Monomorphisms in categories of firm acts

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(Joint research with Ülo Reimaa)
1. Preliminaries

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$$\mu_A : A \otimes S \to A, \quad a \otimes s \mapsto as$$

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is bijective. A semigroup $S$ is called **firm** if the mapping

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is bijective. A right $S$-act $A_S$ is called **unitary** if $\mu_A$ is surjective, i.e. $AS = A$. 
Notation:

\( \text{Act}_S \) — the category of all right \( S \)-acts,
\( \text{UAct}_S \) — the category of all unitary right \( S \)-acts,
\( \text{FAct}_S \) — the category of all firm right \( S \)-acts.

Morphisms in these categories are the right \( S \)-act homomorphisms.
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$\text{FAct}_S \subseteq \text{UAct}_S \subseteq \text{Act}_S$
In a category $\mathcal{A}$, a morphism $f$ is called a **monomorphism** if for any morphisms $u, v$

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**regular mono** $\implies$ **extremal mono** $\implies$ **mono**
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If $S$ is a semigroup then the monomorphisms, the extremal monomorphisms and the regular monomorphisms in $\text{Act}_S$ are precisely the injective homomorphisms.
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Proposition
Let $S$ be a semigroup. Then a morphism $f : A_S \longrightarrow B_S$ in $\text{UAct}_S$ is a monomorphism in $\text{UAct}_S$ if and only if for all $a, b \in A$

\[ f(a) = f(b) \iff \forall s \in S \ as = bs. \]
Corollary

If $S$ is a semigroup and $A_S \in \text{UAct}_S$ then the mapping

\[ \mu_A : A \otimes S \rightarrow A, \ a \otimes s \mapsto as \]

is a monomorphism in $\text{UAct}_S$. 

Example

The category $\text{UAct}_S$ may contain non-injective monomorphisms.

1. Let $S$ be a factorisable semigroup which is not firm (such semigroups exist). Then $S$ is unitary and the mapping $\mu_S : S \otimes S \rightarrow S$ is surjective but not injective. By the previous corollary, it is a monomorphism in $\text{UAct}_S$.

2. Consider a non-trivial right zero semigroup $S$ and a one-element right act $\Theta_S = \{ \theta \}$. The unique morphism $f : S \rightarrow \Theta_S$ is a monomorphism in $\text{UAct}_S$, but it is not injective.
Corollary

If $S$ is a semigroup and $A_S \in \text{UAct}_S$ then the mapping

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Theorem

The following assertions are equivalent for a firm semigroup $S$ and a morphism $f : B_S \longrightarrow A_S$ in $\text{FAct}_S$. 

1. $f$ is a monomorphism.
2. $f$ is an extremal monomorphism.
3. $f$ is a regular monomorphism.
4. For all $a, b \in B_S$, $f(a) = f(b) \Rightarrow \forall s \in S \text{ as } s = bs$.
5. $f = \mu_A(m \otimes 1_S)g$ for a unitary $S$-act $M_S$, an injective homomorphism $m : M_S \longrightarrow A_S$ and an isomorphism $g : B_S \longrightarrow M \otimes S$.
6. $f = h(m \otimes 1_S)g$ for an injective homomorphism $m : M_S \longrightarrow N_S$ and isomorphisms $g : B_S \longrightarrow M \otimes S$, $h : N \otimes S \longrightarrow A_S$. 
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$$f(a) = f(b) \implies \forall s \in S \text{ as } as = bs.$$

5. $f = \mu_{A_S}(m \otimes 1_S)g$ for a unitary $S$-act $M_S$, an injective homomorphism $m : M_S \to A_S$ and an isomorphism $g : B_S \to M_S \otimes S$.

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Proposition

Let $S$ be a firm semigroup and let a firm act $A_S$ have a unitary subact $B_S$ which is not firm. Let $m : B \rightarrow A$ be the inclusion mapping. Then $m \otimes 1_S : B \otimes S \rightarrow A \otimes S$ is a non-injective regular monomorphism in $\text{FAct}_S$. 

Example

Consider the semigroup $S = \{0, a, b, e\}$ given by the following multiplication table:

\[
\begin{array}{c|cccc}
 & 0 & a & b & e \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & \ \\
e & a & b & e & \ \\
\end{array}
\]

Since $S$ has a left identity $e$, it easily follows that $S_S$ is a firm act. Its subact $B_S = \{0, b\}$ is unitary but not firm because $b \cdot 0 = 0 = ba$ while $b \otimes 0 \neq b \otimes a$ in $B \otimes S$.
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3. When are monomorphisms injective?
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Proposition

Let \( S \) be a firm semigroup. Then the following statements are equivalent:

1. monomorphisms in \( \text{UAct}_S \) are injective,
2. \( \mu_A \) is injective for all right \( S \)-acts \( A_S \),
3. every unitary right \( S \)-act is firm (i.e. \( \text{UAct}_S = \text{FAct}_S \)).
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Proposition

Let $S$ be a firm semigroup. Then the following statements are equivalent:

1. monomorphisms in $\text{UAct}_S$ are injective,
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3. every unitary right $S$-act is firm (i.e. $\text{UAct}_S = \text{FAct}_S$).

