Planar graphs are in 1-STRING

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Abstract

We prove that every planar graph is the intersection graph of strings in the plane, such that any two strings intersect at most once.

1 Introduction

A *string* σ is a curve of the plane homeomorphic to a segment. A string σ has two ends, the points of σ that are not ends of σ are *internal points* of σ. Two strings σ_1 and σ_2 *intersect* if they have a common point $p \in \sigma_1 \cap \sigma_2$ and if going around p we successively meet σ_1, σ_2, σ_1, and σ_2. This means that two tangent strings do not intersect. Given a region τ of the plane $\mathcal{P}$, let $\overline{\tau}$ be the region defined by $\mathcal{P} \setminus \tau$.

In this paper, we consider intersection models for planar graphs. A *string representation* of a graph $G = (V, E)$ maps every vertex $v \in V$ to a string σ_v in the plane such that any two vertices are adjacent if and only if their corresponding strings intersect at least once. A graph belongs to the graph class *STRING* if and only if it admits a string representation. Similarly, a *segment representation* of a graph G is a string representation of G in which the strings are segments. A graph belongs to the graph class *SEG* if and only if it admits a segment representation.

These notions were introduced in 1976 by Ehrlich *et al.* [4], who proved the following:

Theorem 1 *[4] Planar graphs are in STRING.*

In his thesis, Scheinerman [10] conjectures a stronger result:

Conjecture 1 *[10] Planar graphs are in SEG.*

Kratochvíl and Matoušek [8] obtained many interesting results about SEG and related graph classes. Independently, Hartman *et al.* [1] and de Fraysseix *et al.* [5] proved Conjecture 1 for bipartite planar graphs. Castro *et al.* [2] proved Conjecture 1 for triangle-free planar graphs. In [7], Grötzsch proved that triangle-free planar graphs are 3-colorable. Observe that, since parallel segments never intersect, a set of parallel segments in a segment representation of a graph induces a stable set of vertices. The construction in [1, 5] (resp. [2]) has the nice property that there are only 2 (resp. 3) possible directions for the segments. So the construction induces a 2-coloring (resp. 3-coloring) of G. In [11], West proposed a stronger version of Conjecture 1 in which only 4 directions are allowed.

Notice that two segments intersect at at most one point, whereas in the construction of Theorem 1, strings may intersect twice. We make another step towards Conjecture 1 by proving that every planar graph admits a *1-string representation*, that is a string representation such that any two strings intersect at most once. A graph belongs to the graph class *1-STRING* if and only if it admits a 1-string representation.

Theorem 2 *Planar graphs are in 1-STRING.*

This answers an open problem of Ossona de Mendez and de Fraysseix [9], which was also mentionned by Kratochvíl.

2 Preliminaries

2.1 Restriction to triangulations

Lemma 1 *Every planar graph is the induced subgraph of some planar triangulation.*

Proof. Let G be a planar graph embedded in the plane, i.e. a plane graph. The graph $h(G)$ is obtained from G by adding in every face f of G a new vertex v_f adjacent to every vertex incident to f in G. Notice that $h(G)$ is also a plane graph and that G is an induced subgraph of $h(G)$. Moreover $h(G)$ is connected, $h(h(G))$ is 2-connected, and $h(h(h(G)))$ is a triangulation. □

Since 1-STRING is a graph class defined by an intersection model, it is closed under taking induced subgraphs. By Lemma 1, it is thus sufficient to prove Theorem 2 for triangulations.

2.2 Definitions

In an embedded planar graph G, the unbounded face of G is called the *outer-face* and every other face of G is an *inner-face* of G. Given an embedded planar graph G, an *outer-vertex* (resp. *outer-edge*) of G is a vertex (resp. edge) of G incident to the outer face. The other vertices (resp. edges) of G are called *inner-vertices* (resp. *inner-edges*) of G. The set of outer-vertices (resp. outer-edges, inner-vertices, and inner-edges) of G is denoted by $V_o(G)$ (resp. $E_o(G)$, $V_i(G)$, and $E_i(G)$). A *near-triangulation* is a planar graph in which all the inner-faces are triangles. An edge uv is a *chord* of some near-triangulation T if u and v are outer-vertices of T and uv is an inner-edge.

Definition 1 *Let $G = (V, E)$ be a graph with a 1-string representation Σ. Given a triplet (a, b, c) of vertices of G, an* (a, b, c)-region *ρ is a region of the plane homeomorphic to the disk and such that (see Figure 1):*

- *for any vertex $v \neq a$, b, and c we have $\rho \cap \sigma_v = \emptyset$*
- *$\rho \cap \sigma_a \cap \sigma_b = \emptyset$, $\rho \cap \sigma_b \cap \sigma_c = \emptyset$, and $\rho \cap \sigma_c \cap \sigma_a = \emptyset$,*
- *$\rho \cap \sigma_b$ and $\rho \cap \sigma_c$ are connected,*
- *$\rho \cap \sigma_a$ has two components,*

- $|\rho \cap \sigma_a| = 3$, $|\rho \cap \sigma_b| = 2$, *and* $|\rho \cap \sigma_c| = 2$,
- *in the boundary of* ρ *we successively intersect* σ_a, σ_a, σ_b, σ_b, σ_c, σ_a, *and* σ_c.

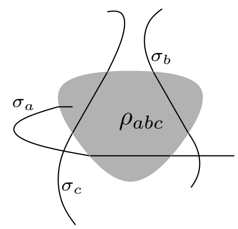

Figure 1: An (a, b, c)-region ρ_{abc}.

Note that according to this definition, in an (a, b, c)-region ρ, one end of the string σ_a is in ρ. When the vertices a, b, and c are not mentionned, we call these regions *face-regions*. Notice that by definition, an (a, b, c)-region, an (a, c, b)-region, a (b, a, c)-region, a (b, c, a)-region, a (c, a, b)-region, and a (c, b, a)-region are pairwise distinct. An region τ of the plane cannot be an (a, b, c)-region and a (c, b, a)-region for example. A region ρ of the plane is an $\{a, b, c\}$*-region* if it is an (a, b, c)-region, an (a, c, b)-region, a (b, a, c)-region, a (b, c, a)-region, a (c, a, b)-region, or a (c, b, a)-region.

Definition 2 *A* strong 1-string representation *of a near-triangulation* T *is a pair* (Σ, R) *such that:*

(1) Σ *is a 1-string representation of* T*,*

(2) R *is a set of disjoint face-regions such that for every inner-face abc of* T*,* R *contains an* $\{a, b, c\}$*-region.*

Definition 3 *A* partial strong 1-string representation *of a near-triangulation* T *is a triplet* (Σ, R, X) *such that*

(1) Σ *is a 1-string representation of* $T \setminus X$ *where* $X \subseteq E_o(T)$ *is a set of outer-edges,*

(2) R *is a set of face-regions such that for every inner-face abc of* T*,* R *contains an* $\{a, b, c\}$*-region.*

Note that in a partial strong 1-string representation (Σ, R, X) of a near-triangulation T, some outer-edges of T do not appear as intersections of two strings of Σ, but for each inner-face of T, there is a corresponding face-region in R.

Definition 4 *A* separating 3-cycle C *of an embedded near-triangulation* T *is a cycle of length* 3 *such that some vertices of* T *lie inside* C *whereas other vertices are outside.*

It is well known that a triangulation is 4-connected if and only if it contains no separating 3-cycle.

Definition 5 *A* W-triangulation *is a 2-connected near-triangulation containing no separating 3-cycle.*

In particular, any 4-connected triangulation is a W-triangulation. Notice that a W-triangulation has no cut vertex, so its outer-edges induce a cycle. The following lemma gives a sufficient condition for a subgraph of a W-triangulation T to be a W-triangulation.

Lemma 2 *Let T be a W-triangulation and consider a cycle C of T. The subgraph defined by C and the edges inside C (according to the embedding of T) is a W-triangulation.*

Proof. Consider the near-triangulation T' induced by some cycle C of T and the edges inside C. By definition, T has no separating 3-cycle and consequently T' does not have any separating 3-cycle. It is then sufficient to show that T' is 2-connected, i.e. T does not have any cut vertex. Consider a vertex v of T, all the faces incident to v are triangles, except at most one (the outer face). Consequently, there exists a path that contains all the neighbors of v, and so $T \setminus v$ is connected. □

Definition 6 *A W-triangulation T is* 3-bounded *if the outer-boundary of T is the union of three paths $(a_1, \ldots, a_p)$, $(b_1, \ldots, b_q)$, and $(c_1, \ldots, c_r)$ that satisfy the following conditions (see Figure 2):*

- $a_1 = c_r$, $b_1 = a_p$, *and* $c_1 = b_q$.
- *the paths are non-trivial, i.e.* $p \geq 2$, $q \geq 2$, *and* $r \geq 2$.
- *there exists no chord $a_i a_j$ (resp. $b_i b_j$, $c_i c_j$), i.e. an edge $a_i a_j$ (resp. $b_i b_j$, $c_i c_j$) with $1 < i+1 < j \leq p$ (resp. $1 < i+1 < j \leq q$, $1 < i+1 < j \leq r$).*

This 3-boundary *of T will be denoted by* $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$.

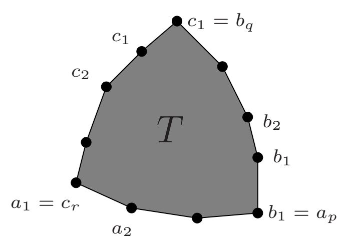

Figure 2: 3-boundary of T.

In the following, we will use the order on the three paths and their directions, i.e. $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ will be different from $(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$-$(a_1, \ldots, a_p)$ and $(a_p, \ldots, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_1)$. The following property describes the shape of a partial strong 1-string representation of a 3-bounded W-triangulation.

Property 1 *A W-triangulation T, 3-bounded by $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$, admits a partial strong 1-string representation (Σ, R, X) contained in a region τ ($\Sigma \cup R \subset \tau$) that satisfies the following properties:*

(a) $X = E_o(G) \setminus \{a_1 a_2\}$,

(b) *τ is a region of the plane homeomorphic to the disk,*

(c) for each inner-vertex v, the intersection of σ_v with the boundary of τ is empty,

(d) for each outer-vertex v, the intersection of σ_v with the boundary of τ is a set containing at most two specific points, the ends of σ_v,

(e) in the boundary of τ we successively meet the ends of $\sigma_{a_2}, \sigma_{a_3}, \ldots, \sigma_{a_p}, \sigma_{b_1}, \ldots, \sigma_{b_q}, \sigma_{c_1}, \ldots, \sigma_{c_r}$.

Notice that for condition (e), we do not precise whether the boundary is traversed clockwise or anticlockwise. This is not necessary since by an axial symmetry of (Σ, R, X) we obtain (Σ', R', X) which has the same properties as (Σ, R, X) with respect to the opposite direction. Note that since $a_p = b_1$, $b_q = c_1$, and $c_r = a_1$, both ends of σ_{b_1} and σ_{c_1} lie on the boundary of τ, but it is not the case for σ_{a_1}.

