Every planar graph is the intersection graph of segments in the plane (extended abstract)

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Abstract. Given a set S of segments in the plane, the *intersection graph* of S is the graph with vertex set S in which two vertices are adjacent if and only if the corresponding two segments intersect. We prove a conjecture of Scheinerman (*PhD Thesis, Princeton University*, 1984) that every planar graph is the intersection graph of some segments in the plane.

1 Introduction

In this paper, we consider intersection models for planar graphs. A *segment model* of a graph G maps every vertex $v \in V(G)$ to a segment $\mathbf{v}$ of the plane so that two segments $\mathbf{u}$ and $\mathbf{v}$ intersect if and only if $uv \in E(G)$. Although this graph family is simply defined, it is not easy to manipulate. Actually, even if this class of graphs is small (there are less than $2^{O(n \log n)}$ such graphs with n vertices [15]) a segment model may be long to encode (in the models of some of these graphs the endpoints of the segments need at least $2^{\sqrt{n}}$ bits to be coded [13]). There are also interesting open problems concerning this class of graphs. For example, we know that deciding whether a graph G admits a segment model is NP-hard [11] but it is still open whether this problem belongs to NP or not. Here we focus on a conjecture proposed by Scheinerman [16], stating that every planar graph has a segment model.

Many work has been done toward this conjecture. Several proofs [3,5,9] have been given for bipartite planar graphs. The case of triangle-free planar graphs was proved by de Castro *et al.* [1] and recently de Fraysseix and Ossona de Mendez [4] proved it for every planar graph that has a 4-coloring in which every induced cycle of length 4 uses at most 3 colors.

Another approach to this problem has been proposed [12,14]. Since it is known [6] that planar graphs are intersection graphs of Jordan arcs in the plane and since two non-parallel segments intersect at most once, it was asked whether planar graphs are intersection graphs of Jordan arcs in the plane if every pair of Jordan arcs $\mathbf{s}_1$ and $\mathbf{s}_2$ intersect at most once and in a non-tangent way (*i.e.* around their intersection point we successively meet $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_1$ and $\mathbf{s}_2$). It was already known when tangent intersection are allowed; indeed every planar graph is the contact graph of touching circles [10]. The authors and Ochem [2] answered positively to this question. This approach of Scheinerman's conjecture was decisive since by improving the proof of this result it yields a proof of Scheinerman's conjecture that we present here. However, the construction we give here does not exactly correspond to a stretching of the strings of the construction given in [2].

The paper is organized as follows. In Section 2 we give some definitions. In particular we define premodels and we outline how to obtain a segment model from a premodel. In Section 3 we describe premodels that exist for 3-bounded W-triangulations, a family of plane graphs including 4-connected triangulations. Then in Section 4 we finally construct segment models for general triangulations, which implies the existence of segment models for general planar graphs.

Due to space limitations, some proofs are omitted and can be found in the full version of the paper attached in appendix.

2 Preliminaries

A *plane graph* is an embedded planar graph. Given a plane graph G, let $V(G)$, $E(G)$ and $F(G)$ be respectively the sets of vertices, edges and inner faces of G. A *near-triangulation* is a plane graph in which every inner face is a triangle. A *triangulation* is a near-triangulation with a triangular outer face. It is easy to see that every planar graph is the induced subgraph of some triangulation. This implies that it is sufficient to consider triangulations. Indeed if a planar graph G is isomorphic to the graph induced by a set $V(G) \subseteq V(T)$ of vertices in a triangulation T, then by removing the segments corresponding to $V(T) \setminus V(G)$ from a segment model of T, we clearly obtain a segment model of G.

In all the paper, the bold notations correspond to geometrical objects like points, segments or lines. For example we will usually denote by $\mathbf{v}$ the segment corresponding to a vertex v and by $(\mathbf{v})$ the line prolonging this segment. Furthermore since we consider finite planar graphs, the segment sets we consider are all finite. Given a segment set S, its set of *representative points* Rep_S is the set that contains the intersection points and the ends of the segments in S. A segment set S is *unambiguous* if every segment $\mathbf{s} \in S$ has distinct endpoints, and if parallel segments of S do not intersect. From now on we use the following definition of model.

Definition 2.1. *Given a segment set S, its* intersection graph G_S *is the graph with vertex set S and where two vertices are adjacent if and only if the corresponding segments intersect. Furthermore if (1) S is unambiguous, if (2) the intersection of any three segments of S is empty, and if (3) every endpoint belongs to exactly one segment, then S is a* model *for any graph G isomorphic to G_S.*

For the proof in Section 4 we need some geometrical structures to represent the triangular inner faces. To each triangular inner face abc we will associate a *face segment*, $\underline{\mathbf{ab}}\mathbf{c}$, $\underline{\mathbf{ac}}\mathbf{b}$ or $\underline{\mathbf{bc}}\mathbf{a}$.

Definition 2.2. *Given an unambiguous segment set S and three pairwise intersecting segments $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, a* face segment $\mathbf{f} = \underline{\mathbf{ab}}\mathbf{c}$ *is a segment $[\mathbf{p}, \mathbf{q}]$ such that:*

- *$\mathbf{p}$ is the intersection point of $\mathbf{a}$ and $\mathbf{b}$, and going around $\mathbf{p}$ we consecutively meet $\mathbf{a}$, $\mathbf{f}$ and $\mathbf{b}$,*
- *$\mathbf{q}$ is an internal point of $\mathbf{c}$ that does not belong to any other segment of S, and*
- *none of its internal points belongs to any segment of S.*

The points $\mathbf{p}$ and $\mathbf{q}$ are respectively called the cross-end *and the* flat-end *of $\underline{\mathbf{ab}}\mathbf{c}$.*

Note that the second item implies that face segments are non-trivial, *i.e.* $\mathbf{p} \neq \mathbf{q}$. Note also that in this definition $\mathbf{a}$ and $\mathbf{b}$ play the same role, so a face segment $\underline{\mathbf{ab}}\mathbf{c}$ is also a face segment $\underline{\mathbf{ba}}\mathbf{c}$ but it is not a face segment $\underline{\mathbf{ac}}\mathbf{b}$.

Definition 2.3. *Given an unambiguous segment set S, two face segments $\mathbf{f}_1$ and $\mathbf{f}_2$ on S are* non-interfering *if one of the following holds:*

- *The segments $\mathbf{f}_1$ and $\mathbf{f}_2$ do not intersect.*
- *The segments $\mathbf{f}_1$ and $\mathbf{f}_2$ have the same cross-end $\mathbf{p}$ and this point is the intersection point of exactly two segments of S, $\mathbf{a}$ and $\mathbf{b}$. Furthermore, one of the lines $(\mathbf{a})$ and $(\mathbf{b})$ separates $\mathbf{f}_1$ and $\mathbf{f}_2$ in distinct half-planes.*

Definition 2.4. *A* full model *of a near triangulation T is a couple $\mathcal{M} = (S, F)$ of segments sets such that:*

- *S is a model of T.*
- *F is a set of non-interfering face segments on S such that for each inner face abc of T, F contains one of the following face segments: $\underline{\mathbf{ab}}\mathbf{c}, \underline{\mathbf{ac}}\mathbf{b}, \underline{\mathbf{bc}}\mathbf{a}$.*
- *$S \cup F$ is unambiguous.*

The next theorem is the main result of the paper.

