Coherency and purity for monoids

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Throughout, $S$ will denote a monoid.

**Finitary condition**
A condition satisfied by all finite monoids.

**Example**
Every element of $S$ has an idempotent power.

Finitary conditions were introduced by **Noether** and **Artin** in the early 20th Century to study rings; they changed the course of algebra entirely.
Coherency for Monoids: a finitary condition

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Finitary conditions were introduced by **Noether** and **Artin** in the early 20th Century to study rings; they changed the course of algebra entirely.

**Example**

Every right congruence on $S$ is finitely generated, i.e. $S$ is **right Noetherian**.

Every right ideal of $S$ is finitely generated, i.e. $S$ is **weakly right Noetherian**.
Coherency for Monoids: the definition

Coherency

This is the (first) finitary condition of importance to us today

Definition

$S$ is right coherent if every finitely generated $S$-subact of every finitely presented right $S$-act is finitely presented.\(^a\)

\(^a\)This definition comes from Wheeler (1976)
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Left coherency is defined dually: $S$ is coherent if it is left and right coherent.
Acts over monoids: $S$-acts

Representation of monoid $S$ by mappings of sets

A **(right) $S$-act** is a set $A$ together with a map

$$A \times S \to A, \ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

Beware: an $S$-act is also called an $S$-set, $S$-system, $S$-action, $S$-operand, or $S$-polygon.

Let $Act-S$ denote the class of all $S$-acts.
Acts over monoids: $S$-acts

Standard definitions/Elementary observations

- $S$-acts form a variety of universal algebras, to which we may apply the usual notions of subalgebra ($S$-subact), morphism ($S$-morphism), congruence, etc.
- $S$-acts and $S$-morphisms form a category, $\text{Act-}S$.
- We have usual definitions of free, projective, injective, etc. including variations on flat.
- Free $S$-acts are disjoint unions of copies of $S$. 
Acts over monoids: \( S \)-acts

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- We have usual definitions of \textit{free}, \textit{projective}, \textit{injective}, etc. including variations on \textit{flat}.

- Free \( S \)-acts are \textbf{disjoint unions of copies of} \( S \).

- \( A \) is \textbf{finitely presented} if

\[
A \cong F_S(X)/\rho
\]

for some finitely generated free \( S \)-act \( F_S(X) \) and finitely generated congruence \( \rho \).
Right coherent monoids
First observations

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**Theorem:** Normak (77)

If S is right noetherian then S is right coherent.

**Example:** Fountain (92)

There is a monoid S which is weakly right noetherian but which is not right coherent.
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**Example:** Fountain (92)

There is a monoid $S$ which is weakly right noetherian but which is not right coherent.

Let us call the example above the **Fountain monoid**; it is made up of a group and a 4 element nilpotent semigroup.
Why is coherency interesting?

- The definition is natural, and fits with that for rings.
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- It has connections with products and ultraproducts of flat left \(S\)-acts (Bulman-Fleming and McDowell, G, Sedaghatjoo).
- Coherency is related to purity (more later).
Why is coherency interesting?

**Theorem:** Wheeler (1976); G (1986), Ivanov (1992)

The following are equivalent for a monoid $S$:

1. $S$ is right coherent;
2. the existentially closed $S$-acts form an axiomatisable class;
3. the first-order theory of $S$-acts has a model companion.
Let $A$ be an $S$-act. An **equation** over $A$ has the form

$$xs = xt, \quad xs = yt \text{ or } xs = a$$

where $x, y$ are variables, $s, t \in S$ and $a \in A$. 
Equations over $S$-acts

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**Consistency**

A set of equations is consistent if it has a solution in some $S$-act $B \supseteq A$. 
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**Consistency**

A set of equations is **consistent** if it has a solution in some $S$-act $B \supseteq A$.

**Absolutely pure and almost pure**

$A$ is **absolutely pure** (**almost pure**) if every finite consistent set of equations over $A$ (in 1 variable) has a solution in $A$.

*absolutely pure = algebraically closed*
*almost pure = 1-algebraically closed*
Equations and inequations over $S$-acts

Let $A$ be an $S$-act. An **inequation** over $A$ has the form

$$xs \neq xt, \; xs \neq yt \; \text{or} \; xs \neq a$$

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Let $A$ be an $S$-act. An inequation over $A$ has the form

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A set of equations and inequations is **consistent** if it has a solution in some $S$-act $B \supseteq A$. 
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A set of equations and inequations is **consistent** if it has a solution in some $S$-act $B \supseteq A$.

**Existentially closed and 1-existentially closed**

$A$ is **existentially closed** (1-existentially closed) if every finite consistent set of equations and inequations over $A$ (in 1 variable) has a solution in $A$. 
Why is coherency interesting?