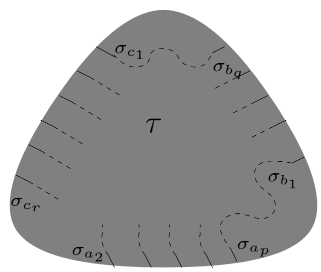

Figure 3: Property 1

Due to its length, the proof of Property 1 is in Appendix A.

3 Proof in the general case

Theorem 3 *Each embedded triangulation T admits a strong 1-string representation* (Σ, R).

Proof. We prove this result by induction on the number of separating 3-cycles. Notice that any triangulation T is 3-connected, and that if T has no separating 3-cycle, then T is 4-connected and is a W-triangulation. Consequently, if T is a 4-connected triangulation whose outer-vertices are a, b, and c, then T is a 3-bounded W-triangulation and (a, b)-(b, c)-(c, a) is a 3-boundary of T. By Property 1, T admits a partial strong 1-string representation (Σ, R, X), with $X = \{bc, ca\}$, that is contained in a region τ $(\Sigma \cup R \subset \tau)$. Furthermore, in the boundary of τ we successively meet the ends of $\sigma_b, \sigma_b, \sigma_c, \sigma_c, \sigma_a$. To obtain a strong 1-string representation of T, it is sufficient (since $X = \{bc, ca\}$) to extend σ_a, σ_b, and σ_c outside of τ in order to obtain an intersection with σ_a and σ_c and with σ_b and σ_c, as depicted on Figure 4.

Suppose now that T is a triangulation that contains at least one separating 3-cycle. Consider a separating 3-cycle (a, b, c) such that there is no separating 3-cycle in the subgraph T' that lies inside the cycle (a, b, c) (according to the embedding of T). Note that T' is a 4-connected triangulation.

Let T_1 be the triangulation obtained by removing all the vertices that lie inside the cycle (a, b, c). Let T_2 be the subgraph of T induced by all the vertices of T that lie inside the cycle (a, b, c). Note that the vertices $a, b,$ and c belong to T_1 but not to T_2. In T_1, the cycle (a, b, c) is a face of the triangulation and is no more a separating 3-cycle. By induction hypothesis, T_1 admits a strong 1-string representation (Σ_1, R_1). In the strong 1-string representation (Σ_1, R_1)

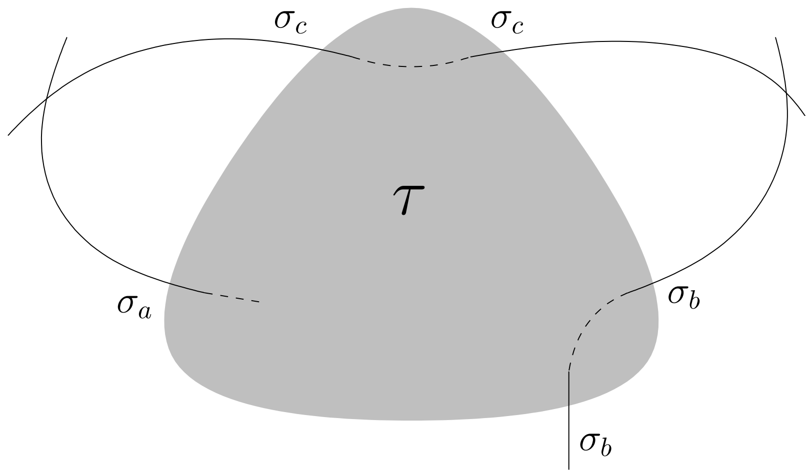

Figure 4: Strong 1-string-representation of T from $(\Sigma, R, X) \subset \tau$.

of T_1, there exists a face-region ρ_{abc} corresponding to the face abc. W.l.o.g., say that ρ_{abc} is an (a, b, c)-region, as depicted on Figure 5.

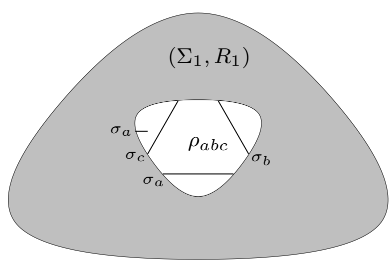

Figure 5: In the strong 1-string representation (Σ_1, R_1) of T_1, the (a, b, c)-region ρ_{abc}.

Since T' is a triangulation, for each vertex v of T', there exists a cycle $(v_1, \ldots, v_n)$ in T' whose vertices are exactly the neighbors of v. Suppose that the vertex a (resp. b and c) has exactly one neighbor v that lies inside (a, b, c). Then there exists a cycle (b, v, c) (resp. (a, v, c) and (a, v, b)) in T' and consequently v is a neighbor of a, b, and c in T'. Suppose that there exists another vertex w in T', then w lies either inside the cycle (a, v, b), inside (a, v, c), or inside (b, v, c) and then one of this cycle is a separating 3-cycle. This is impossible by definition of the cycle (a, b, c). So we can distinguish two cases (see Figure 6), (A) the case where the vertices a, b, and c have a common neighbor inside (a, b, c) and where $T' = K_4$, and (B) the case where each of the vertices a, b, and c have at least two neighbors inside (a, b, c).

Case (A): The vertices a, b, and c have a common neighbor inside (a, b, c) and $T' = K_4$. To obtain a strong 1-string representation (Σ, R) of T, we need to define a string σ_v that corresponds to v. Since $E(T) \setminus E(T_1) = \{va, vb, vc\}$ this string σ_v has to intersect the strings $\sigma_a, \sigma_b, \sigma_c$ that corresponds respectively to the vertices a, b, c. Moreover, we also need to define three disjoint face-regions $\rho_{acv}, \rho_{vbc}, \rho_{vab}$ that correspond respectively to the faces acv, vbc, vab. In our construction, this string σ_v and these three face-regions $\rho_{acv}, \rho_{vbc}, \rho_{vab}$

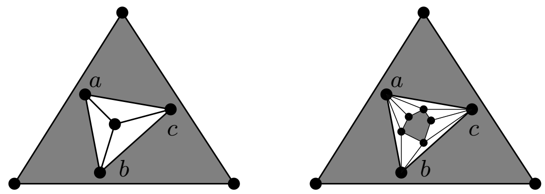

Figure 6: The cases (A) and (B).

are drawn inside the region ρ_{abc}. This construction appears on Figure 7.

Since (Σ_1, R_1) is a strong 1-string representation of T_1 and since $\sigma_v, \rho_{acv}, \rho_{vbc}, \rho_{vab}$ are drawn inside ρ_{abc}, $(\Sigma \cup \{\sigma_v\}, R \setminus \{\rho_{abc}\} \cup \{\rho_{acv}, \rho_{vbc}, \rho_{vab}\}$ is a strong 1-string representation of T.

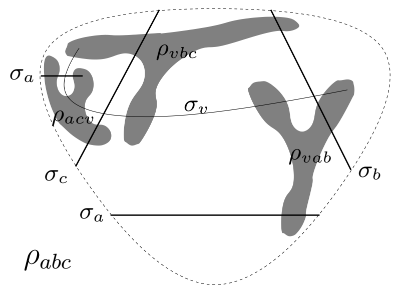

Figure 7: Case (A): Modifications inside ρ_{abc}.

Case (B): Each of the vertices a, b, and c have at least two neighbors inside (a, b, c). Suppose now that a (resp. b and c) has at least two neighbors in T' that lie inside the cycle (a, b, c).

There exists a cycle $(c, a_1, \ldots, a_p, b)$ (resp. $(a, b_1, \ldots, b_q, c)$ and $(b, c_1, \ldots, c_r, a)$) in T' whose vertices are exactly the neighbors of a (resp. b and c). We already know that $p > 1, q > 1, r > 1$ and that $a_p = b_1$, $b_q = c_1$, and $c_r = a_1$. Moreover, since b_1 and c (resp. c_1 and a, and a_1 and b) are the only two common neighbors of a and b (resp. b and c, and a and c) in T' (else there would be a separating 3-cycle) then $(a_1, \ldots, a_p = b_1, \ldots, b_q = c_1, \ldots, c_r = a_1)$ is a cycle. This implies from Lemma 2 that T_2 is a W-triangulation.

Suppose that there exists an edge $a_i a_j$ (resp. $b_i b_j$, $c_i c_j$) with $1 < i + 1 < j \le p$ (resp. $1 < i + 1 < j \le q$, $1 < i + 1 < j \le r$). Then, the cycle (a, a_i, a_j) (resp. (b, b_i, b_j), (c, c_i, c_j)) would be a separating 3-cycle of T'. Consequently, T_2 is a 3-bounded W-triangulation and since the face region ρ_{abc} in (Σ_1, R_1) is an (a, b, c)-region (not an (b, a, c) or an (c, a, b)-region), let us consider the 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ of T_2. With respect to this 3-boundary, T_2 has a partial strong 1-string representation (Σ_2, R_2, X_2), with $X_2 = E_o \setminus \{a_1 a_2\}$ (c.f. Property 1). Let τ_2 be the region of the plane homeomorphic to the disk containing this representation.

Let $\sigma_a^1, \sigma_b^1, \sigma_c^1$ be the strings of Σ_1 corresponding respectively to the vertices a, b, and c in the strong 1-string representation of the triangulation T_1. By symmetry, one can suppose that in the boundary of ρ_{abc}, one can find anticlockwise $\sigma_a^1, \sigma_a^1, \sigma_b^1, \sigma_b^1, \sigma_c^1, \sigma_a^1, \sigma_c^1$.

Let $\sigma^2_{a_2},\ldots,\sigma^2_{a_p}=\sigma^2_{b_1},\sigma^2_{c_1},\ldots,\sigma^2_{c_r}=\sigma^2_{a_1}$ be the strings corresponding respectively to the vertices $a_2,\ldots,a_p=b_1,\ldots b_q=c_1,\ldots c_r=a_1$ in the partial strong 1-string representation of T_2. Again, by symmetry, one can suppose that in the boundary of τ_2 one can find anticlockwise the ends of $\sigma^2_{a_2},\ldots,\sigma^2_{a_p},\sigma^2_{b_1},\ldots,\sigma^2_{b_q},\sigma^2_{c_1},\ldots,\sigma^2_{c_r}$. W.l.o.g., one can suppose that one can insert the region τ_2 in the center of the face-region ρ_{abc} (see Figure 8).

To obtain a strong 1-string representation (Σ,R) of T, we need to extend the strings $\sigma^2_{a_2},\ldots,\sigma^2_{a_p},\sigma^2_{b_1},\ldots,\sigma^2_{b_q},\sigma^2_{c_1},\ldots,\sigma^2_{c_r}$ to obtain intersections that correspond to the edges in the set $E(T)\backslash(E(T_1)\cup(E(T_2)\backslash X_2))=\{aa_i\mid i\in[1,p]\}\cup\{bb_i\mid i\in[1,q]\}\cup\{cc_i\mid i\in[1,r]\}\cup\{a_ia_{i+1}\mid i\in[2,p-1]\}\cup\{b_ib_{i+1}\mid i\in[1,q-1]\}\cup\{c_ic_{i+1}\mid i\in[1,r-1]\}$. Let us denote $\sigma_{a_2},\ldots,\sigma_{a_p}=\sigma_{b_1},\sigma_{c_1},\ldots,\sigma_{c_r}=\sigma_{a_1}$ the extensions of the strings $\sigma^2_{a_2},\ldots,\sigma^2_{a_p}=\sigma^2_{b_1},\sigma^2_{c_1},\ldots,\sigma^2_{c_r}=\sigma^2_{a_1}$. We also need to define face regions for the faces in the set $\{abb_1,aca_1,bcc_1\}\cup\{aa_ia_{i+1}\mid i\in[1,p-1]\}\cup\{bb_ib_{i+1}\mid i\in[1,q-1]\}\cup\{cc_ic_{i+1}\mid i\in[1,r-1]\}$.