Theorem 2.5. *Every triangulation T has a full model $\mathcal{M} = (S, F)$.*

2.1 Premodels

In our proofs, we use a different kind of model. The main difference with full models is that more than two segments of S can intersect in a same point.

In the following, we consider a segment set S and a set F of non-interfering face segments on S, where $S \cup F$ is unambiguous. Let us denote the segments of S (resp. F) by $\mathbf{s}_1, \mathbf{s}_2, \ldots$ (resp. $\mathbf{f}_1, \mathbf{f}_2, \ldots$). Given a representative point $\mathbf{p}$, its *incidence sequence* $\mathcal{I}(\mathbf{p})$ is the undirected circular sequence of segments (from $S \cup F$) we meet by going around $\mathbf{p}$. This sequence is undirected because it will make no difference going clockwise or anti-clockwise. By extension, the *partial topological incidence sequence* of $\mathbf{p}$, $\mathcal{I}^*(\mathbf{p})$ is the sequence obtained in the following way. Prolong every segment that ends at $\mathbf{p}$ and consider its new incidence sequence.

Then replace every occurrence of $\mathbf{s}_i$ and $\mathbf{f}_i$ that was not in $\mathcal{I}(\mathbf{p})$ before by $(\mathbf{s}_i)$ and $(\mathbf{f}_i)$. It is clear that $\mathcal{I}(\mathbf{p})$ is a subsequence of $\mathcal{I}^*(\mathbf{p})$ (*i.e.* $\mathcal{I}(\mathbf{p}) \subseteq \mathcal{I}(\mathbf{p})$). We say that $\mathcal{I}(\mathbf{p})$ is of the form $([\mathbf{r}_1], \mathbf{r}_2, \ldots, \mathbf{r}_k)$ for $\mathbf{r}_i \in S \cup F$, if either $\mathcal{I}(\mathbf{p}) = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k)$, $\mathcal{I}(\mathbf{p}) = (\mathbf{r}_2, \ldots, \mathbf{r}_k)$, or $\mathcal{I}(\mathbf{p}) \subseteq ((\mathbf{r}_1), \mathbf{r}_2, \ldots, \mathbf{r}_k) \subseteq \mathcal{I}^*(\mathbf{p})$.

Let us define types for the representative points, depending on their incidence sequence. These types are not always entirely determined by the incidence sequence and we will have to assign a type (among the possible ones) to each representative point. Furthermore, these types are in correspondence with some graphs we also describe here.

- A point is a *segment end* if its incidence sequence is $(\mathbf{s}_1)$. The corresponding graph is the single vertex s_1.
- A point is a *flat face segment end* if its incidence sequence is $(\mathbf{s}_1, \mathbf{f}_1, \mathbf{s}_1)$. The corresponding graph is the single vertex s_1.
- A point may be a *crossing* if it has an incidence sequence of the form $(\mathbf{s}_1, [\mathbf{f}_1], \mathbf{s}_2, [\mathbf{f}_2], \mathbf{s}_1, [\mathbf{s}_2])$ or $(\mathbf{s}_1, [\mathbf{f}_1], \mathbf{s}_2, \mathbf{s}_1, [\mathbf{f}_2], \mathbf{s}_2)$. The corresponding graph is the edge $s_1 s_2$.
- A point may be a *path–*$(s_1, s_2, \ldots, s_k)$*–point* with $k \geq 2$, if it has an incidence sequence of the form $(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k, (\mathbf{s}_1), (\mathbf{s}_2))$ (See Figure 1). Such a typed point is in correspondence with *path–*$(s_1, s_2, \ldots, s_k)$, the graph with vertex set $\{s_1, \ldots, s_k\}$ and edge set $\{s_i s_{i+1} \mid 1 \leq i < k\}$.

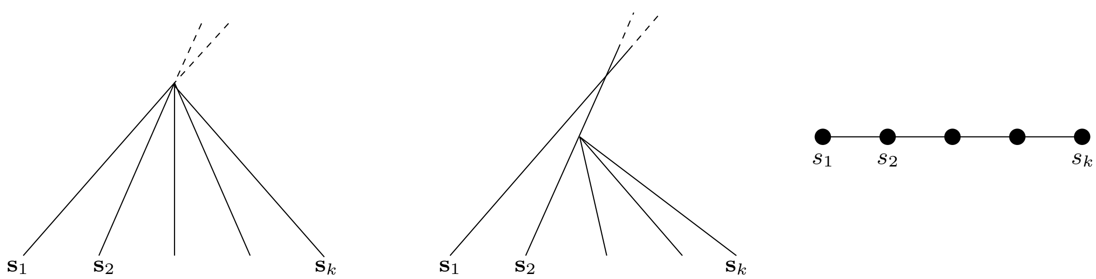

Fig. 1. A path–$(s_1, s_2, \ldots, s_k)$–point, its partial realization, and its corresponding graph

- A point may be a *fan–*$s_1 \triangleleft (s_2, \ldots, s_k)$*–point* with $k \geq 2$, if it has an incidence sequence of the form $(\mathbf{s}_1, [\mathbf{f}_1], \mathbf{s}_2, \ldots, \mathbf{s}_k, (\mathbf{s}_1), [\mathbf{f}_1], (\mathbf{s}_2))$ (See Figure 2), with $\mathbf{f}_1 = \underline{\mathbf{s_1 s_2}}\mathbf{x}$. Note that since $\mathbf{f}_1$ is a face segment it occurs at most once in the incidence sequence. Such a typed point is in correspondence with *fan–*$s_1 \triangleleft (s_2, \ldots, s_k)$, the graph with a vertex s_1 dominating a path $(s_2, \ldots, s_k)$.