Corollary

Monomorphisms in $\text{FAct}_S$ are injective if

1. for every $s, t \in S$, there exists $u \in S$ such that $s = su$ and $t = tu$, or
2. $S$ is nilpotent.
A left $S$-act $sM$ is called **flat** if for every injective homomorphism $f : B_S \rightarrow A_S$ of right $S$-acts the mapping $f \otimes 1_M : B \otimes M \rightarrow A \otimes M$ is injective.
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**Proposition**

Let $S$ be a firm semigroup. Monomorphisms in $FAct^S$ are injective if and only if $S$ is flat.

**Corollary**

If $S$ is a left absolutely flat semigroup then monomorphisms in $FAct^S$ are injective.

**Corollary**

If $S$ is an inverse semigroup then monomorphisms in $FAct^S$ are injective.
A left $S$-act $\_M$ is called **flat** if for every injective homomorphism $f : B_S \rightarrow A_S$ of right $S$-acts the mapping $f \otimes 1_M : B \otimes M \rightarrow A \otimes M$ is injective. A semigroup $S$ is said to be **left absolutely flat** if all left $S$-acts are flat.

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**Corollary**

If $S$ is a left absolutely flat semigroup then monomorphisms in $\text{FAct}_S$ are injective.
A left $S$-act $SM$ is called **flat** if for every injective homomorphism $f : BS \to AS$ of right $S$-acts the mapping $f \otimes 1_M : BM \to AM$ is injective. A semigroup $S$ is said to be **left absolutely flat** if all left $S$-acts are flat.

**Proposition**

*Let $S$ be a firm semigroup. Monomorphisms in $\text{FAct}_S$ are injective if and only if $S$ is flat.*

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4. The lattice of subobjects

In a category \( \mathcal{A} \), subobjects of an object \( A \) are defined as equivalence classes of monomorphisms \( f : B \to A \) with respect to the equivalence relation \( \equiv \) defined by

\[
  f \equiv g \iff f = gh \text{ for some isomorphism } h.
\]

\[
\begin{array}{c}
A \\
\uparrow f \quad \downarrow g \\
B \quad \quad \quad \quad C \\
\downarrow h
\end{array}
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We will denote such an equivalence class by \( \overline{f} \) and the class of subobjects of \( A \) in the category \( \mathcal{A} \) by \( \text{Sub}_{\mathcal{A}}(A) \).
4. The lattice of subobjects

In a category $\mathcal{A}$, subobjects of an object $A$ are defined as equivalence classes of monomorphisms $f : B \rightarrow A$ with respect to the equivalence relation $\equiv$ defined by

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We will denote such an equivalence class by $\bar{f}$ and the class of subobjects of $A$ in the category $\mathcal{A}$ by $\text{Sub}_\mathcal{A}(A)$. On $\text{Sub}_\mathcal{A}(A)$ one can define a partial order by

$$\bar{f} \leq \bar{g} \iff f = gh \text{ for some morphism } h.$$
Theorem (González-Férez and Marín, 2010)

In the category of firm modules over an associative ring, the lattices of subobjects are modular.
For $A_S \in \text{FAct}_S$ we write

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By a subact of an act $A_S$ we mean a subset $B \subseteq A$ such that $bs \in B$ for each $b \in B$ and $s \in S$. 

Theorem: Let $A_S$ be a firm right act over a firm semigroup $S$. Then $S(A)$ is a poset which is a modular lattice, isomorphic to the lattice $U(A)$. So the study of monomorphisms in $\text{FAct}_S$ reduces to the study of unitary subacts of firm acts.
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By a \textbf{subact} of an act \( A_S \) we mean a subset \( B \subseteq A \) such that \( bs \in B \) for each \( b \in B \) and \( s \in S \). We denote by \( \mathcal{U}(A) \) the set of unitary subacts of a firm act \( A_S \). It is easy to see that \( \mathcal{U}(A) \) is a modular lattice where meets are intersections, joins are unions, and the order is given by inclusion.
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**Theorem**

Let $A_S$ be a firm right act over a firm semigroup $S$. Then $S(A)$ is a poset which is a modular lattice, isomorphic to the lattice $\mathcal{U}(A)$.

So the study of monomorphisms in $\text{FAct}_S$ reduces to the study of unitary subacts of firm acts.
References