The construction of (Σ,R) appears on Figure 8. Let $\Sigma=\Sigma_1\cup\Sigma_2\backslash\{\sigma^2_{a_2},\ldots,\sigma^2_{a_p},\sigma^2_{b_2},\ldots,\sigma^2_{b_q},\sigma^2_{c_2},\ldots,\sigma^2_{c_r}\}\cup\{\sigma_{a_2},\ldots,\sigma_{a_p},\sigma_{b_2},\ldots,\sigma_{b_q},\sigma_{c_2},\ldots,\sigma_{c_r}\}$ and $R=R_1\setminus\{\rho_{abc}\}\cup R_2\cup\{\rho_{aca_1},\rho_{c_1bc},\rho_{b_1ab},\rho_{a_2a_1a}\}\cup\{\rho_{a_{i+1}aa_i}\mid i\in[2,p-1]\}\cup\{\rho_{b_{i+1}bb_i}\mid i\in[1,q-1]\}\cup\{\rho_{c_{i+1}cc_i}\mid i\in[1,r-1]\}$.

Since (Σ_1,R_1) is a strong 1-string representation of T_1 and (Σ_2,R_2,X_2) is a partial strong 1-string representation of T_2, it is clear that (Σ,R) is a strong 1-string representation of T.

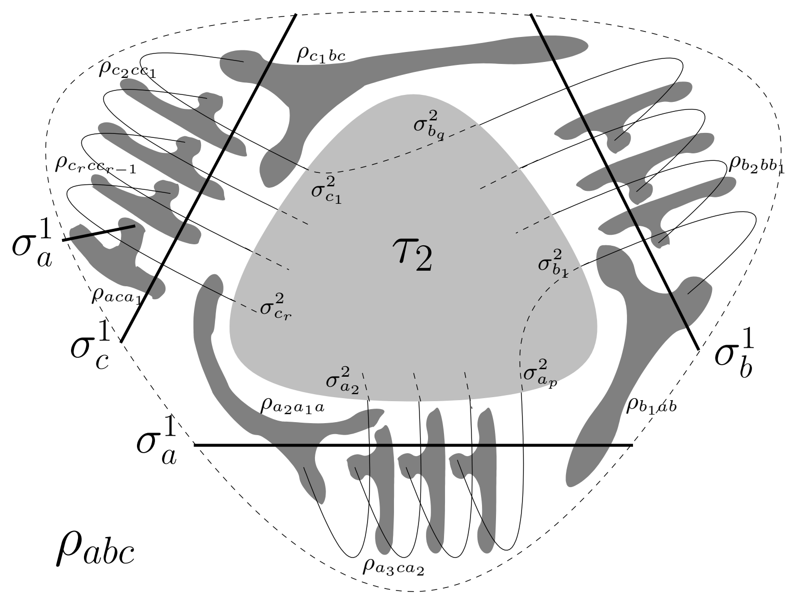

Figure 8: Case (B): Modifications inside ρ_{abc}.

Consequently, every triangulation admits a strong 1-string representation, which proves Theorem 3 and then Theorem 2. □

4 Conclusion

One can wonder whether the method we use in this paper that is based on Whitney's decomposition can be used to prove that any planar graph admits a segment representation. This would need strong conditions on the way (a,b,c)-region are represented to use the same inductive scheme.

Another interesting question is whether this result holds for other surfaces. For exemple, does any graph embedded in an oriented surface $\mathbb{S}_g$ have a 1-string representation in $\mathbb{S}_g$?

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A Proof of Property 1.

Before proving Property 1, we give some definitions and we present Property 2. Consider a 3-bounded W-triangulation $T \neq K_3$ whose boundary is $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ such that T does not contain any chord $a_i b_j$ or $a_i c_j$.

Let $D \subseteq V_i(T)$ be the set of inner-vertices of T that are adjacent to some vertex a_i with $i > 1$.

Since T has at least 4 vertices, no separating 3-cycle, and no chord a_ia_j, a_ib_j, or a_ic_j, then a_1 and a_2 (resp. b_1 and b_2) have exactly one common neighbor in $V(T) \setminus \{c_1\}$ (resp. $V(T) \setminus \{a_1\}$) that will be denoted a (resp. d_1).

Since there is no chord a_ia_j, a_ib_j, or a_ic_j, for each vertex a_i with $i \in [2, p-1]$ (resp. a_p), all the neighbors of a_i (resp. a_p) except a_{i-1} and a_{i+1} (resp. a_{p-1} and b_2) are in D. Since for each $i \in [2, p]$, there is a path between the neighbors of a_i, and since the vertices a_i and a_{i+1} have a common neighbor in D, then the set D induces a connected graph. Since a is in D, the set $D \cup \{a_1\}$ also induces a connected graph.

The *adjacent path* of T with respsect to the 3-boundary $(a_1, \dots, a_p)$-$(b_1, \dots, b_q)$-$(c_1, \dots, c_r)$ is the shortest path linking d_1 and a_1 in $T[D \cup \{a_1\}]$ (the graph induced by $D \cup \{a_1\}$). This path will be denoted $(d_1, d_2, \dots, d_s, a_1)$.

Observation 1 *There exists neither an edge d_id_j with $2 \le i+1 < j \le s$, nor an edge a_1d_i with $1 \le i < s$. Otherwise $(d_1, d_2, \dots d_s)$ is not the shortest path between d_1 and a_1.*

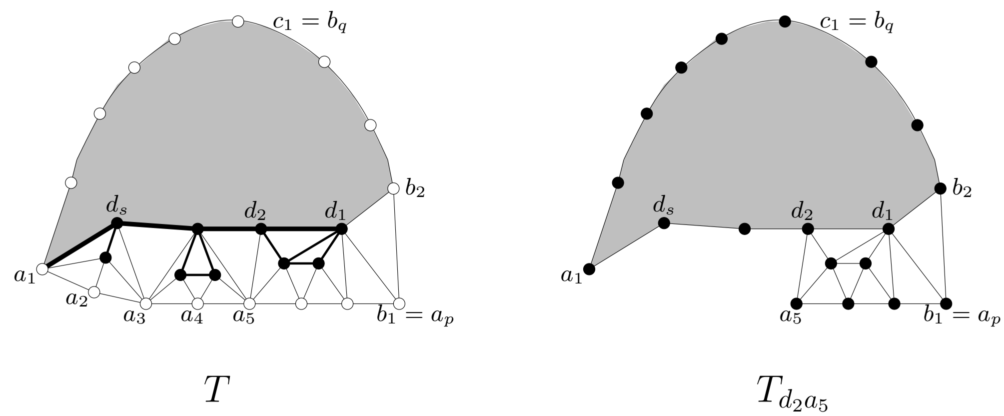

Figure 9: the adjacent path of T and the graph $T_{d_2a_5}$.

For each edge $d_xa_y \in E(T)$ with $x \in [1, s]$ and $y \in [2, p]$, we define the graph $T_{d_xa_y}$. Since $D \subseteq V_i(T)$, $C = (a_1, d_s, \dots, d_x, a_y, \dots, a_p, b_2, \dots, b_q, c_2, \dots, c_r)$ is a cycle. The graph $T_{d_xa_y}$ is the graph lying inside the cycle C (see Figure 9).

From Lemma 2, the graph $T_{d_xa_y}$ is a W-triangulation.

Property 2 *Consider a 3-bounded W-triangulation T with a 3-boundary $(a_1, \dots, a_p)$-$(b_1, \dots, b_q)$-$(c_1, \dots, c_r)$ that does not have any chord a_ib_j or a_ic_j and with an adjacent path $(d_1, d_2, \dots, d_s, a_1)$.*

For each edge $d_xa_y \in E(T)$, the graph $T_{d_xa_y}$ admits a partial strong 1-string representation (Σ, R, X) contained in a region τ ($\Sigma \cup R \subset \tau$) that satisfies the following properties:

(a) $X = E_o(G) \setminus \{d_xa_y\}$,

(b) τ is a region of the plane homeomorphic to the disk,

(c) for each inner-vertex v, the intersection of σ_v with the boundary of τ is empty,

(d) for each outer-vertex v different from d_x and a_y, the intersection of σ_v with the boundary of τ is a set containing at most two specific points, the ends of σ_v,

(e) *the intersection of* d_x *with the boundary of* τ *is a set containing exactly two internal points of* σ_{d_x}*. Furthermore,* $\sigma_{d_x} \cap \overline{\tau}$ *is connected.*

(f) *the intersection of* a_y *with the boundary of* τ *is a set containing exactly two internal points of* σ_{a_y} *and at least one end of* σ_{a_y} *(two when* $a_y = a_p$*). Furthermore,* $\sigma_{a_y} \cap \overline{\tau}$ *is connected.*

(g) *in the boundary of* τ *we successively meet the ends of* $\sigma_{a_y}, \ldots, \sigma_{a_p}, \sigma_{b_1}, \ldots, \sigma_{b_q}, \sigma_{c_1}, \ldots, \sigma_{c_r}, \sigma_{d_s}, \ldots, \sigma_{d_{x+1}}$*, and then we successively meet internal points of* $\sigma_{d_x}, \sigma_{a_y}, \sigma_{d_x}$*, and* σ_{a_y}*.*

The last condition implies that σ_{d_x} and σ_{a_y} intersect inside $\overline{\tau}$.

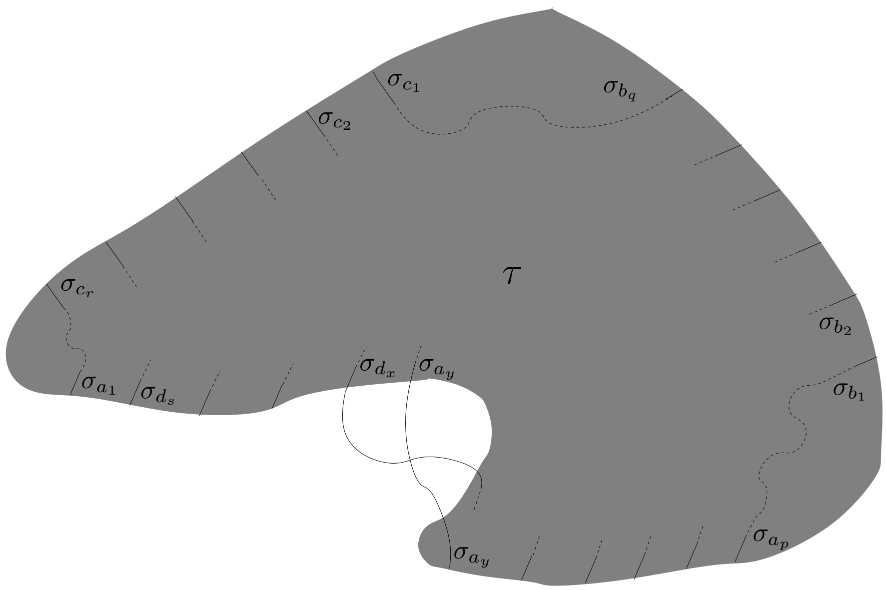

Figure 10: Property 2.