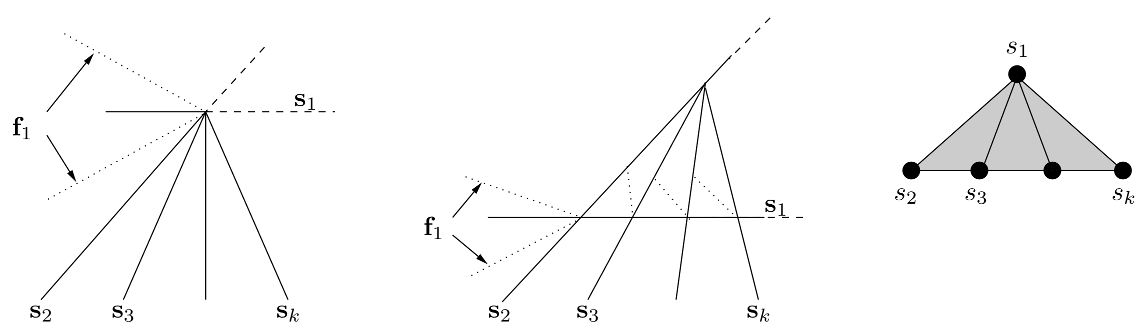

Fig. 2. A fan–$s_1 \triangleleft (s_2, \ldots, s_k)$–point, its partial realization , and its corresponding graph

In fact, there are three more kinds of special points that are not detailed here but can be found in the full version of the paper.

Actually, the graphs we considered here are plane graphs, and their inner faces are the grey faces in the figures. As in [4], we need a bipartite digraph to describe the constraints between segments and representative points.

Definition 2.6. *Given a segment set R, the* constraints digraph $Const_R$ *is the bipartite digraph with vertex sets R and Rep_R, and where $\mathbf{r} \in R$ and $\mathbf{p} \in Rep_R$ are linked if and only if $\mathbf{p} \in \mathbf{r}$. More precisely, there is an arc from $\mathbf{p}$ to $\mathbf{r}$ if $\mathbf{p}$ is an endpoint of $\mathbf{r}$, otherwise (when $\mathbf{p}$ is an internal point of $\mathbf{r}$) the arc goes from $\mathbf{r}$ to $\mathbf{p}$.*

Informally this graph describes the fact that the position of a segment is determined by its endpoints, and determines the position of its internal representative points.

Definition 2.7. *Given a segment set S, a set F of non-interfering face segments on S and a function τ that assigns a type to each representative point, the triple $\mathcal{M} = (S, F, \tau)$ is a premodel of a near-triangulation T if the following holds:*

- *The set $S \cup F$ is unambiguous and the digraph $Const_{S\cup F}$ is acyclic.*
- *A vertex $a \in V(T)$ if and only if $\mathbf{a} \in S$.*
- *An edge $ab \in E(T)$ if and only if $\mathbf{a}$ and $\mathbf{b}$ intersect in a point $\mathbf{p}$ such that the graph corresponding to $\tau(\mathbf{p})$ contains the edge ab.*
- *A face $abc \in F(T)$ if and only if one of the following holds:*
  - *either there exists a face segment $\underline{\mathbf{ab}}\mathbf{c}$, $\underline{\mathbf{ac}}\mathbf{b}$ or $\underline{\mathbf{bc}}\mathbf{a}$ in F,*
  - *or, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ intersect in a point $\mathbf{p}$ such that abc is an inner face of the graph corresponding to $\tau(\mathbf{p})$.*

Note that a premodel $\mathcal{M} = (S, F, \tau)$ of a near-triangulation T has a bounded number of representative points. There are at most $2|V(T)|$ segment ends, at most $F(T)$ flat face segment ends, and at most $E(T)$ points of another type (since each of them corresponds to at least one edge of T).

Remark 2.8. If a premodel $\mathcal{M} = (S, F, \tau)$ of a near-triangulation T has $2|V(T)| + |F(T)| + |E(T)|$ representative points, then (S, F) is a full model of T.

2.2 Local Perturbations

In this subsection we describe how to transform a premodel $\mathcal{M} = (S, F, \tau)$ of a near triangulation T into a full model $\mathcal{M}' = (S', F')$ of T. In the following the segments denoted by $\mathbf{r}_i$ are segments of $S \cup F$. Let us define three basic moves: prolonging, gliding and traversing.

Lemma 2.9 (prolonging). *Consider a premodel $\mathcal{M} = (S, F, \tau)$ of a near triangulation T with an intersection point $\mathbf{p}$ which is the end of a segment $\mathbf{s}_1 \in S$. If for every segment $\mathbf{s}_2 \in S$ that has an end in $\mathbf{p}$, there is no directed path from $\mathbf{s}_2$ to $\mathbf{s}_1$ in $Const_{S\cup F}$, it is possible to prolong $\mathbf{s}_1$ across $\mathbf{p}$ without creating a cycle in $Const_{S'\cup F}$ (where S' is the new segment set). Furthermore, if the type $\tau(\mathbf{p})$ is still applicable to $\mathbf{p}$ then (S', F, τ) remains a premodel of T.*

Remark 2.10. Consider a premodel $\mathcal{M} = (S, F, \tau)$ with a point $\mathbf{p}$ that is the intersection of exactly two segments from S, $\mathbf{s}_1$ and $\mathbf{s}_2$. By prolonging all the segments that end at $\mathbf{p}$ we obtain a segment set S' such that $Const_{S'\cup F}$ remains acyclic.

A segment set R is *flexible* if every representative point $\mathbf{p}$ is internal for at most two segments of R. Note that according to the defined types for every premodel $\mathcal{M} = (S, F, \tau)$, the set $S \cup F$ is flexible.

Definition 2.11. *A* move *of a segment set $R = \{\mathbf{r}_i = [\mathbf{a}_i, \mathbf{b}_i] \mid 1 \le i \le |R|\}$ is a segment set R' such that $R' = \{\mathbf{r}'_i = [\mathbf{a}'_i, \mathbf{b}'_i] \mid 1 \le i \le |R|\}$. An* interpolation *of this move is a continuous function defined for $t \in [0, 1]$ that gives a move R^t of R such that $R^0 = R$ and $R^1 = R'$.*

Lemma 2.12 (gliding). *Consider a flexible and unambiguous segment set R such that $Const_R$ is acyclic, and a representative point $\mathbf{p}$ of R. If the segments $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i$ are consecutive around $\mathbf{p}$, if all the segments $\mathbf{r}_2, \ldots, \mathbf{r}_i$ have an end at $\mathbf{p}$ and are in the same half-plane delimited by $(\mathbf{s}_1)$ (See Figure 3), and if in $Const_R$ the vertex $\mathbf{r}_1$ cannot be reached from any $\mathbf{r}_j$ with $2 \le j \le i$, then there exists a move R' with an interpolation R^t such that for every $t \in]0, 1]$:*

