matrix,
(ii) $Q(A) = \{x \geq 0 : \ Ax \geq \mathbf{1}\}$ is not an integral polyhedron,
(iii) For every $i = 1, \ldots, n$, both $Q(A) \cap \{x : \ x_i = 0\}$ and $Q(A) \cap \{x : \ x_i = 1\}$ are integral polyhedra.

If A is *mni*, the clutter $\mathcal{C}(A)$ is also called *mni*. Equivalently, a clutter $\mathcal{C}$ is *mni* if it is not ideal but all its proper minors are ideal.

For $t \geq 2$ integer, let $\mathcal{J}_t$ denote the clutter with $t + 1$ vertices and edges corresponding, respectively, to the points and lines of the finite degenerate projective plane. Namely, $V(\mathcal{J}_t) = \{0, \ldots, t\}$, and $E(\mathcal{J}_t) = \{\{1, \ldots, t\}, \{0, 1\}, \{0, 2\}, \ldots, \{0, t\}\}$.

A matrix A is *isomorphic* to a matrix B if B can be obtained from A by a permutation of rows and a permutation of columns.

Let J denote a square matrix all of whose entries are 1's, and let I be the identity matrix. Given a *mni* matrix A, let $\bar{x}$ be a nonintegral extreme point of the polyhedron $Q(A) = \{x \geq 0 : \ Ax \geq \mathbf{1}\}$, that is, at least one component $\bar{x}_j$ of $\bar{x}$ is fractional. The maximum row submatrix $\bar{A}$ of A such that $\bar{A}\bar{x} = \mathbf{1}$ is called a *core* of A. So A has a core for each nonintegral extreme point of $Q(A)$.

THEOREM 3.20. (Lehman [**39**]) *Let A be a mni matrix and $B = b(A)$. Then*
- (i) *A has a unique core $\bar{A}$ and B has a unique core $\bar{B}$;*
- (ii) *$\bar{A}$ and $\bar{B}$ are square matrices;*
- (iii) *Either A is isomorphic to $M(\mathcal{J}_t)$, $t \geq 2$, or the rows of $\bar{A}$ and $\bar{B}$ can be permuted so that*
$$\bar{A}\bar{B}^T = J + dI$$
for some positive integer d.

Lehman's proof of this theorem is rather terse. Seymour [**55**], Padberg [**48**] and Gasparyan, Preissmann and Sebö [**34**] give more accessible presentations of Lehman's proof. In the next section, we present a proof of Lehman's theorem following Padberg's polyhedral point of view. Before proving Theorem 3.20, we present some of its consequences.

Bridges and Ryser [**3**] studied square matrices Y, Z that satisfy the matrix equation $YZ = J + dI$.

THEOREM 3.21. (Bridges and Ryser [**3**]) *Let Y and Z be $n \times n$ 0,1 matrices such that $YZ = J + dI$ for some positive integer d. Then*
- (i) *each row and column of Y has the same number r of ones, each row and column of Z has the same number s of ones with $rs = n + d$,*
- (ii) *$YZ = ZY$,*

PROOF. It is straightforward to check that $(J + dI)^{-1} = \frac{1}{d}I - \frac{1}{d(n+d)}J$. Hence
$$YZ = J + dI \Rightarrow YZ(\frac{1}{d}I - \frac{1}{d(n+d)}J) = I \Rightarrow Z(\frac{1}{d}I - \frac{1}{d(n+d)}J)Y = I$$
$$\text{i.}e. \qquad ZY = \frac{1}{n+d}ZJY + dI = \frac{1}{n+d}\mathbf{s}\mathbf{r}^T + dI$$
where $\mathbf{s} = Z\mathbf{1}$ and $\mathbf{r} = Y^T\mathbf{1}$.

It follows that, for each i and j, $n + d$ divides $r_i s_j$. On the other hand, the trace of the matrix ZY is equal to the trace of YZ, which is $n(d+1)$. This implies $\frac{1}{n+d}(\sum_1^n s_i r_i) = n$ and, since $s_i > 0$ and $r_i > 0$, we have $r_i s_i = n+d$. Now consider distinct i, j. Since $r_i s_i = r_j s_j = n + d$ and $n + d$ divides $r_i s_j$ and $r_j s_i$, it follows that $r_i = r_j$ and $s_i = s_j$. Therefore, all columns of Z have the same sum s and all rows of Y have the same sum r. Furthermore, $ZY = J + dI$ and, by symmetry, all columns of Y have the same sum and all rows of Z have the same sum. □

Theorems 3.20 and 3.21 have the following consequence.