We now prove Properties 1 and 2.

Theorem 4 *Property 1 (resp. Property 2) holds for any W-triangulation* T *(resp.* $T_{d_x a_y}$*).*

This theorem implies Property 1 which is used in the proof of Theorem 2. Although Property 2 is not used in the proof of Theorem 2, we need it to prove Property 1. Indeed, we prove these two properties by doing a "crossed" induction.

Proof. The proof of Theorem 4 uses a decomposition of triangulations defined by Whitney in [12] and recently used by the second author in [6]. We prove Theorem 4 by induction on the number of edges of T or $T_{d_x a_y}$. For the initial step we prove the following lemma.

Lemma 3 *Property 1 (resp. Property 2) holds for any W-triangulation* T *(resp.* $T_{d_x a_y}$*) with* $|E(T)| \leq 3$ *(resp.* $|E(T_{d_x a_y})| \leq 3$*).*

Proof. There is only one W-triangulation with at most 3 edges, the graph K_3. This implies that there is no W-triangulation $T_{d_x a_y}$ with at most 3 edges, so Property 2 obviously holds for any W-triangulation $T_{d_x a_y}$ with at most 3 edges.

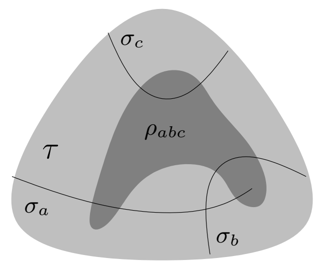

Figure 11: Initial case for Theorem 4.

For Property 1, we have to consider all the possibles 3-boundaries of K_3. All these 3-boundaries are equivalent. Let $V(K_3) = \{a, b, c\}$ and consider the 3-boundary (a, b)-(b, c)-(c, a). In the Figure 11 there is a partial strong 1-string representation (Σ, R, X) of K_3 contained in τ and with $\Sigma = \{\sigma_a, \sigma_b, \sigma_c\}$, $R = \{\rho_{abc}\}$, and $X = \{bc, ac\}$.

□

We now prove the inductive step with the following lemma.

Lemma 4 *For any integer $m > 3$, Property 1 holds for any W-triangulation T such that $|E(T)| < m$ and Property 2 holds for any W-triangulation $T_{d_x a_y}$ such that $|E(T_{d_x a_y})| < m$, then Property 1 and Property 2 respectively holds for any W-triangulation T or $T_{d_x a_y}$ such that $|E(T)| = m$ and $|E(T_{d_x a_y})| = m$.*

Proof. We first prove that if the conditions of Lemma 4 are satisfied, then Property 1 holds for any W-triangulations T such that $|E(T)| = m$. We then prove that it is also the case for Property 2 with any W-triangulations $T_{d_x a_y}$ such that $|E(T_{d_x a_y})| = m$.

Case 1: Proof of Property 1 for a W-triangulation T such that $|E(T)| = m$. Let $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$ be the 3-boundary of T considered. We distinguish different cases according to the existence of a chord $a_i b_j$ or $a_i c_j$ in T. We successively consider the case where there is a chord $a_1 b_j$, with $1 < j < q$, the case where there is a chord $a_i b_j$, with $1 < i < p$ and $1 < j \leq q$, and the case where there is a chord $a_i c_j$, with $1 < i \leq p$ and $1 < j < r$. We then finish with the case where there is no chord $a_i b_j$, with $1 \leq i \leq p$ and $1 \leq j \leq q$ (by definition of 3-boundary, T has no chord $a_1 b_q$, $a_i b_1$, or $a_p b_j$), and no chord $a_i c_j$, with $1 \leq i \leq p$ and $1 \leq j \leq r$ (by definition of 3-boundary, T has no chord $a_p c_1$, $a_i c_r$, or $a_1 c_j$).

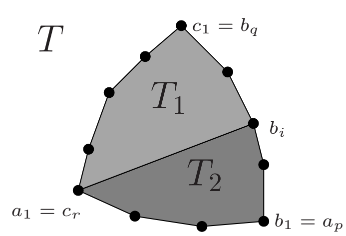

Figure 12: Case 1.1: Chord $a_1 b_i$.

Case 1.1: There is a chord a_1b_j, with $1 < j < q$ (see Figure 12). Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, b_i, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_1, a_2, \ldots, b_1, b_i, a_1)$). By Lemma 2, T_1 and T_2 are W-triangulations. Since T has no chord a_xa_y, b_xb_y, or c_xc_y, (b_ic_r)-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_i)$ (resp. $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_i)$-(b_ia_1)) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $a_1a_2 \notin E(T_1)$ (resp. $c_1c_2 \notin E(T_2)$), T_1 (resp. T_2) has less edges then T, Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries. Let (Σ_1, R_1, X_1) (resp. (Σ_2, R_2, X_2)) be the partial strong 1-string representations contained in the region τ_1 (resp. τ_2) obtained for T_1 (resp. T_2). In Figure 13 we show how to associate this two representations to obtain (Σ, R, X), a partial strong 1-string representation of T that satisfies Property 1. Notice that the boundary of τ_1 is traversed anticlockwise and the boundary of τ_2 is traversed clockwise.

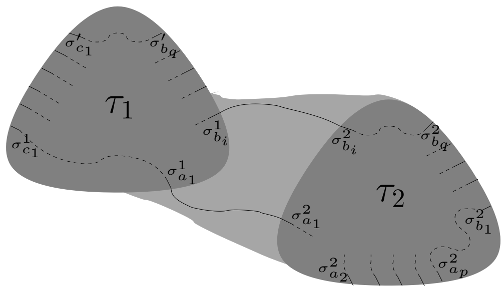

Figure 13: Case 1.1: (Σ, R, X).

We can easily check that (Σ, R, X) is as expected:

- Σ is a 1-string representation: Since $(E(T_1) \setminus X_1) \cap E(T_2) \setminus X_2) = \emptyset$, there is no pair of strings cossing each other more than once.
- Σ is a 1-string representation of $T \setminus X$ with $X = E_o(T) \setminus \{a_1a_2\}$: Indeed, $(T_1 \setminus X_1) \cup T_2 \setminus X_2) = T \setminus X$.
- (Σ, R) is "strong": Each inner-face of T is an inner-face in T_1 or T_2 and the regions τ_1 and τ_2 are disjoint (so the face-regions in τ_1 are disjoint from the face-regions in τ_2).
- We see in Figure 13 that conditions (b), (c), (d), and (e) of Property 1 are satisfied.

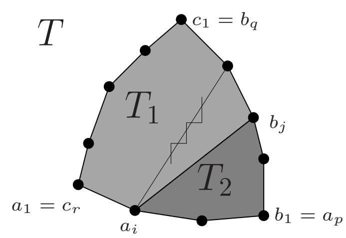

Figure 14: Case 1.2: Chord a_ib_j.

Case 1.2: There is a chord a_ib_j, with $1 < i < p$ and $1 < j \leq q$ (see Figure 14). If there are several chords a_ib_j, we consider one which maximizes j, i.e. such that there is no chord $a_i b_k$ with $j < k \leq q$. Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, a_2, \ldots, a_i, b_j, \ldots, b_q, c_2, \ldots, c_r)$ (resp. $(a_i, \ldots, a_p, b_2, \ldots, b_j, a_i)$). By Lemma 2, T_1 and T_2 are W-triangulations. Since T has no chord $a_x a_y$, $b_x b_y$, $c_x c_y$, or $a_i b_k$ with $k > j$, $(a_1, \ldots, a_i)$-$(a_i, b_j, \ldots, b_q)$-$(c_1, \ldots, c_r)$ (resp. (a_i, b_j)-$(b_j, \ldots, b_1)$-$(a_p, \ldots, a_i)$) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $b_1 b_2 \notin E(T_1)$ (resp. $a_1 a_2 \notin E(T_2)$), T_1 (resp. T_2) has less edges then T, Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries. Let (Σ_1, R_1, X_1) (resp. (Σ_2, R_2, X_2)) be the partial strong 1-string representations contained in the region τ_1 (resp. τ_2) obtained for T_1 (resp. T_2). In Figure 15 we show how to associate this two representations to obtain (Σ, R, X), a partial strong 1-string representation of T that satisfies Property 1. Notice that the boundary of τ_1 is traversed clockwise and the boundary of τ_2 is traversed anticlockwise.

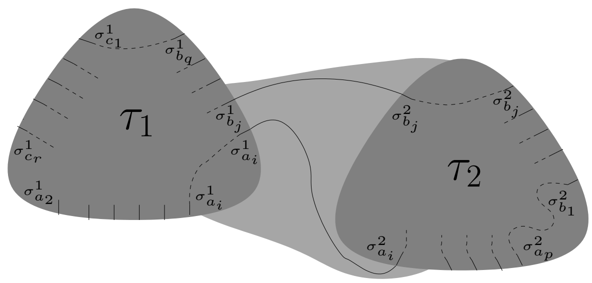

Figure 15: Case 1.2: (Σ, R, X).

As in Case 1.1, we easily check that (Σ, R, X) is correct.

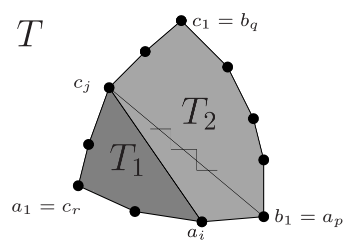

Figure 16: Case 1.3: Chord $a_i c_j$.

Case 1.3: There is a chord $a_i c_j$, with $1 < i \leq p$ and $1 < j < r$ (see Figure 16). If there are several chords $a_i c_j$, we consider one which maximizes i, i.e. such that there is no chord $a_k c_j$ with $i < k < r$. Let T_1 (resp. T_2) be the subgraph of T that lies inside the cycle $(a_1, a_2, \ldots, a_i, c_j, \ldots, c_r)$ (resp. $(c_j, a_i, \ldots, a_p, b_2, \ldots, b_q, c_2, \ldots, c_j)$). By Lemma 2, T_1 and T_2 are W-triangulations. Since T has no chord $a_x a_y$, $b_x b_y$, $c_x c_y$ ou $a_k c_j$ avec $k > i$, $(a_1, \ldots, a_i)$-(a_i, c_j)-$(c_j, \ldots, c_r)$ (resp. $(c_j, a_i, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_j)$) is a 3-boundary of T_1 (resp. T_2). Furthermore, since $b_1 b_2 \notin E(T_1)$ (resp. $a_1 a_2 \notin E(T_2)$), T_1 (resp. T_2) has less edges then T, Property 1 holds for T_1 and T_2 with the mentioned 3-boundaries. Let (Σ_1, R_1, X_1) (resp. (Σ_2, R_2, X_2)) be the partial strong 1-string representations contained in the region τ_1 (resp. τ_2) obtained for T_1 (resp. T_2). In Figure 17 we show how to associate this two representations to obtain (Σ, R, X), a partial strong 1-string representation of T that satisfies Property 1.