- *The set R^t is unambiguous and $Const_{R^t}$ is acyclic.*
- *The point $\mathbf{p}$ splits into two representative points $\mathbf{p}_1^t$ and $\mathbf{p}_2^t$, which incidence sequence are respectively $(\mathbf{r}_1^t, \mathbf{r}_2^t, \ldots, \mathbf{r}_i^t, \mathbf{r}_1^t)$ and the incidence sequence of $\mathbf{p}$ without the occurrences of $\mathbf{r}_2^t, \ldots, \mathbf{r}_i^t$.*
- *For every representative point $\mathbf{q} \neq \mathbf{p}$ of R there is a representative point $\mathbf{q}^t$ in R^t with exactly the same topological incidence sequence.*
- *There is no other representative point (i.e. $|Rep_{R^t}| = |Rep_R| + 1$).*
- *Every segment $\mathbf{r}^t \in R^t$ (resp. representative point $\mathbf{q}^t \in Rep_{R^t}$) that is not reachable from any $\mathbf{p}_1^t$ in $Const_R^t$ is static, that is $\mathbf{r}^t = \mathbf{r}$ (resp. $\mathbf{q}^t = \mathbf{q}$).*

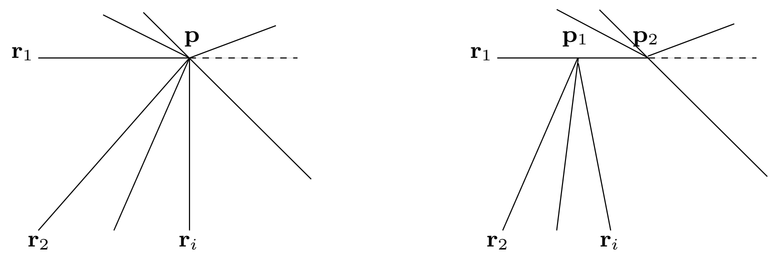

Fig. 3. gliding of $\mathbf{r}_2, \ldots, \mathbf{r}_i$ on $\mathbf{r}_1$.

Lemma 2.13 (traversing). *Consider a flexible and unambiguous segment set R such that $Const_R$ is acyclic, and a representative point $\mathbf{p}$ of R which incidence sequence is $(\mathbf{r}_1, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_j, \mathbf{r}_1, \mathbf{r}_{j+1}, \ldots, \mathbf{r}_k, \mathbf{r}_i)$ with $2 < i \leq j \leq k$ (See Figure 4). There exists a move R' with an interpolation R^t such that for every $t \in]0,1]$:*

- *The set R^t is unambiguous and $Const_{R^t}$ is acyclic.*
- *The point $\mathbf{p}$ splits into i representative points $\mathbf{p}_l^t$, for $1 \leq l \leq i$, which incidence sequence are $(\mathbf{r}_i^t, \mathbf{r}_2^t, \ldots, \mathbf{r}_i^t)$ for $l = 1$, $(\mathbf{r}_1^t, \mathbf{r}_l^t, \mathbf{r}_1^t, \mathbf{r}_l^t)$ for $1 < l < i$, and $(\mathbf{r}_1^t, \mathbf{r}_i^t, \ldots, \mathbf{r}_j^t, \mathbf{r}_1^t, \mathbf{r}_{j+1}, \ldots, \mathbf{r}_k, \mathbf{r}_i^t)$ for $l = i$.*
- *For every representative point $\mathbf{q} \neq \mathbf{p}$ of R there is a representative point $\mathbf{q}^t$ in R^t with exactly the same topological incidence sequence.*
- *There is no other representative point (i.e. $|Rep_{R^t}| = |Rep_R| + i - 1$).*
- *Every segment $\mathbf{r} \in R$ (resp. representative point $\mathbf{q} \in Rep_R$) that is not reachable from $\mathbf{p}_i^t$ in $Const_{R^t}$ is static, that is $\mathbf{r}^t = \mathbf{r}$ (resp. $\mathbf{q}^t = \mathbf{q}$).*

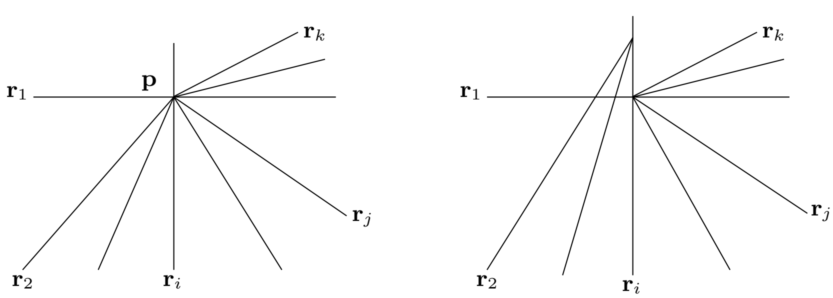

Fig. 4. traversing

Given an intersection point $\mathbf{p}$ in a premodel $\mathcal{M} = (S, F, \tau)$ of T, a *partial realization* of $\mathbf{p}$ is an operation that combines a basic move at $\mathbf{p}$ and the addition of new face segments (eventually none), and that yields another premodel $\mathcal{M}' = (S', F', \tau')$ of T. A simple example of a partial realization at $\mathbf{p}$ is prolonging a segment $\mathbf{s}$ across $\mathbf{p}$, choosing $\mathbf{s}$ in such a way that $\tau(\mathbf{p})$ still applies and that the constraints digraph remains acyclic. Such a partial realization is called a *maximization* of $\mathbf{p}$, and if $\mathbf{p}$ is already internal in two segments we say that this point is *maximal*. In a premodel, we say that a point $\mathbf{p}$ is *simple* if it is either a segment end, a flat face segment end, or a maximal point without any segment of S ending here (at $\mathbf{p}$). Otherwise, we say that this point is *special*.

Proposition 2.14. *Consider a premodel $\mathcal{M} = (S, F, \tau)$ of a near-triangulation T. Every special point $\mathbf{p}$ of $\mathcal{M}$ that is maximal admits a partial realization.*

Proof. Note that since $\mathbf{p}$ is special and maximal there are at least three segments from S intersecting at $\mathbf{p}$. We distinguish different cases according to the type of $\mathbf{p}$.

If this point is a path–$(s_1, s_2, \ldots, s_k)$–point we do a gliding of $\{\mathbf{s}_3, \ldots, \mathbf{s}_k\}$ on $\mathbf{s}_2$ to a new representative point $\mathbf{q}$ (by Lemma 2.12 since $\mathbf{p}$ is not an end of $\mathbf{s}_2$). Let $\mathbf{p}$ and $\mathbf{q}$ be respectively typed as the crossing point of $\mathbf{s}_1$ and $\mathbf{s}_2$, and as a path–$(s_2, \ldots, s_k)$–point (See Figure 1). Under these conditions the gliding keeps the constraints digraph acyclic and preserves the topological incidence sequence of the other representative points (so that their type can remain unchanged). Thus, since the graph that corresponded to $\mathbf{p}$ (the path $(s_1, \ldots, s_k)$) is the union of the graphs corresponding to $\mathbf{p}$ and to $\mathbf{q}$, we are done.