COROLLARY 3.22. *Let A be a mni matrix nonisomorphic to $M(\mathcal{J}_t)$. Then it has a non-singular row submatrix $\bar{A}$ with exactly r ones in every row and column. Moreover, rows of A not in $\bar{A}$ have at least $r + 1$ ones.*

This implies the next result, which is a restatement of Theorem 3.19.

COROLLARY 3.23. *Let A be a $0, 1$ matrix. $Q(A) = \{x \in R^n_+ : Ax \geq \mathbf{1}\}$ is an integral polyhedron if and only if $\min\{wx : x \in Q(A)\}$ has an integral optimal solution for all $w \in \{0, 1, \infty\}^n$.*

Note that Theorem 1.9 mentioned in the introduction follows from Corollary 3.23.

Proof of Lehman's Theorem

Let A be an $m \times n$ *mni* matrix, $\bar{x}$ a nonintegral extreme point of $Q(A) = \{x \in R^n_+ : Ax \geq \mathbf{1}\}$ and $\bar{A}$ a core of A. That is, $\bar{A}$ is the maximal row submatrix of A such that $\bar{A}\bar{x} = \mathbf{1}$. For simplicity of notation, assume that $\bar{A}$ corresponds to the first p rows of A, i.e. the entries of $\bar{A}$ are a_{ij} for $i = 1, \dots, p$ and $j = 1, \dots, n$. Since A is *mni*, every component of $\bar{x}$ is nonzero. Therefore $p \geq n$ and $\bar{A}$ has no row or column containing only 0's or only 1's.

The following easy result will be applied to the bipartite representation G of the 0,1 matrix $J - \bar{A}$ where J denotes the $p \times n$ matrix of all 1's, namely ij is an edge of G if and only if $a_{ij} = 0$, for $1 \leq i \leq p$ and $1 \leq j \leq n$. Let $d(u)$ denote the degree of node u.

Lemma 3.24. (de Bruijn and Erdös [**24**]) *Let $(I \cup J, E)$ be a bipartite graph with no isolated node. If $|I| \geq |J|$ and $d(i) \geq d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$, then $|I| = |J|$ and $d(i) = d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$.*

Proof. $|I| = \sum_{i \in I}(\sum_{j \in N(i)} \frac{1}{d(i)}) \leq \sum_{i \in I} \sum_{j \in N(i)} \frac{1}{d(j)} = \sum_{j \in J} \sum_{i \in N(j)} \frac{1}{d(j)} = |J|$. Now the hypothesis $|I| \geq |J|$ implies that equality holds throughout. So $|I| = |J|$ and $d(i) = d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$. □

The key to proving Lehman's theorem is the following lemma. We refer the reader to Sections 8.1 to 8.5 in Chapter 8 of Schrijver [**52**] for any undefined term related to polyhedra.

Lemma 3.25. *$p = n$ and, if $a_{ij} = 0$ for $1 \leq i, j \leq n$, then row i and column j of $\bar{A}$ have the same number of ones.*

Proof. Let x^j be defined by

$$x^j_k = \begin{cases} \bar{x}_k & \text{if} \quad k \neq j \\ 1 & \text{if} \quad k = j \end{cases}$$

and let F_j be the face of $Q(A) \cap \{x_j = 1\}$ of smallest dimension that contains x^j. Since A is *mni*, F_j is an integral polyhedron. The proof of the lemma will follow unexpectedly from computing the dimension of F_j.

The point x^j lies at the intersection of the hyperplanes in $\bar{A}x = \mathbf{1}$ such that $a_{kj} = 0$ (at least $n - \sum_{k=1}^p a_{kj}$ such hyperplanes are independent since $\bar{A}$ has rank n) and of the hyperplane $x_j = 1$ (independent of the previous hyperplanes). It follows that

$$dim(F_j) \leq n - (n - \sum_{k=1}^{p} a_{kj} + 1) = \sum_{k=1}^{p} a_{kj} - 1$$

Choose a row a^i of $\bar{A}$ such that $a_{ij} = 0$. Since $x^j \in F_j$, it is greater than or equal to a convex combination of extreme points b^ℓ of F_j, say $x^j \geq \sum_{\ell=1}^t \gamma_\ell b^\ell$, where $\gamma > 0$ and $\sum \gamma_\ell = 1$.

$$1 = a^i x^j \geq \sum_{\ell=1}^{t} \gamma_\ell a^i b^\ell \geq 1 \tag{3.9}$$

Therefore, equality must hold throughout. In particular $a^i b^\ell = 1$ for $\ell = 1, \dots, t$. Since b^ℓ is a 0,1 vector, it has exactly one nonzero entry in the set of columns k where $a_{ik} = 1$. Another consequence of the fact that equality holds in (3.9) is that