Notice that the boundary of τ_1 is traversed clockwise and the boundary of τ_2 is traversed anticlockwise.

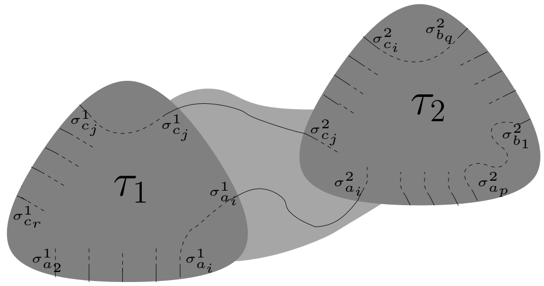

Figure 17: Case 1.3: (Σ, R, X).

As in Case 1.1, we easily check that (Σ, R, X) is correct.

Case 1.4: There is no chord a_ib_j, with $1 \leq i \leq p$ and $1 \leq j \leq q$, and no chord a_ic_j, with $1 \leq i \leq p$ and $1 \leq j \leq r$ (see Figure 18). In this case we consider the adjacent path $(d_1, \ldots, d_s, a_1)$ (see Figure **??**) of T with respect to its 3-boundary, $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Consider the edge d_sa_y, with $1 < y \leq p$, which minimizes y. This edge exists since, by definition of d_s, d_s is adjacent to some vertex a_y with $y > 1$. The W-triangulation $T_{d_sa_y}$ having less edges than T ($a_1a_2 \notin E(T_{d_sa_y})$), Proprerty 2 holds for $T_{d_sa_y}$. Let (Σ', R', X') be the partial strong 1-string representations contained in the region τ' obtained for $T_{d_sa_y}$.

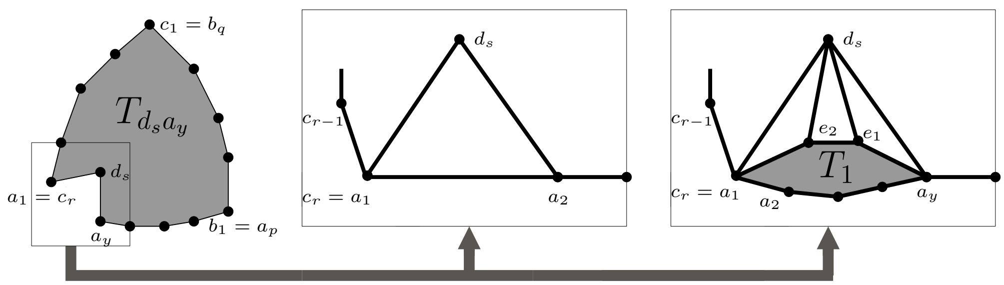

Figure 18: Case 1.4: No chord a_ib_j or a_ic_j.

Now we distinguish two cases according to the position of a_y, the first is when $y = 2$ and the second is when $y > 2$.

Case 1.4.1: $y = 2$ (see Figure 19). In Figure 19, starting from (Σ', R', X'), we show how to extend the string $\sigma'_{a_1} \in \Sigma'$ and how to draw the (a_1, a_2, d_s)-region $\rho_{a_1a_2d_s}$ to obtain (Σ, R, X), a partial strong 1-string representation of T that satisfies Property 1. Here we have $\Sigma = (\Sigma' \setminus \{\sigma'_{a_1}\}) \cup \{\sigma_{a_1}\}$, with σ_{a_1} being the extension of σ'_{a_1}, $R = R' \cup \{\rho_{a_1a_2d_s}\}$, and $X = E_o(T) \setminus \{a_1a_2\}$.

We check that (Σ, R, X) is correct:

- Σ is a 1-string representation: Since $a_1d_s \notin E(T_{d_sa_2}) \setminus X'$ (resp. $a_1a_2 \notin E(T_{d_sa_2}) \setminus X'$),

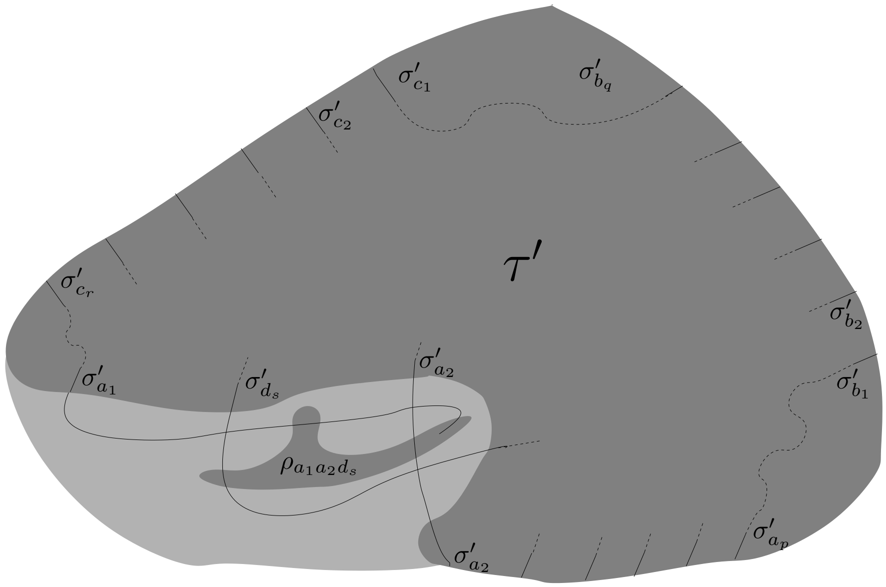

Figure 19: Case 1.4.1.

the two strings σ_{a_1} and σ_{d_s} (resp. σ_{a_1} and σ_{a_2}) intersect only once, in $\tau \cap \overline{\tau'}$. So there is no pair of strings cossing each other more than once.

- Σ is a 1-string representation of $T \setminus X$ with $X = E_o(T) \setminus \{a_1a_2\}$: Indeed, $(E(T_{d_sa_2}) \setminus X') \cup \{a_1d_s, a_1a_2\} = E(T) \setminus X$.
- (Σ, R) is "strong": The only inner-face of T that is not an inner-face in $T_{d_sa_2}$ is $a_1a_2d_s$. Since the regions τ' and $\rho_{a_1a_2d_s}$ are disjoint, all the face-regions of $R = R' \cup \{\rho_{a_1a_2d_s}\}$ are disjoint.
- We see in Figure 19 that conditions (b), (c), (d), and (e) of Property 1 are satisfied.

Case 1.4.2: $y > 2$ (see Figure 20). Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of d_s strictly inside the cycle $(d_s, a_1, a_2, \ldots, a_y)$, going "from right to left" (see Figure 20). By minimality of y we have $e_i \neq a_j$, for all $1 \leq i \leq t$ and $1 \leq j \leq y$.

Let T_1 be the subgraph of T that lies inside the cycle $(a_1, \ldots, a_y, e_1, \ldots, e_t, a_1)$. By Lemma 2, T_1 is a W-triangulation. Since the W-triangulation T has no separating 3-cycle (d_s, a_y, e_i) or (d_s, e_i, e_j), there exists no chord a_ye_i or e_ie_j in T_1. So (a_2, a_1)-$(a_1, e_t, \ldots, e_1, a_y)$-$(a_y, \ldots, a_2)$ is a 3-boundary of T_1. Finally, since T_1 has less edges than T ($a_1d_s \notin E(T_1)$), Property 1 holds for T_1 with respect to the mentionned 3-boundary. Let (Σ_1, R_1, X_1) be the partial strong 1-string representations contained in the region τ_1 obtained for T_1.

In Figure 20, starting from (Σ', R', X') and (Σ_1, R_1, X_1), we show how to join the strings $\sigma'_{a_1} \in \Sigma'$ and $\sigma^1_{a_1} \in \Sigma_1$ (resp. $\sigma'_{a_y} \in \Sigma'$ and $\sigma^1_{a_y} \in \Sigma_1$), how to extend the strings $\sigma^1_{e_i} \in \Sigma^1$, for $1 \leq i \leq t]$, and how to draw the face-regions $\rho_{a_ye_1d_s}$, $\rho_{e_ta_1d_s}$, and $\rho_{e_ie_{i-1}d_s}$, for $2 \leq i \leq t$, in order to obtain (Σ, R, X), a partial strong 1-string representation of T that satisfies Property 1. Here we have $\Sigma = (\Sigma' \setminus \{\sigma'_{a_1}, \sigma'_{a_y}\}) \cup (\Sigma_1 \setminus (\{\sigma^1_{a_y}, \sigma^1_{a_1}\} \cup \{\sigma^1_{e_i} \mid i \in [1,t]\})) \cup \{\sigma_{a_1}, \sigma_{a_y}\} \cup \{\sigma_{e_i} \mid i \in [1,t]\}$, with σ_{a_1} (resp. σ_{a_y}) being the junction of σ'_{a_1} and $\sigma^1_{a_1}$ (resp. σ'_{a_y} and $\sigma^1_{a_y}$), the strings

σ_{e_i} being the extensions of the strings $\sigma^1_{e_i} \in \Sigma_1$, $R = R' \cup R_1 \cup \{\rho_{a_y e_1 d_s}, \rho_{e_t a_1 d_s}\} \cup \{\rho_{d_s e_i e_{i-1}} \mid i \in [2,t]\}$ and $X = E_o(T) \setminus \{a_1 a_2\}$.

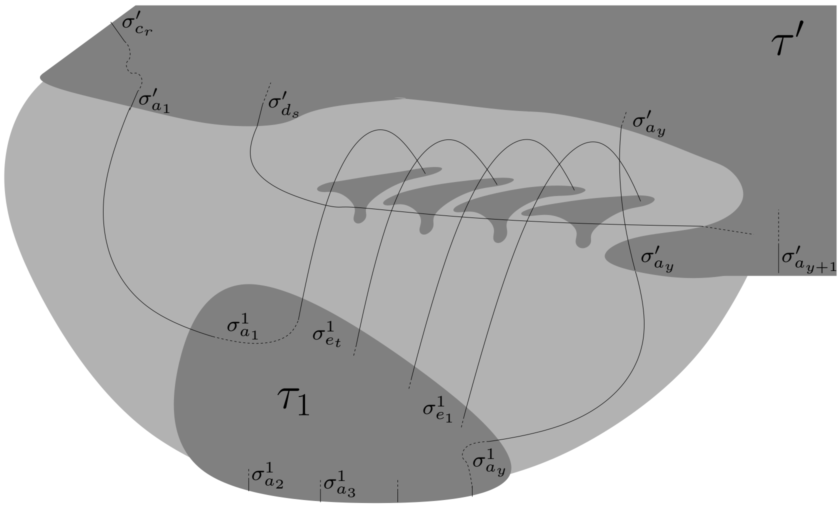

Figure 20: Case 1.4.2.