If this point is a fan–$s_1 \lhd (s_2, \ldots, s_k)$–point we do a traversing of $\{\mathbf{s}_3, \ldots, \mathbf{s}_k\}$ along $\mathbf{s}_2$ and through $\mathbf{s}_1$ to a new representative point $\mathbf{q}$. We add the face segments $\underline{\mathbf{s_1 s_i s_{i-1}}}$, with $3 \leq i \leq k$, and we let $\mathbf{q}$ be typed as a path–$(s_2, \ldots, s_k)$–point (See Figure 2). Under these conditions the traversing keeps the constraints digraph acyclic and preserves the topological incidence sequence of the other representative points. Thus since the graph that corresponded to $\mathbf{p}$ (the fan–$s_1 \lhd (s_2, \ldots, s_k)$) is the union of the graphs corresponding to the new crossing points, to the new face segments, to $\mathbf{p}$ and to $\mathbf{q}$, we are done.

For the other kinds of types, we refer to the full version of the paper. This concludes the proof of the proposition. □

Now let us note that any partial realization increases the number of representative points. Since a premodel with the maximum number of representative points is a full model (*Cf.* Remark 2.8), we have the following corollary.

Corollary 2.15. *Any premodel $\mathcal{M} = (S, F, \tau)$ of a near-triangulation T admits a sequence of partial realizations that yield a full model $\mathcal{M}' = (S', F')$ of T.*

3 The case of 4-connected triangulations.

Let T be a near-triangulation. A *chord* of T is an edge not incident to the outer face but which ends are on the outer face. A *separating 3-cycle* C is a cycle of length 3 such that some vertices of T lie inside C whereas other vertices are outside. It is well known that a triangulation is 4-connected if and only if it contains no separating 3-cycle.

Definition 3.1. *A* W-triangulation *T is a 2-connected near-triangulation containing no separating 3-cycle. Such a W-triangulation is* 3-bounded *if its outer boundary is the union of three paths, $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_q)$, and $(c_1, \ldots, c_r)$, that satisfy the following conditions (see Figure 5):*

- *$a_1 = c_r$, $b_1 = a_p$, and $c_1 = b_q$.*
- *the paths are non-trivial (*i.e. *$p \geq 2$, $q \geq 2$, and $r \geq 2$).*
- *there exists no chord $a_i a_j$, $b_i b_j$, or $c_i c_j$.*

This 3-boundary *of T will be denoted by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$.*

In the following, we will use the order on the three paths and their directions, *i.e.* $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ will be different from $(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$-$(a_1, \ldots, a_p)$ and $(a_p, \ldots, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_1)$.

Property 1 *Consider any W-triangulation T 3-bounded by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$.*

(1) If $p = 2$ (see Figure 6, left), for any triangle $\mathbf{BCD}$, there exists a premodel $\mathcal{M} = (S, F, \tau)$ of T contained in the triangle $\mathbf{BCD}$ such that
  - *every special point $\mathbf{p}$ of $\mathcal{M}$ is a point of $b_q = c_1 = [\mathbf{BC}]$, $a_2 = b_1 = [\mathbf{BD}]$ or $c_r = a_1 = [\mathbf{CD}]$,*
  - *$\mathbf{B}$ is a path–$(b_1, b_2, \ldots, b_q)$–point,*

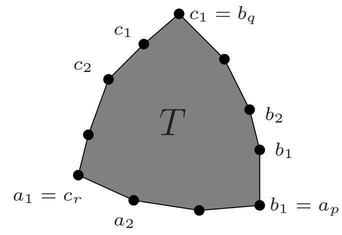

Fig. 5. A 3-bounded W-triangulation T.

- **C** *is a path–*$(c_1, c_2, \dots, c_r)$*–point,*
- **D** *is a fan–*$a_2\lhd(d_1, \dots, d_s, a_1)$*–point (where* $d_1, d_2, \dots, d_s$ *are inner vertices of* T*) such that there is a face segment incident only if* $s = 0$ *(i.e.,* **D** *is a fan–*$a_2\lhd(a_1)$*).*

(2) If $p > 2$ *(see Figure 6, right), for any triangle* **ABC** *there exists a point* **D** *inside this triangle and a premodel* $\mathcal{M} = (S, F, \tau)$ *of* T *contained in the polygon* **ABCD** *such that*
- *every special point* $\mathbf{p}$ *of* $\mathcal{M}$ *is a point of* $a_p = b_1 = [\mathbf{AB}]$*,* $b_q = c_1 = [\mathbf{BC}]$*,* $[\mathbf{CD}]$ *(that is contained in* $a_1 = c_r$*) or* $[\mathbf{AD}]$ *(that is contained in* a_2*),*
- **A** *is a path–*$(a_2, \dots, a_p)$*–point.*
- **B** *is a path–*$(b_1, b_2, \dots, b_q)$*–point,*
- **C** *is a path–*$(c_1, c_2, \dots, c_r)$*–point,*
- **D** *is the crossing point of* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *(with possibly one face segment incident to it corresponding to the inner face of* T *incident to* a_1a_2*),*

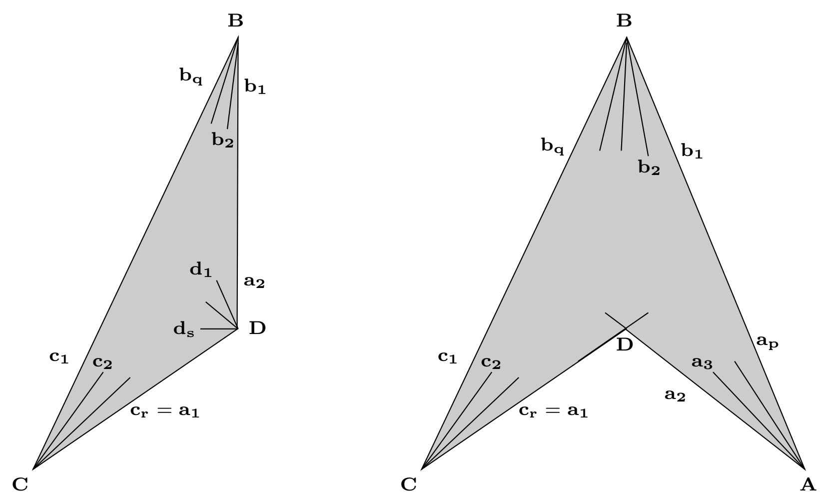

Fig. 6. Property 1 for one W-triangulation T with $p = 2$ and one with $p > 2$.

Note that in both cases, at most one face segment is incident to **D**, since a_1a_2 is incident to exactly one inner face of T. Furthermore since path–points cannot have incident face segments, there is no face segment incident to $\mathbf{A}, \mathbf{B}, \mathbf{C}$ (resp. $\mathbf{B}, \mathbf{C}$) when $p > 2$ (resp. $p = 2$).

This property is the core of our construction and its proof can be found in the full version of the paper. Our proof is based on a decomposition of 4-connected triangulations already used in [2,7,18].