Theorem: G (1986)

The following are equivalent for a monoid $S$:

1. $S$ is right coherent;
2. the existentially closed $S$-acts $\mathcal{E}$ form an axiomatisable class;
3. the first-order theory of $S$-acts has a model companion;
4. the 1-existentially closed $S$-acts $\mathcal{E}_1$ form an axiomatisable class;
5. the absolutely pure $S$-acts $\mathcal{A}$ form an axiomatisable class;
6. the almost pure $S$-acts $\mathcal{A}_1$ form an axiomatisable class.
The following monoids are right coherent:
Which monoids are right coherent?


Theorem:

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- the free monoid on $X$;
- the free left restriction monoid on $X$. 
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- the free commutative monoid on $X$;
- the free monoid on $X$;
- the free left restriction monoid on $X$.

Theorem:
The free inverse monoid on $X$ for $|X| \geq 2$ is not right coherent.
Completely right pure monoids

**Definition:** A monoid is completely right pure if every $S$-act is absolutely pure.

Clearly

\[ A \subseteq A_1 \subseteq \text{Act}-S. \]

**Theorem: G (1991)**

A monoid $S$ is completely right pure if and only if all $S$-acts are almost pure, i.e.

\[ \text{Act}-S = A_1 \iff \text{Act}-S = A. \]
The fact $\text{Act-}S = A_1 \iff \text{Act-}S = A_1$ enabled me to characterise *completely right pure monoids (1991)* in a way analogous to that of Skornjakov (1979), and Fountain (1974) and Isbell (1972) for *completely right injective monoids*.

**A Question**

Does there exist a monoid $S$ and an $S$-act $A$ such that $A$ is almost pure but not absolutely pure??????
Purity: Absolute purity vs almost purity
The Question: does $A = A_1$ for every monoid $S$?

Theorem: G, Yang Dandan, Salma Shaheen (2016)

Let $S$ be a finite monoid and let $A$ be an almost pure $S$-act. Then $A$ is absolutely pure.

Consequently: $A = A_1$ is a finitary condition.
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Let $S$ be a finite monoid and let $A$ be an almost pure $S$-act. Then $A$ is absolutely pure.

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Theorem: G, Yang Dandan (2016/7)
Let $S$ be a right coherent monoid and let $A$ be an almost pure $S$-act. Then $A$ is absolutely pure.

That is,

$$S \text{ right coherent } \implies A = A_1$$
Purity: Absolute purity vs almost purity
New Question: does $A = A_1$ if and only if $S$ is right coherent?
Purity: Absolute purity vs almost purity
New Question: does $\mathcal{A} = \mathcal{A}_1$ if and only if $S$ is right coherent?

Counterexample **G, Yang Dandan (2017)**

No! The Fountain Monoid is an example of a non-coherent monoid such that $\mathcal{A} = \mathcal{A}_1$. 
Purity: Absolute purity vs almost purity
The Question: does $A = A_1$ for every monoid $S$?

For an $S$-act $A$ we can build canonical absolutely pure (almost pure) extensions $A(\aleph_0)$ ($A(1)$).

**Proposition G: 2017**

The following are equivalent for a monoid $S$:

1. every almost pure $S$-act is absolutely pure;
2. for every **finitely generated subact of every finitely presented** $S$-act $A$, we have $A(1)$ is a retract of $A(\aleph_0)$. 
• Does there exist a monoid $S$ and an $S$-act $A$ such that $A$ is almost pure but not absolutely pure? Use the last result to write down a condition on chains of right congruences such that every almost pure $S$-act is absolutely pure; **now find an $S$ not satisfying this condition.**
Questions, Questions...!

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- What happens for rings and modules? Are the almost pure (1-algebraically closed) absolutely pure (algebraically closed)?
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- Determine exact connections of right coherency with products-ultraproducts of flat **left** $S$-acts.
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- Other finitary conditions arise from model theoretic considerations of $S$-acts; many open questions remain!
Why am I interested in coherency?

Purity: absolute purity vs almost purity

Purity properties may be reformulated as weak injectivity properties. Injectivity may be reformulated as a stronger purity property.

**Definition:** A monoid is completely right injective (completely right pure) if every $S$-act is injective (absolutely pure).

**Fountain (1974), Isbell (1972) (following work of Skornjakov (69) and others)**

Characterised completely right injective monoids in terms of right ideals and elements.
Completely right injective monoids

**Theorem: Skornjakov (1969)**

A monoid $S$ is completely right injective if $S$ has a left zero and $S$ satisfies (*) for any right ideal $I$ of $S$ and right congruence $\rho$ on $S$, there is an $s \in I$ such that for all $u, v \in S$, $w \in I$, $sw \rho w$ and if $u \rho v$ then $su \rho sv$.

**Theorem: Fountain (1974)**

A monoid $S$ is completely right injective if and only if $S$ has a zero, and each right ideal $I$ has an idempotent generator $e$ such that, for each pair of elements $a, b \in S \setminus I$, we have $a'ea \mathcal{R} b'eb$ for all $a' \in V(a)$, $b' \in V(b)$ implies that $a'ea = b'eb$ for all $a' \in V(a)$, $b' \in V(b)$. 