We check that (Σ, R, X) is correct:

- Σ is a 1-string representation: Since the edges a_1e_t, a_1d_s, a_ye_1, e_ie_{i+1}, and e_id_s are not in $(E(T_{d_s a_y}) \setminus X') \cup (E(T_1) \setminus X_1)$ there is no two strings intersecting more than once.
- Σ is a 1-string representation of $T \setminus X$ with $X = E_o(T) \setminus \{a_1a_2\}$: Indeed, $E(T) \setminus X = (E(T_{d_s a_y}) \setminus X') \cup (E(T_1) \setminus X_1) \cup \{a_ye_1, e_ta_1, d_sa_1\} \cup \{e_ie_{i-1} \mid i \in [2,t]\} \cup \{d_se_i \mid i \in [1,t]\}$.
- (Σ, R) is "strong": The only inner-faces of T that are not inner-faces in $T_{d_s a_y}$ or T_1 are $a_1e_td_s$, $a_ye_1d_s$, and the faces $e_ie_{i-1}d_s$, for $2 \le i \le t$. Since the regions τ', τ_1, $\rho_{a_y e_1 d_s}$, $\rho_{e_t a_1 d_s}$, and $\rho_{e_i e_{i-1} d_s}$, for $2 \le i \le t$, are all disjoint, all the face-regions of R are disjoint.
- We see in Figure 20 that conditions (b), (c), (d), and (e) of Property 1 are satisfied.

This completes the study of Case 1. So, Property 1 holds for any W-triangulation T such that $|E(T)| = m$.

Case 2: Proof of Property 2 for any W-triangulation $T_{d_x a_y}$ such that $|E(T_{d_x a_y})| = m$. Recall that the W-triangulation $T_{d_x a_y}$ is a subgraph of a W-triangulation T with 3-boundary $(a_1, \ldots, a_p)$-$(b_1, \ldots, b_q)$-$(c_1, \ldots, c_r)$. Moreover, T has no chord a_ib_j or a_ic_j and its adjacent path is $(d_1, \ldots, d_s, a_1)$, avec $s \ge 1$.

When $d_xa_y \neq d_1a_p$ we define the couple of integers $(z,w) \neq (x,y)$, with $1 \le z \le x$ and $y \le w \le p$, such that there is an edge $d_za_w \in E(T_{d_x a_y})$ (there is at least one such edge, d_1a_p). Within all the possibles couples $(z,w) \neq (x,y)$, we consider the one that maximizes z and then minimizes w. Since the vertex d_{x-1} is by definition adjacent to some vertex a_i we observe that, by maximality of z, we have $z = x$ or $x - 1$.

We distinguish five cases. First we consider the case where $d_xa_y = d_1a_p$ (Case 2.1). When $d_xa_y \neq d_1a_p$ the cases depend on the edge d_za_w. When $z = x$ we have the case where $w = y+1$

(Case 2.2) and the case where $w > y + 1$ (Case 2.4), and when $z = x - 1$ we have the case where $w = y$ (Case 2.3) and the case where $w > y$ (Case 2.5).

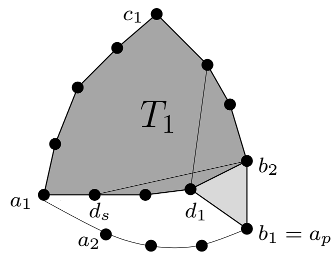

Figure 21: Case 2.1: $T_{d_x a_y} = T_{d_1 a_p}$.

Case 2.1: $d_x a_y = d_1 a_p$ **(see Figure 21).** Let T_1 be the subgraph of $T_{d_1 a_p}$ that lies inside the cycle $(a_1, d_s, \ldots, d_1, b_2, \ldots, b_q, c_2, \ldots, c_r)$. By Lemma 2, T_1 is a W-triangulation. This W-triangulation has no chord $b_i b_j$, $c_i c_j$, $d_i d_j$, or $a_1 d_j$. We consider two cases according to the existence of an edge $d_1 b_i$ with $2 < i \leq q$.

- If T_1 has no chord $d_1 b_i$ then $(d_1, b_2, \ldots, b_q)$-$(c_1, \ldots, c_r)$-$(a_1, d_s, \ldots, d_1)$ is a 3-boundary of T_1.

- If T_1 has a chord $d_1 b_i$, with $2 < i \leq q$, note that $q > 2$ and that there cannot be a chord $b_2 a_1$ or $b_2 d_j$, with $1 < j \leq s$ (this would violate the planarity of $T_{d_x a_y}$, see Figure 21) So in this case, $(b_2, d_1, \ldots, d_s, a_1)$-$(c_r, \ldots, c_1)$-$(b_q, \ldots, b_2)$ is a 3-boundary of T_1.

Finally, since T_1 is a W-triangulation with less edges than $T_{d_1 a_p}$, Property 1 holds for T_1 with respect to at least one of the two mentionned 3-boundaries. Whichever 3-boundary we consider, we obtain a partial strong 1-string representation (Σ_1, R_1, X_1) of T_1 with the same properties:

- $X_1 = E_o(T) \setminus \{d_1 b_2\}$,

- $\Sigma_1 \cup R_1$ is contained in a regoin τ_1 homeomorphic to the disk,

- in the boundary of τ_1 we successively meet the ends of $\sigma^1_{d_1}, \ldots, \sigma^1_{d_s}, \sigma^1_{a_1}, \sigma^1_{c_r}, \ldots, \sigma^1_{c_1}, \sigma^1_{b_q}, \ldots, \sigma^1_{b_2}$ (in the clockwise or in the anticlockwise sense).

In Figure 22 we modify (Σ_1, R_1, X_1), by extending the strings $\sigma^1_{d_1}$ and $\sigma^1_{b_2} \in \Sigma^1$ and by adding a new string σ_{a_p} and a new face region $\rho_{d_1 b_2 a_p}$. This leads to (Σ, R, X), a partial strong 1-string representation of $T_{d_1 a_p}$ that satisfies Property 2. Here we have $X = E_o(T_{d_1 a_p}) \setminus \{d_1 a_p\}$, $R = R_1 \cup \{\rho_{d_1 b_2 a_p}$, and $\Sigma = (\Sigma_1 \setminus \{\sigma^1_{d_1}, \sigma^1_{b_2}\}) \cup \{\sigma_{d_1}, \sigma_{b_2}, \sigma_{a_p}\}$, the strings σ_{d_1} and σ_{b_2} being the extensions of the strings $\sigma^1_{d_1}$ and $\sigma^1_{b_2} \in \Sigma_1$.

We check that (Σ, R, X) is correct:

- Σ is a 1-string representation: It is clear that there is no two strings intersecting more than once.

- Σ is a 1-string representation of $T_{d_1 a_p} \setminus X$: Indeed, $E(T_{d_1 a_p}) \setminus X = (E(T_1) \setminus X_1) \cup \{a_p d_1, a_p b_2\}$.

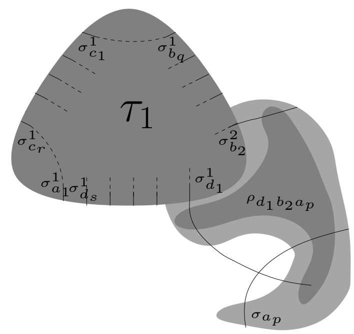

Figure 22: Case 2.1: (Σ, R, X).

- (Σ, R) is "strong": The only inner-face of $T_{d_1 a_p}$ that is not an inner-face of T_1 is $d_1 a_p b_2$. Since the regions τ_1 and $\rho_{d_1 a_p b_2}$ are disjoint, all the face-regions of R are disjoint.
- We see in Figure 22 that conditions (b), (c), (d), (e), (f), and (g) of Property 2 are satisfied.

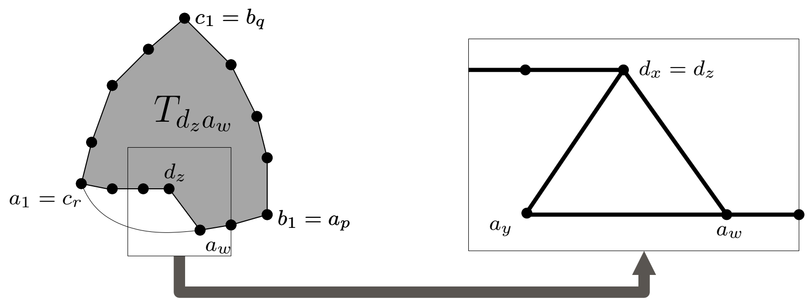

Figure 23: Case 2.2: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x$ and $w = y + 1$.

Case 2.2: $T_{d_x a_y} \neq T_{d_1 a_p}$**,** $z = x$ **and** $w = y + 1$ **(see Figure 23).** By Lemma 2, $T_{d_z a_w}$ is a W-triangulation. Since $T_{d_z a_w}$ has less edges than $T_{d_x a_y}$ ($d_x a_y \notin E(T_{d_z a_w})$), Property 2 holds for $T_{d_z a_w}$. Let (Σ', R', X') be the partial strong 1-string representation of $T_{d_z a_w}$ contained in the region τ' with $X' = E_o(T_{d_z a_w}) \setminus \{d_z a_w\}$.

In Figure 24 we modify (Σ', R', X'), by extending the string $\sigma'_{a_w} \in \Sigma'$ and by adding a new string σ_{a_y} and a new face region $\rho_{a_y a_w d_x}$. This leads to (Σ, R, X), a partial strong 1-string representation of $T_{d_x a_y}$ that satisfies Property 2. Here we have $X = E_o(T_{d_x a_y}) \setminus \{d_x a_y\}$, $R = R' \cup \{\rho_{a_y a_w d_x}$, and $\Sigma = (\Sigma' \setminus \{\sigma'_{a_w}\}) \cup \{\sigma_{a_w}, \sigma_{a_y}\}$, the string σ_{a_w} being the extension $\sigma^1_{a_w} \in \Sigma'$.

We check that (Σ, R, X) is correct:

- Σ is a 1-string representation: It is clear that there is no two strings intersecting more than once.
- Σ is a 1-string representation of $T_{d_x a_y} \setminus X$: Indeed, $E(T_{d_x a_y}) \setminus X = (E(T_{d_z a_w}) \setminus X') \cup \{d_z a_w\}$.

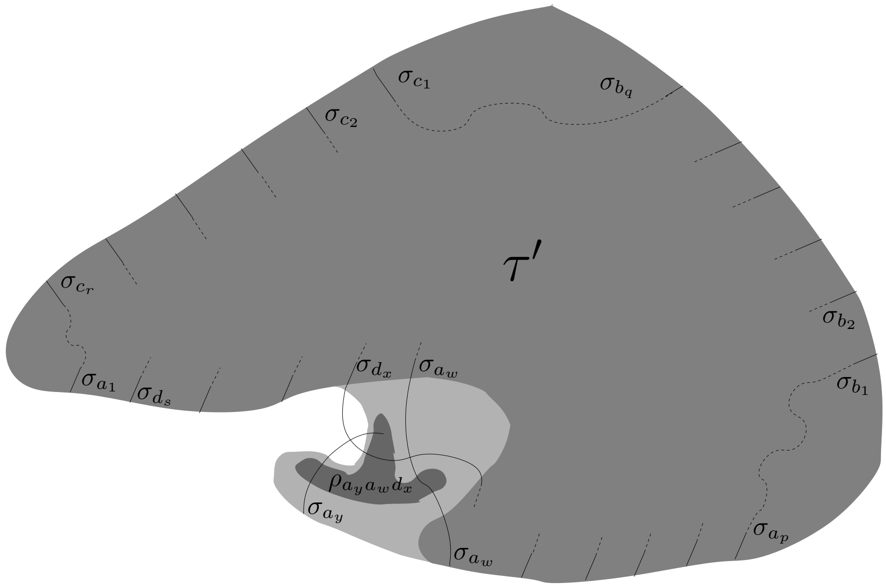

Figure 24: Case 2.2: (Σ, R, X).