4 Proof of Theorem 2.5

We prove that every triangulation T has a full model (S, F) by induction on the number k of separating 3-cycles in T. If $k = 0$ the triangulation T is a W-triangulation 3-bounded by (a, b)-(b, c)-(c, a), where a, b

and c are the vertices on its outer-boundary. Then Property 1 provides us a premodel $\mathcal{M} = (S, F, \tau)$ of T and by Corollary 2.15 we obtain a full model (S', F') of T.

If $k \geq 1$, let $C = (a, b, c)$ be a 3-cycle such that the triangulation T' induced by the vertices on and inside C does not contain any separating 3-cycle. Let T_1 be the triangulation obtained by removing all the vertices that lie strictly inside the cycle C. Let T_2 be the subgraph of T induced by all the vertices of T that lie strictly inside the cycle C. By definition of C, T_2 is either (A) a single vertex v or (B) a W-triangulation (see Figure 7). In T_1, the cycle C delimits a face and is no more a separating 3-cycle. Since T_1 has one separating 3-cycle

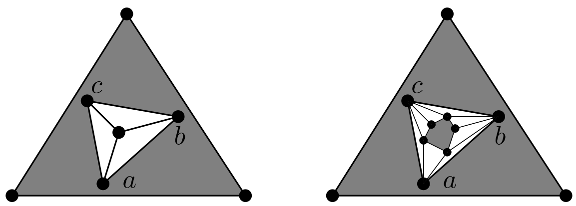

Fig. 7. The cases (A) and (B).

less than T, the induction hypothesis implies that T_1 admits a full model $\mathcal{M} = (S, F)$. Since abc is an inner face of T_1 there is a corresponding face segment, say $\mathbf{a\underline{c}b}$, in F and let respectively $\mathbf{B}$ and $\mathbf{C}$ be its flat end and its cross end. Note that there might be an other face segment incident to $\mathbf{C}$. If it exists we denote it $\mathbf{a\underline{c}d}$ since it would correspond to a face acd adjacent to the edge ac in T_1. Since F is non-interfering we know that ($\mathbf{a}$) or ($\mathbf{c}$) separate $\mathbf{a\underline{c}b}$ and $\mathbf{a\underline{c}d}$ in distinct half-planes. Here we assume, without loss of generality that the line ($\mathbf{a}$) separates them. Now let $\epsilon > 0$ be a real such that for every representative point $\mathbf{p} \in Rep_{S \cup F} \setminus \{\mathbf{B}, \mathbf{C}\}$ we have $dist(\mathbf{p}, \mathbf{a\underline{c}b}) > \epsilon$, and let the region $\mathcal{R}_\epsilon$ be the set of points at distance at most ϵ from $\mathbf{a\underline{c}b}$. The definition of ϵ implies that (1) the only segments intersecting $\mathcal{R}_\epsilon$ are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a\underline{c}b}$ and eventually $\mathbf{a\underline{c}d}$ if it exists; and that (2) the endpoints of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (resp. the flat end of $\mathbf{a\underline{c}d}$) are not in $\mathcal{R}_\epsilon$. Since there is no inner face abc in T we remove $\mathbf{a\underline{c}b}$ from F and we add some segments and face segments in $\mathcal{R}_\epsilon$ to obtain a full model of the whole T.

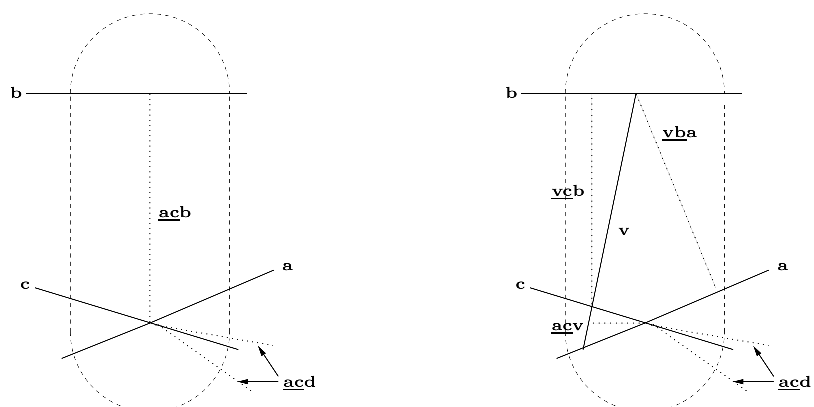

Fig. 8. Case (A): Modifications inside $\mathcal{R}_\epsilon$.

Case (A): T_2 is a single vertex v. Since $\mathbf{a\underline{c}b}$ and $\mathbf{a\underline{c}d}$ (if it exists) are non-interfering, it is easy to draw in the region $\mathcal{R}_\epsilon$ a segment $\mathbf{v}$ that only intersect $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$; and three face segments $\mathbf{\underline{vb}a}$, $\mathbf{\underline{vc}b}$, and $\mathbf{\underline{ac}v}$ such that the set $\{\mathbf{\underline{vb}a}, \mathbf{\underline{vc}b}, \mathbf{\underline{ac}v}, \mathbf{a\underline{c}d}\}$ is non-interfering (see Figure 8). Now it is clear that from the model $\mathcal{M}$ of T_1 we have added a segment for v, three crossings for va, vb and vc, removed the face segment of acb, and added the face segments of vba, acv and vcb; thus we have a full model of T.

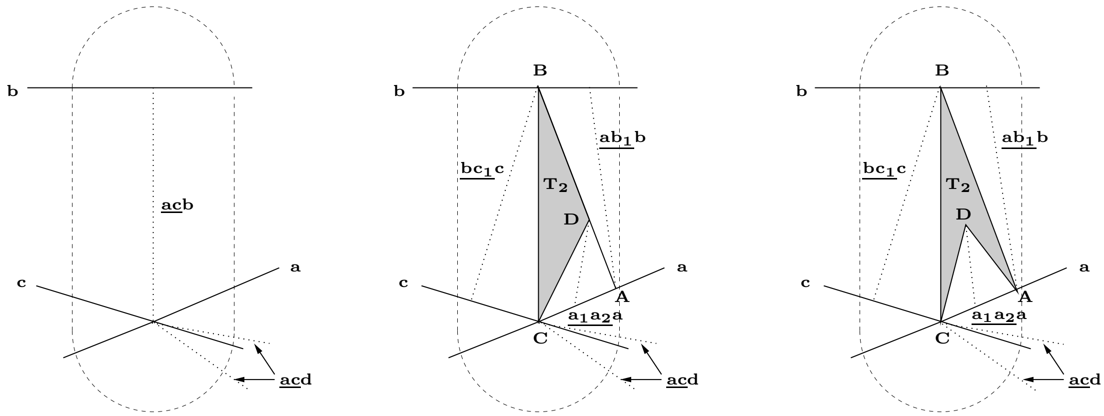

Fig. 9. Case (B): Modifications inside $\mathcal{R}_\epsilon$.