$x^j_k = \sum_{\ell=1}^t \gamma_\ell b^\ell_k$ for every k where $a_{ik} = 1$. Now, since $x^j_k > 0$ for all k, it follows that F_j contains at least $\sum_{k=1}^n a_{ik}$ linearly independent points b^ℓ, i.e.

$$dim(F_j) \geq \sum_{k=1}^n a_{ik} - 1.$$

Therefore, $\sum_{k=1}^n a_{ik} \leq \sum_{k=1}^p a_{kj}$ for all i, j such that $a_{ij} = 0$.

Now Lemma 3.24 applied to the bipartite representation of $J - \bar{A}$ implies that $p = n$ and

$$\sum_{k=1}^n a_{ik} = \sum_{k=1}^n a_{kj} \text{ for all } i, j \text{ such that } a_{ij} = 0.$$

□

LEMMA 3.26. *$\bar{x}$ has exactly n adjacent extreme points in $Q(A)$, all with 0,1 coordinates.*

PROOF. By Lemma 3.25, exactly n inequalities of $A\bar{x} \geq \mathbf{1}$ hold with equality, namely $\bar{A}\bar{x} = \mathbf{1}$. In the polyhedron $Q(A)$, an edge adjacent to $\bar{x}$ is defined by $n-1$ of the n equalities in $\bar{A}x = \mathbf{1}$. Moving along such an edge from $\bar{x}$, at least one of the coordinates decreases. Since $Q(A) \in R^n_+$, this implies that $\bar{x}$ has exactly n adjacent extreme points on $Q(A)$. Suppose $\bar{x}$ has a nonintegral adjacent extreme point $\bar{x}'$. Since A is mni, $0 < \bar{x}'_j < 1$ for all j. Let $\bar{A}'$ be the $n \times n$ nonsingular submatrix of A such that $\bar{A}'\bar{x}' = \mathbf{1}$. Since $\bar{x}$ and $\bar{x}'$ are adjacent on $Q(A)$, $\bar{A}$ and $\bar{A}'$ differ in only one row. W.l.o.g. assume that $\bar{A}'$ corresponds to rows 2 to $n+1$. Since A contains no dominating row, there exists j such that $a_{1j} = 0$ and $a_{n+1,j} = 1$. Since $\bar{A}'$ cannot contain a column with only 1's, $a_{ij} = 0$ for some $2 \leq i \leq n$. But now, Lemma 3.24 is contradicted with row i and column j in either $\bar{A}$ or $\bar{A}'$. □

Lemma 3.26 has the following implication. Let $\bar{B}$ denote the $n \times n$ 0,1 matrix whose rows are the extreme points of $Q(A)$ adjacent to $\bar{x}$. By Remark 3.4(i), $\bar{B}$ is a submatrix of B. By Lemma 3.26, $\bar{B}$ satisfies the matrix equation

$$\bar{A}\bar{B}^T = J + D$$

where J is the matrix of all 1's and D is a diagonal matrix with positive diagonal entries $d_1, \ldots, d_n$.

LEMMA 3.27. *Either*

(i) *$\bar{A} = \bar{B}$ are isomorphic to $M(\mathcal{J}_t)$, for $t \geq 2$, or*
(ii) *$D = dI$, where d is a positive integer.*

PROOF. Consider the bipartite representation G of the 0,1 matrix $J - \bar{A}$.

Case 1: G is connected.

Then it follows from Lemma 3.25 that

$$\sum_k a_{ik} = \sum_k a_{kj} \text{ for all } i, j. \tag{3.10}$$

Let α denote this common row and column sum.

$$(n + d_1, \ldots, n + d_n) = \mathbf{1}^T(J + D) = \mathbf{1}^T\bar{A}\bar{B}^T = (\mathbf{1}^T\bar{A})\bar{B}^T = \alpha\mathbf{1}^T\bar{B}^T$$

Since there is at most one d, $1 \leq d < \alpha$, such that $n + d$ is a multiple of α, all d_i must be equal to d, i.e. $D = dI$.

Case 2: G is disconnected.

Let $q \geq 2$ denote the number of connected components in G and let

$$\bar{A} = \begin{pmatrix} K_1 & & \mathbf{1} \\ & \dots & \\ \mathbf{1} & & K_q \end{pmatrix}$$

where K_t are 0,1 matrices, for $t = 1, \dots, q$. It follows from Lemma 3.25 that the matrices K_t are square and $\sum_k a_{ik} = \sum_k a_{kj} = \alpha_t$ in each K_t.