- (Σ, R) is "strong": The only inner-face of $T_{d_x a_y}$ that is not an inner-face of $T_{d_z a_w}$ is $d_x a_y a_w$. Since the regions τ' and $\rho_{d_x a_y a_w}$ are disjoint, all the face-regions of R are disjoint.
- We see in Figure 24 that conditions (b), (c), (d), (e), (f), and (g) of Property 2 are satisfied.

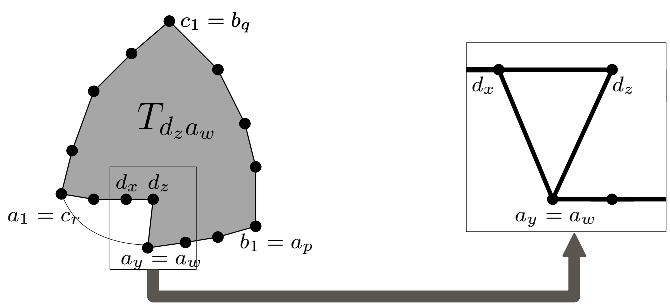

Figure 25: Case 2.3: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x - 1$ and $w = y$.

Case 2.3: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x - 1$ and $w = y$ (see Figure 25). By Lemma 2, $T_{d_z a_w}$ is a W-triangulation. Since $T_{d_z a_w}$ has less edges than $T_{d_x a_y}$ ($d_x a_y \notin E(T_{d_z a_w})$), Property 2 holds for $T_{d_z a_w}$. Let (Σ', R', X') be the partial strong 1-string representation of $T_{d_z a_w}$ contained in the region τ' with $X' = E_o(T_{d_z a_w}) \setminus \{d_z a_w\}$.

In Figure 26, we modify (Σ', R', X') by extending the string $\sigma'_{d_x} \in \Sigma'$ and by adding a new face region $\rho_{d_x a_y d_w}$. This leads to (Σ, R, X), a partial strong 1-string representation of

$T_{d_xa_y}$ that satisfies Property 2. Here we have $X = E_o(T_{d_xa_y}) \setminus \{d_xa_y\}$, $R = R' \cup \{\rho_{d_xa_yd_w}$, and $\Sigma = (\Sigma' \setminus \{\sigma'_{d_x}\}) \cup \{\sigma_{d_x}\}$, the string σ_{d_x} being the extension $\sigma^1_{d_x} \in \Sigma'$.

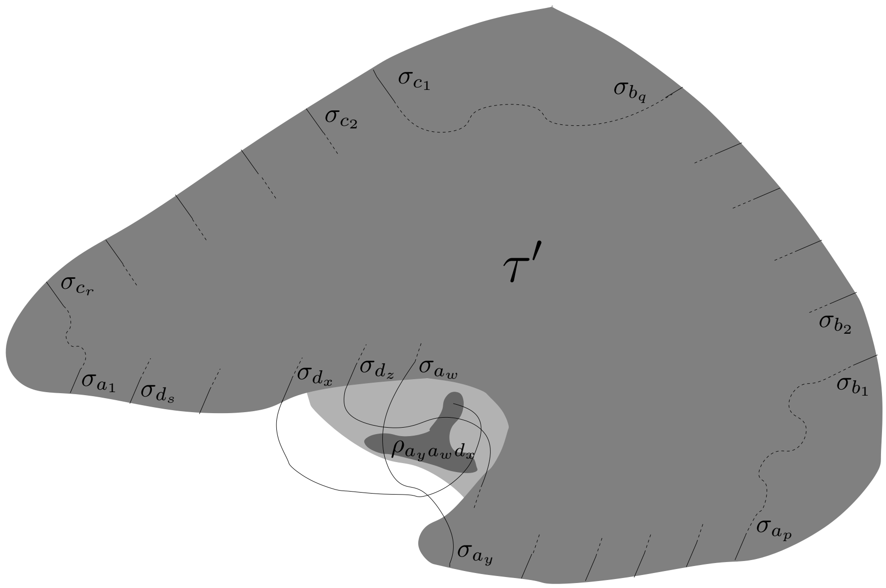

Figure 26: Case 2.3: (Σ, R, X).

We check that (Σ, R, X) is correct:

- Σ is a 1-string representation: Since the edges d_xd_z and d_xa_y are not in $(E(T_{d_za_w}) \setminus X')$ there is no two strings intersecting more than once.
- Σ is a 1-string representation of $T_{d_xa_y} \setminus X$: Indeed, $E(T_{d_xa_y}) \setminus X = (E(T_{d_za_w}) \setminus X') \cup \{d_xd_z, d_xa_y\}$.
- (Σ, R) is "strong": The only inner-face of $T_{d_xa_y}$ that is not an inner-face of $T_{d_za_w}$ is $d_xd_za_y$. Since the regions τ' and $\rho_{d_xd_za_y}$ are disjoint, all the face-regions of R are disjoint.
- We see in Figure 26 that conditions (b), (c), (d), (e), (f), and (g) of Property 2 are satisfied.

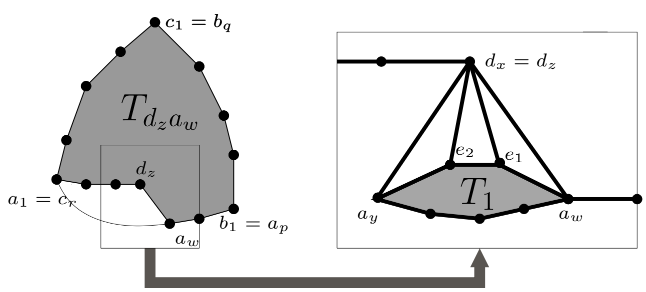

Figure 27: Case 2.4: $T_{d_xa_y} \neq T_{d_1a_p}$, $z = x$ and $w > y + 1$.

Case 2.4: $T_{d_xa_y} \neq T_{d_1a_p}$**,** $z = x$ **and** $w > y+1$ **(see Figure 27).** By Lemma 2, $T_{d_za_w}$ is a W-triangulation. Since $T_{d_za_w}$ has less edges than $T_{d_xa_y}$ ($d_xa_y \notin E(T_{d_za_w})$), Property 2 holds for $T_{d_za_w}$. Let (Σ', R', X') be the partial strong 1-string representation of $T_{d_za_w}$ contained in the region τ' with $X' = E_o(T_{d_za_w}) \setminus \{d_za_w\}$.

Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of d_x strictly inside the cycle $(d_x, a_y, \ldots, a_w)$, going "from right to left" (see Figure 27). Since there is no chord a_ia_j we have $t > 0$. Furthermore by minimality of w we have $e_i \neq a_j$, for all $1 \leq i \leq t$ and $y \leq j \leq w$. Let T_1 be the subgraph of $T_{d_xa_y}$ that lies inside the cycle $(a_y, \ldots, a_w, e_1, \ldots, e_t, a_y)$. By Lemma 2, T_1 is a W-triangulation. Since the W-triangulation $T_{d_xa_y}$ has no separating 3-cycle (d_x, a_w, e_i) or (d_x, e_i, e_j), there exists no chord a_we_i or e_ie_j in T_1. With the fact that $t > 0$, we know that (e_t, a_y)-$(a_y, \ldots, a_w)$-$(a_w, e_1, \ldots, e_t)$ is a 3-boundary of T_1. Finally, since T_1 has less edges than $T_{d_xa_y}$ ($d_xa_y \notin E(T_1)$), Property 1 holds for T_1 with respect to the mentionned 3-boundary. Let (Σ_1, R_1, X_1) be the partial strong 1-string representations contained in the region τ_1 obtained for T_1.

In Figure 28, starting from (Σ', R', X') and (Σ_1, R_1, X_1), we show how to join the strings $\sigma'_{a_w} \in \Sigma'$ and $\sigma^1_{a_w} \in \Sigma_1$, how to extend the string $\sigma^1_{a_y} \in \Sigma^1$ and the strings $\sigma^1_{e_i} \in \Sigma^1$, for $1 \leq i \leq t$, and how to draw the face-regions $\rho_{a_ye_td_x}$, $\rho_{e_1a_wd_x}$, and $\rho_{e_ie_{i-1}d_x}$, for $2 \leq 2 \leq t$, in order to obtain (Σ, R, X), a partial strong 1-string representation of $T_{d_xa_y}$ that satisfies Property 2. Here we have $\Sigma = (\Sigma' \setminus \{\sigma'_{a_w}\}) \cup (\Sigma_1 \setminus (\{\sigma^1_{a_i} \mid i \in [y,w]\} \cup \{\sigma^1_{e_i} \mid i \in [1,t]\})) \cup \{\sigma_{a_i} \mid i \in [y,w]\} \cup \{\sigma_{e_i} \mid i \in [1,t]\}$, with σ_{a_w} being the junction of σ'_{a_w} and $\sigma^1_{a_w}$, the strings σ_{a_i} (resp. σ_{e_i}) being the extensions of the strings $\sigma^1_{a_i} \in \Sigma_1$ (resp. $\sigma^1_{e_i} \in \Sigma_1$), $R = R' \cup R_1 \cup \{\rho_{e_1a_wd_x}, \rho_{a_ye_td_x}\} \cup \{\rho_{d_se_te_{t-1}} \mid i \in [2,t]\}$ and $X = E_o(T) \setminus \{d_xa_y\}$.

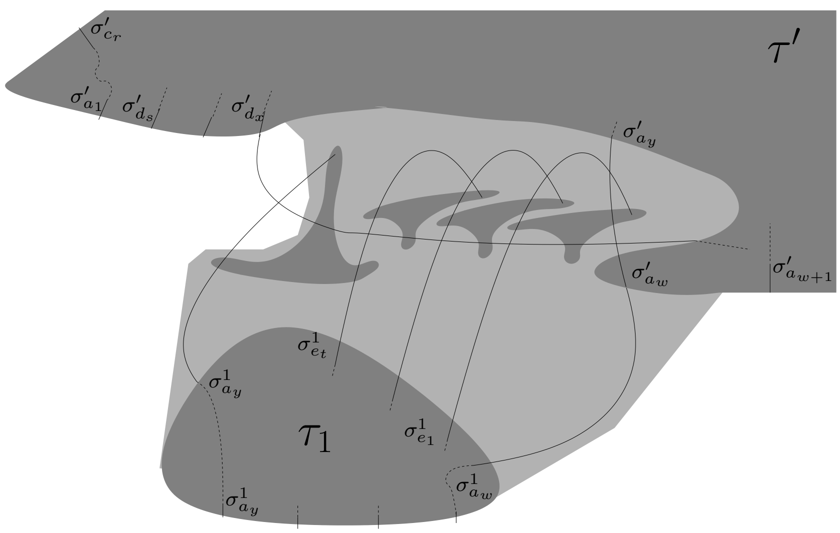

Figure 28: Case 2.4: (Σ, R, X).