Case (B): T_2 is a W-triangulation. Let $a_1, a_2, \ldots, a_p$ be the neighbors of a inside the cycle (a, b, c) going from c to b excluded. Similarly let $b_1, b_2, \ldots, b_q$ (resp. $c_1, c_2, \ldots, c_r$) be the neighbors of b (resp. c) inside the cycle (a, b, c) going from a to c (resp. from b to a) excluded. It is clear that $a_1 = c_r$, $b_1 = a_p$, and $c_1 = b_q$. Furthermore, since there is no separating 3-cycle inside C, we have that:

– p, q, and $r \geq 2$.
– $(a_1, a_2, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_r)$ is a cycle, thus T_2 is a W-triangulation.
– T_2 has no chord $a_x a_y$, $b_x b_y$, or $c_x c_y$ with $y > x + 1$.

Thus T_2 is a W-triangulation 3-bounded by $(a_1, a_2, \ldots, a_p)$-$(b_1, b_2, \ldots, b_q)$-$(c_1, c_2, \ldots, c_r)$. Here we choose this particular 3-boundary because of the assumption that $(\mathbf{a})$ separates $\underline{\mathbf{ac}}\mathbf{b}$ and $\underline{\mathbf{ac}}\mathbf{d}$ (if it exists). We now apply Property 1 with respect to this 3-boundary and this implies that if $p = 2$ (resp. $p > 2$) then T_2 has a premodel $\mathcal{M}' = (S', F', \tau')$ inside the triangle **BCD** (resp. the polygon **ABCD**), where **A** is a point of $\mathbf{a} \cap \mathcal{R}_\epsilon$ (See Figure 9) and **D** is an internal point of $[\mathbf{A}, \mathbf{B}]$ (resp. a point strictly inside **ABC**). If $p = 2$ we prolong $\mathbf{b}_1 = [\mathbf{BD}]$ across **D** until reaching **A** and note that since all the special points lie on **BCD**, Lemma 2.9 implies that the constraints digraph of $\mathcal{M}'$ remains acyclic. Note also that according to the definition of $\mathcal{R}_\epsilon$, the full model $\mathcal{M}$ and the premodel $\mathcal{M}'$ only intersect at **A**, **B** and **C**. Now we are going to merge $\mathcal{M}$ and $\mathcal{M}'$ in order to construct a premodel $\mathcal{M}^* = (S^*, F^*, \tau^*)$ of the whole T. To do this, let $S^* = S \cup S'$ and $F^* = (F \setminus \underline{\mathbf{ac}}\mathbf{b}) \cup F' \cup \{\underline{\mathbf{a_1a_2}}\mathbf{a}, \underline{\mathbf{ab_1}}\mathbf{b}, \underline{\mathbf{bc_1}}\mathbf{c}\}$; where $\underline{\mathbf{a_1a_2}}\mathbf{a}$ goes from **D** to a point of $[\mathbf{A}, \mathbf{C}]$, $\underline{\mathbf{ab_1}}\mathbf{b}$ goes from **A** to a point of $\mathbf{b} \cap \mathcal{R}_\epsilon$, and $\underline{\mathbf{bc_1}}\mathbf{c}$ goes from **B** to a point of $\mathbf{c} \cap \mathcal{R}_\epsilon$ (See Figure 9). Observe that F^* is non-interfering, in particular we see that $\underline{\mathbf{a_1a_2}}\mathbf{a}$ does not interfere with another face segment **f** at **D**, since **f** would be inside **ABCD**. We now define τ^* as follows. Let **A** be a fan–$a\triangleleft(a_p, \ldots, a_2)$–point, let **B** be a fan–$b\triangleleft(b_q, \ldots, b_1)$–point, and let **C** be a fan–$c\triangleleft(a, c_r, \ldots, c_1)$–point. If $p > 2$ the point **D** remains the crossing point of a_1 and a_2, even with its new incident face segment. If $p = 2$ the point **D** was either a fan–$a_2\triangleleft(d_1, \ldots, d_s, a_1)$–point (for some vertices $d_1, \ldots, d_s$) or a fan–$a_2\triangleleft(a_1)$–point. In the first case let **D** be a fan–$a_2\triangleleft(a_1, d_s, \ldots, d_1)$–point (possible since it has no incident face segment in $\mathcal{M}'$). In the second case let **D** be the crossing point of a_1 and a_2 with one or two incident face segments. Note that in both case the graph corresponding to **D** remains unchanged. For the other representative points of $\mathcal{M}^*$ let their type remain as in $\mathcal{M}$ or $\mathcal{M}'$.

We now verify that $\mathcal{M}^*$ is a premodel of T.

- It is clear that $S^* \cup F^*$ is unambiguous and we show here that $Const_{S^* \cup F^*}$ is acyclic. Indeed this digraph arises from the union of $Const_{S \cup F}$ and $Const_{S' \cup F'}$ (where S' has a segment a_2 prolonged until **A** when $p = 2$) by adding the vertices corresponding to the new face segments and their flat end point, and adding the arcs incident to these vertices. But since the face segments have out-degree zero in the constraints digraphs, there is no cycle in $Const_{S^* \cup F^*}$ passing through a face segment. Thus a cycle would be in the union of $Const_{S \cup F}$ and $Const_{S' \cup F'}$. These two digraph being acyclic, this cycle should successively pass through a segment of $Const_{S' \cup F'}$, through one of the points **A**, **B** and **C**, and through a segment of $Const_{S \cup F}$. But this is impossible since in $Const_{S' \cup F'}$ the only points that intersect $\mathcal{M}$, **A**, **B** and **C**, have in-degree zero.

- Since $V(T)$ is the disjoint union of $V(T_1)$ and $V(T_2)$ we have that a vertex $v \in V(T)$ if and only if $\mathbf{v} \in S^*$.
- Note that $E(T) = E(T_1) \cup E(T_2) \cup \{aa_1 = ac_r\} \cup \{aa_2, \ldots, aa_p\} \cup \{bb_1, \ldots, bb_q\} \cup \{cc_1, \ldots, cc_r\}$, that **A** was not a representative point in $\mathcal{M}$ (resp. was either an end point or a path–$(a_2, \ldots, a_p)$–point in $\mathcal{M}'$) and that now it is a fan–$a \lhd (a_p, \ldots, a_2)$–point, that **B** was a flat face segment end in $\mathcal{M}$ (resp. was a path–$(b_1, \ldots, b_q)$–point in $\mathcal{M}'$) and that now it is a fan–$b \lhd (b_q, \ldots, b_1)$–point that **C** was the crossing point of **a** and **c** in $\mathcal{M}$ (resp. was a path–$(c_1, \ldots, c_r)$–point in $\mathcal{M}'$) and that now it is a fan–$c \lhd (a, c_r, \ldots, c_1)$–point. Since the other representative points remain with the same corresponding graphs, one can easily check (see Figure 10) that $E(T)$ is exactly the set of edges induces by $\mathcal{M}^*$.
- Note that $F(T) = (F(T_1) \setminus acb) \cup F(T_2) \cup \{a_1a_2a, ab_1b, bc_1c\} \cup \{aa_ia_{i+1} \mid 2 \le i < p\} \cup \{bb_ib_{i+1} \mid 1 \le i < p\} \cup \{cc_ic_{i+1} \mid 2 \le i < p\} \cup \{acc_r\}$. According to the face segments added in F^* (the ones in $F^* \setminus (F \cup F')$), the faces induced by **A**, **B** and **C**, and since the other representative points remain with the same corresponding graphs, one can easily check (see Figure 10) that $F(T)$ is exactly the set of faces induced by $\mathcal{M}^*$.