Suppose first that $\bar{A}$ has no row with $n-1$ ones. Then every K_t has at least two rows and columns. We claim that, for every j, k, there exist i, l such that $a_{ij} = a_{ik} = a_{lj} = a_{lk} = 1$. The claim is true if $q \geq 3$ or if $q = 2$ and j, k are in the same component (simply take two rows i, l from a different component). So suppose $q = 2$, column j is in K_1 and column k is in K_2. Since no two rows are identical, we must have $\alpha_1 \geq 1$, i.e. $a_{ij} = 1$ for some row i of K_1. Similarly, $a_{lk} = 1$ for some row l of K_2. The claim follows.

For each row b of $\bar{B}$, the vector $\bar{A}b^T$ has an entry greater than or equal to 2, so there exist two columns j, k such that $b_j = b_k = 1$. By the claim, there exist rows a_i and a_l of $\bar{A}$ such that $a_i b^T \geq 2$ and $a_l b^T \geq 2$, contradicting the fact that $\bar{A}b^T$ has exactly one entry greater than 1.

Therefore $\bar{A}$ has a row with $n-1$ ones. Now it is routine to check that $\bar{A}$ is isomorphic to $M(\mathcal{J}_t)$, for $t \geq 2$. □

To complete the proof of Theorem 3.20, it only remains to show that the core $\bar{A}$ is unique and that $\bar{B}$ is a core of B and is unique.

If $\bar{A} = M(\mathcal{J}_t)$ for some $t \geq 2$, then the fact that A has no dominated rows implies that $A = \bar{A}$. Thus $B = \bar{B} = M(\mathcal{J}_t)$. So, the theorem holds in this case.

If $\bar{A}\bar{B}^T = J + dI$ for some positive integer d, then, by Theorem 3.21, all rows of $\bar{A}$ contain r ones. Therefore, $\bar{x}_j = \frac{1}{r}$, for $j = 1, \dots, n$. The feasibility of $\bar{x}$ implies that all rows of A have at least r ones, and Lemma 3.25 implies that exactly n rows of A have r ones. Now $Q(A)$ cannot have a nonintegral extreme point $\bar{x}'$ distinct from $\bar{x}$, since the above argument applies to $\bar{x}'$ as well. Therefore A has a unique core $\bar{A}$. Since $\bar{x}$ has exactly n neighbors in $Q(A)$ and they all have s components equal to one, the inequality $\sum_1^n x_i \geq s$ is valid for the 0,1 points in $Q(A)$. This shows that every row of B has at least s ones and exactly n rows of B have s ones. Since B is *mni*, $\bar{B}$ is the unique core of B. □

Examples of *mni* Clutters

Let $Z_n = \{0, \dots, n-1\}$. We define addition of elements in Z_n to be addition modulo n. Let $k \leq n-1$ be a positive integer. For each $i \in Z_n$, let C_i denote the subset $\{i, i+1, \dots, i+k-1\}$ of Z_n. Define the *circulant* clutter $\mathcal{C}_n^k$ by $V(\mathcal{C}_n^k) = Z_n$ and $E(\mathcal{C}_n^k) = \{C_0, \dots, C_{n-1}\}$.

Lehman [**38**] gave three infinite classes of minimally nonideal clutters: $\mathcal{C}_n^2$, $n \geq 3$ odd, their blockers, and the degenerate projective planes $\mathcal{J}_n$, $n \geq 2$.

Conjecture 3.28. (Cornuéjols and Novick [**23**]) *There exists n_0 such that, for $n \geq n_0$, all mni matrices have a core isomorphic to $\mathcal{C}_n^2$, $\mathcal{C}_n^{\frac{n+1}{2}}$ for $n \geq 3$ odd, or $\mathcal{J}_n$, for $n \geq 2$.*

However, there exist several known "small" *mni* matrices that do not belong to any of the above classes. For example, Lehman [**38**] noted that $\mathcal{F}_7$ is *mni*. $\mathcal{F}_7$ is the clutter with 7 vertices and 7 edges corresponding to points and lines of the

Fano plane (finite projective geometry on 7 points):

$$M(\mathcal{F}_7)=\begin{pmatrix}1&1&0&1&0&0&0\\0&1&1&0&1&0&0\\0&0&1&1&0&1&0\\0&0&0&1&1&0&1\\1&0&0&0&1&1&0\\0&1&0&0&0&1&1\\1&0&1&0&0&0&1\end{pmatrix}$$

Let K_5 denote the complete graph on five nodes and let $\mathcal{O}_{K_5}$ denote the clutter whose vertices are the edges of K_5 and whose edges are the odd cycles of K_5 (the triangles and the pentagons). Seymour [**54**] noted that $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, and $\mathcal{C}_9^2$ with the extra edge $\{3,6,9\}$ are *mni*.