We check that (Σ, R, X) is correct:

- Σ is a 1-string representation: Since the edges d_xa_y, a_we_1, e_ie_{i+1}, and d_xe_i are not in $(E(T_{d_xa_y}) \setminus X') \cup (E(T_1) \setminus X_1)$ there is no two strings intersecting more than once.

- Σ is a 1-string representation of $T_{d_xa_y} \setminus X$ with $X = E_o(T_{d_xa_y}) \setminus \{d_xa_y\}$: Indeed, $E(T_{d_xa_y}) \setminus X = (E(T_{d_za_w}) \setminus X') \cup (E(T_1) \setminus X_1) \cup \{a_we_1, d_xa_y\} \cup \{e_ie_{i-1} \mid i \in [2,t]\} \cup \{d_xe_i \mid$

$i \in [1, t]\}$.

- (Σ, R) is "strong": The only inner-faces of $T_{d_x a_y}$ that are not inner-faces in $T_{d_z a_w}$ or T_1 are $d_x a_y e_t$, $d_x a_w e_1$, and the faces $d_x e_i e_{i-1}$, for $2 \leq i \leq t$. Since the regions τ', τ_1, $\rho_{d_x a_y e_t}$, $\rho_{d_x a_w e_1}$, and $\rho_{d_x e_i e_{i-1}}$, for $2 \leq i \leq t$, are all disjoint, all the face-regions of R are disjoint.
- We see in Figure 28 that conditions (b), (c), (d), (e), (f), and (g) of Property 2 are satisfied.

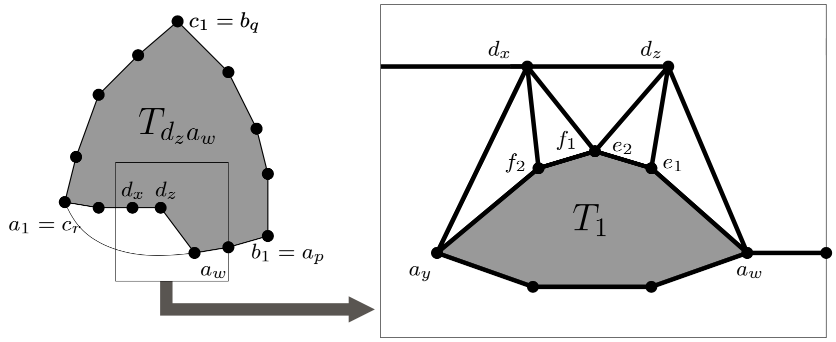

Figure 29: Case 2.5: $T_{d_x a_y} \neq T_{d_1 a_p}$, $z = x - 1$ and $w > y$.

Case 2.5: $d_x a_y \neq d_1 a_p$, $z = x - 1$ and $w > y$ (see Figure 29). By Lemma 2, $T_{d_z a_w}$ is a W-triangulation. Since $T_{d_z a_w}$ has less edges than $T_{d_x a_y}$ ($d_x a_y \notin E(T_{d_z a_w})$), Property 2 holds for $T_{d_z a_w}$. Let (Σ', R', X') be the partial strong 1-string representation of $T_{d_z a_w}$ contained in the region τ' with $X' = E_o(T_{d_z a_w}) \setminus \{d_z a_w\}$.

Let us denote $e_1, e_2, \ldots, e_t$ the neighbors of d_z strictly inside the cycle $(d_z, d_x, a_y, \ldots, a_w, d_z)$, going "from right to left" (see Figure 29). By maximality of z, there is no edge $d_x a_w$, so $t > 0$. Let us denote $f_1, \ldots, f_u$ the neighbors of d_x strictly inside the cycle $(d_x, a_y, \ldots, a_w, d_z)$, going "from right to left" (see Figure 29). Note that $f_1 = e_t$ and that by minimality of w, there is no edge $d_z a_y$, so $u > 0$.

By minimality of w we have $e_i \neq a_j$ (resp. $f_i \neq a_j$), for all $1 \leq i \leq t$ (resp. $1 \leq i \leq u$) and $y \leq j \leq w$. Let T_1 be the subgraph of $T_{d_x a_y}$ that lies inside the cycle $(a_y, \ldots, a_w, e_1, \ldots, e_t, f_2, \ldots, f_u, a_y)$. By Lemma 2, T_1 is a W-triangulation. Since the W-triangulation $T_{d_x a_y}$ has no separating 3-cycle (d_z, a_w, e_i), (d_z, e_i, e_j), (d_x, f_i, f_j), or (d_x, f_i, a_y), there exists no chord $a_w e_i$, $e_i e_j$, $f_i f_j$, or $f_i a_y$ in T_1. With the fact that $t > 0$ and $u > 0$, we know that $(f_1, f_2, \ldots, f_u, a_y)$-$(a_y, \ldots, a_w)$-$(a_w, e_1, \ldots, e_t)$ is a 3-boundary of T_1. Finally, since T_1 has less edges than $T_{d_x a_y}$ ($d_x a_y \notin E(T_1)$), Property 1 holds for T_1 with respect to the mentionned 3-boundary. Let (Σ_1, R_1, X_1) be the partial strong 1-string representations contained in the region τ_1 obtained for T_1.

In Figure 30, starting from (Σ', R', X') and (Σ_1, R_1, X_1), we show how to join the strings $\sigma'_{a_w} \in \Sigma'$ and $\sigma^1_{a_w} \in \Sigma_1$, how to extend the string $\sigma'_{d_x} \in \Sigma'$, $\sigma^1_{a_y} \in \Sigma^1$ the strings $\sigma^1_{e_i} \in \Sigma^1$, for $1 \leq i \leq t$, and the strings $\sigma^1_{f_i} \in \Sigma^1$, for $2 \leq i \leq u$, and how to draw the face-regions $\rho_{d_z a_w e_1}$, $\rho_{d_z e_i e_{i-1}}$, for $2 \leq i \leq t$, $\rho_{d_z d_x e_t}$, $\rho_{d_x f_i f_{i-1}}$, for $2 \leq i \leq u$, and $\rho_{d_x a_y f_u}$ in order to obtain (Σ, R, X), a partial strong 1-string representation of $T_{d_x a_y}$ that satisfies Property 2. Here we have $\Sigma = (\Sigma' \setminus \{\sigma'_{d_x}, \sigma'_{a_w}\}) \cup (\Sigma_1 \setminus (\{\sigma^1_{a_i} \mid i \in [y, w]\} \cup \{\sigma^1_{e_i} \mid i \in [1, t]\} \cup \{\sigma^1_{e_i} \mid i \in [2, u]\})) \cup \{\sigma_{a_i} \mid$

$i \in [y,w]\} \cup \{\sigma_{e_i} \mid i \in [1,t]\} \cup \{\sigma_{e_i} \mid i \in [2,u]\}$, with σ_{a_w} being the junction of σ'_{a_w} and $\sigma^1_{a_w}$, the strings σ_{a_i} (resp. σ_{e_i} or σ_{f_i}) being the extensions of the strings $\sigma^1_{a_i} \in \Sigma_1$ (resp. $\sigma^1_{e_i}$ or $\sigma^1_{f_i} \in \Sigma_1$), $R = R' \cup R_1 \cup \{\rho_{d_z a_w e_1}, \rho_{d_z d_x e_t}, \rho_{d_x a_y f_u}\} \cup \{\rho_{d_z e_i e_{i-1}} \mid i \in [2,t]\} \cup \{\rho_{d_x f_i f_{i-1}} \mid i \in [2,u]\}$, and $X = E_o(T) \setminus \{d_x a_y\}$.

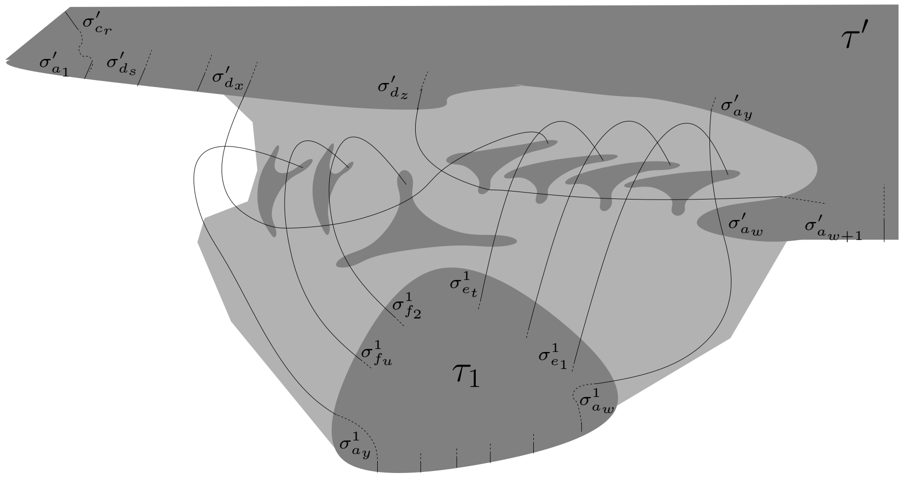

Figure 30: Case 2.5: (Σ, R, X).

We check that (Σ, R, X) is correct:

- Σ is a 1-string representation: Since the edges $d_z e_i$ with $1 \le i \le t$, $d_x d_z$, $a_w e_1$, $e_i e_{i-1}$ with $2 \le i \le t$, $d_x f_i$ with $1 \le i \le u$, $d_x a_y$, $f_i f_{i-1}$ with $3 \le i \le u$, and $f_u a_y$ are not in $(E(T_{d_x a_y}) \setminus X') \cup (E(T_1) \setminus X_1)$ there is no two strings intersecting more than once.

- Σ is a 1-string representation of $T_{d_x a_y} \setminus X$ with $X = E_o(T_{d_x a_y}) \setminus \{d_x a_y\}$: Indeed, $E(T_{d_x a_y}) \setminus X = (E(T_{d_z a_w}) \setminus X') \cup (E(T_1) \setminus X_1) \cup \{d_x a_y, d_x d_z, a_w e_1, a_y f_u\} \cup \{d_z e_i \mid i \in [1,t]\} \cup \{d_x f_i \mid i \in [1,u]\} \cup \{e_i e_{i-1} \mid i \in [2,t]\} \cup \{f_i f_{i-1} \mid i \in [2,u]\}$.

- (Σ, R) is "strong": The only inner-faces of $T_{d_x a_y}$ that are not inner-faces in $T_{d_z a_w}$ or T_1 are $d_z a_w e_1$, $d_z e_i e_{i-1}$ for $2 \le i \le t$, $d_z d_x e_t$ $d_x f_i f_{i-1}$ for $2 \le i \le u$, and $d_x a_y f_u$. Since the regions τ', τ_1, $\rho_{d_z a_w e_1}$, $\rho_{d_z e_i e_{i-1}}$ for $2 \le i \le t$, $\rho_{d_z d_x e_t}$ $\rho_{d_x f_i f_{i-1}}$ for $2 \le i \le u$, and $\rho_{d_x a_y f_u}$ are all disjoint, all the face-regions of R are disjoint.

- We see in Figure 30 that conditions (b), (c), (d), (e), (f), and (g) of Property 2 are satisfied.

This completes the study of Case 2. So, Property 2 holds for any W-triangulation $T_{d_x a_y}$ such that $|E(T_{d_x a_y})| = m$. This completes the proof of Lemma 4. □

This completes the proof of Theorem 4. □