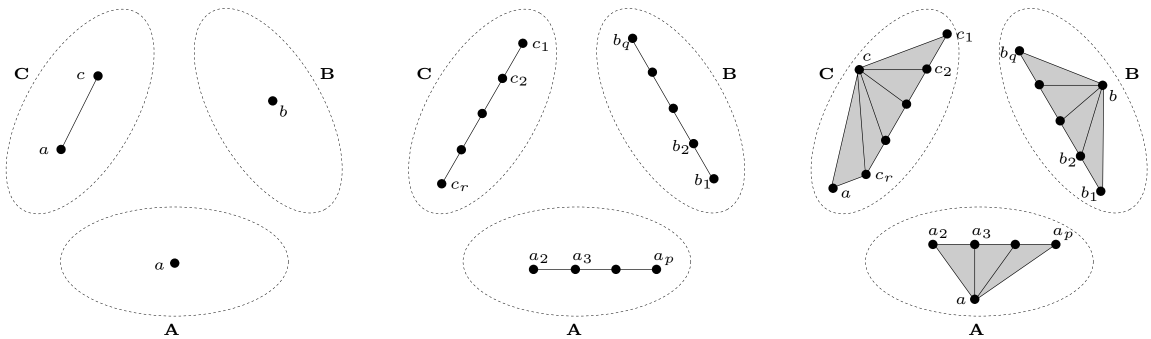

Fig. 10. The graphs corresponding to **A**, **B** and **C** in $\mathcal{M}$ (left), $\mathcal{M}'$ (center) and $\mathcal{M}^*$ (right).

Finally since T has a premodel $\mathcal{M}^*$, Corollary 2.15 implies that it has a full model, proving Theorem 2.5. □

5 Conclusion

West conjectures that every planar graph is the intersection graph of segments using only four directions [17]. Furthermore if the segment set is unambiguous, parallel segments induce a stable set, and the four directions would correspond to a four coloring of the planar graph. This conjecture is true for some families of planar graphs. Indeed, every bipartite planar graph has a representation with two directions [9,3,5] and every triangle free planar graph (that is 3-colorable by Grötzsch's theorem) has a representation with three directions [1].

De Fraysseix and Ossona de Mendez proposed [4] the following generalization of Scheinerman's Conjecture : "Every planar linear hypergraph is the intersection hypergraph of segments in the plane.", where a linear hypergraphs is an hypergraph such that two hyperedges intersect in at most one vertex. This generalization does not holds since the second author found a counterexample [8].

In our proof we need the constraints digraph to be acyclic in order to perform local perturbations on the segment set, like gliding or traversing. We wonder whether this condition is necessary: is it always possible to do local perturbations in any flexible segment set R (with possibly cycles in $Const_R$)? The flexibility of R is required since Pappus's construction gives us a segment set with only one point that is internal in 3 segments, and such that some glidings are impossible.

References

1. N. de Castro, F. Cobos, J.C. Dana, A. Márquez, and M. Noy. Triangle-free planar graphs as segment intersection graphs. *J. Graph Algorithms Appl.*, 6(1):7–26, 2002.
2. J. Chalopin, D. Gonçalves, and P. Ochem. Planar graphs are in 1-STRING. *Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, 609–617, 2007.
3. J. Czyzowicz, E. Kranakis, and J. Urrutia. A simple proof of the representation of bipartite planar graphs as the contact graphs of orthogonal straight line segments. *Inform. Process. Lett.*, 66(3):125–126, 1998.
4. H. de Fraysseix and P. Ossona de Mendez. Representations by Contact and Intersection of Segments. *Algorithmica*, 47:453–463, 2007.
5. H. de Fraysseix, P. Ossona de Mendez, and J. Pach. Representation of planar graphs by segments. *Intuitive geometry (Szeged, 1991), Colloq. Math. Soc. János Bolyai*, 63:109–117, 1994.
6. G. Ehrlich, S. Even, and R.E. Tarjan. Intersection Graphs of Curves in the Plane. *J. Combin. Theory. Ser. B* 21:8–20, 1976.
7. D. Gonçalves. Edge-Partition of Planar Graphs into two Outerplanar Graphs. *Proceedings of the 37th Annual ACM Symposium on Theory of Computing*, 504–512, 2005.
8. D. Gonçalves. A Planar linear hypergraph whose edges cannot be represented as straight line segments. *European J. Combin.* 30 (2009), pp. 280-282
9. I.B.-A. Hartman, I. Newman, R. Ziv. On grid intersection graphs. *Discrete Math.*, 87(1):41–52, 1991.
10. P. Koebe. Kontaktprobleme der Konformen Abbildung. *Ber. Verh. Sachs. Akademie der Wissenschaften Leipzig, Math.-Phys. Klasse*, 88 (1936), 141–164.
11. J. Kratochvíl. String graphs. II. Recognizing String Graphs is NP-hard. *J. Combin. Theory. Ser. B*, 52:67–78, 1991.
12. J. Kratochvíl. Geometric representations of Graphs. Graduate Course, Barcelona, april 2005. retrieved Jan. 2006 at `http://kam.mff.cuni.cz/~honza/ig.ps`.
13. J. Kratochvíl and J. Matoušek. Intersection Graphs of Segments. *J. Combin. Theory. Ser. B*, 62:180–181, 1994.
14. P. Ossona de Mendez. retrieved Mar. 2006 at `http://www.ehess.fr/centres/cams/person/pom/langfr/research.html`.
15. J. Pach and J. Solymosi Crossing Patterns of Segments. *J. Combin. Theory. Ser. A*, 96:316–325, 2001.
16. E.R. Scheinerman. Intersection classes and multiple intersection parameters of graphs. *PhD Thesis, Princeton University*, 1984.
17. D. West. Open problems. *SIAM J. Discrete Math. Newslett.*, 2(1):10–12, 1991.
18. H. Whitney. A theorem on graphs. *Ann. of Math. (2)*, 32(2):378–390, 1931.