Ding [**25**] found the following *mni* clutter: $V(\mathcal{D}_8)=\{1,\dots,8\}$ and

$$E(\mathcal{D}_8)=\{\{1,2,6\},\{2,3,5\},\{3,4,8\},\{4,5,7\},\{2,5,6\},\{1,6,7\},\{4,7,8\},\{1,3,8\}\}.$$

Cornuéjols and Novick [**23**] characterized the *mni* circulant clutters $\mathcal{C}_n^k$. They showed that the following ten clutters are the only *mni* $\mathcal{C}_n^k$ for $k\geq 3$:

$$\mathcal{C}_5^3,\ \mathcal{C}_8^3,\ \mathcal{C}_{11}^3,\ \mathcal{C}_{14}^3,\ \mathcal{C}_{17}^3,\ \mathcal{C}_7^4,\ \mathcal{C}_{11}^4,\ \mathcal{C}_9^5,\ \mathcal{C}_{11}^6,\ \mathcal{C}_{13}^7.$$

Independently, Qi [**50**] discovered $\mathcal{C}_9^5$ and $\mathcal{C}_{11}^6$ and Ding [**25**] discovered $\mathcal{C}_8^3$.

Let $\mathcal{T}_{K_5}$ denote the clutter whose vertices are the edges of K_5 and whose edges are the triangles of K_5 (interestingly, $M(\mathcal{T}_{K_5})$ is also the node-node adjacency matrix of the Petersen graph). It can be shown that $\mathcal{T}_{K_5}$, $core(b(\mathcal{T}_{K_5}))$ and their blockers are *mni*. Often, when a *mni* clutter $\mathcal{H}$ has the property that $core(\mathcal{H})$ and $core(b(\mathcal{H}))$ are also *mni*, many more *mni* clutters can be constructed from $\mathcal{H}$ and from $b(\mathcal{H})$, see [**23**]. For example, Cornuéjols and Novick [**23**] have constructed more than one thousand mni clutters from $\mathcal{T}_{K_5}$. More results can be found in [**46**].

Lütolf and Margot [**42**] designed a computer program that enumerates possible cores of minimally nonideal matrices. It first enumerates the square 0,1 matrices Y, Z that satisfy the matrix equation $YZ=J+dI$, and then checks that the covering polyhedron has a unique nonintegral extreme point. Lütolf and Margot [**42**] enumerated all square *mni* matrices of dimension at most 12×12 and found 20 such matrices (previously, only 15 were known); they found 13 new square *mni* matrices of dimensions 14×14 and 17×17; and they found 38 new nonsquare *mni* matrices with 11, 14 and 17 columns with nonisomorphic cores. The overwhelming majority of these examples have $d=1$: Only three cores with $d=2$ are known (namely $\mathcal{F}_7,\mathcal{T}_{K_5}$ and the core of its blocker) and none with $d\geq 3$.

4. Conclusion

In these notes we considered the min-max equation (1.1)=(1.2) and the max-min equation (1.3)=(1.4). We focused on the cases where the polyhedra in (1.1) and (1.3) are integral. When this occurs, the matrix A is called ideal and perfect respectively. There are striking similarities between perfect and ideal matrices, such as Theorems 1.11 and 3.19. But there are also important differences. For instance, if A is perfect then (1.4) has an integral optimal solution for every integral w, whereas if A is ideal it is not true that (1.3) has an integral optimal solution for every integral w (e.g. $A=Q_6$ and $w=1$). The Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour and Thomas together with Chvátal's connection between perfect graphs and perfect matrices imply an explicit characterization of all minimally imperfect matrices whereas the situation for minimally nonideal matrices looks more complex, due to the existence of numerous individual examples (see

Section 3.6). For perfect matrices, there is a beautiful polyhedral theorem (Theorem 1.11), an "excluded structure" theorem (the Strong Perfect Graph Theorem) and a decomposition theorem (Theorem 2.15). For ideal matrices, no decomposition theorem is known and there is no excluded structure theorem that provides an explicit list of minimally nonideal matrices, although Theorem 3.20 provides a partial answer. Finally, there is a nice polyhedral theorem for ideal matrices (Theorem 3.19). The counterpart of this polyhedral theorem for matrices with the MFMC property is still open (Conjecture 1.7).

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