Injective hulls for S-posets

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Contents

1. Introduction: Results of injective hulls on posets

1967, Banaschewski B., Bruns G., Categorical construction of the MacNeille completion, *Arch. Math.*

Theorem 1 (Banaschewski B., Bruns G.) For a partially ordered set P, T.A.E.:

1. P is complete;
2. P is strictly injective;
3. P is a retract of every extension;
4. P has no essential extensions.

1. Introduction: Results of injective hulls on posets

Theorem 2(Banaschewski B., Bruns G.) For a partially ordered set P, T.A.E.:

1. E is a MacNeille completion of P;
2. E is an essential, strictly injective extension of P;
3. E is a strictly injective extension of P not containing any properly smaller such extension of P;
4. E is an essential extension of P not containing in any properly larger such extension of P.

1970, Burns G., Lakser H., injective hulls of semilattices. *Bulletin*

Theorem 3 (Burns G., Lakser H.)
A semilattice S is injective iff it is complete and satisfying

$$a \bigwedge \bigvee M = \bigvee (a \bigwedge m \mid m \in M), \qquad (1)$$

for all $a \in S$, $M \subseteq S$.

1970, Burns G., Lakser H., injective hulls of semilattices. *Bulletin*

$$I_D(S) = \{A \subseteq S \mid A = A\downarrow; \quad M \subseteq A \text{ with } \bigvee M \text{ exists and satisfying (1)} \Rightarrow \bigvee M \in A\}.$$

Theorem 4 (Burns G., Lakser H.)

Injective hulls of a semilattice S is $I_D(S)$.

1. Introduction: Results of injective hulls on certain S-systems over a semilattice

1972, Johnson C.S., J.R., McMorris F.R., injective hulls on certain S-systems over a semilattice. *Proc. Amer. Math. Soc.*

Theorem 5 (Johnson C.S., J.R., McMorris F.R.)

Injective hulls of an S-system M_S is $I_D(M_S)$.

1974, Schein B.M. Injectives in certain classes of semigroups. *Semigroup Forum*.

Theorem 6. (1974 Schein) The category of semigroups has only trivial injectives.

2012, Lambek J., Barr M., Kennison J.F. and Raphael R.,
Injective hulls of partially ordered monoids. *Theory Appl. Categ.*

The category of po-monoids

Partially ordered monoids (**po-monoid**) with *submultiplicative* order-preserving mappings, i.e., an order-preserving mapping $\phi : A \to B$ of po-monoids satisfying

$$\phi(1) = 1,$$

$$\phi(a)\phi(b) \leq \phi(ab),$$

for all $a, b \in A$.

2012, Lambek J., Barr M., Kennison J.F. and Raphael R.,
Injective hulls of partially ordered monoids. *Theory Appl. Categ.*

Theorem 7 (2012 Lambek, Barr, Kennison and Raphael)
A po-monoid is injective iff it is a quantale.

Definition 1 (Mulvey C.J., 1986)

A quantale is a complete lattice Q equipped with an associative binary operation satisfying

$$a(\bigvee M) = \bigvee\{am \mid m \in M\}, \quad (\bigvee M)a = \bigvee\{ma \mid m \in M\}$$

for any $a \in Q,\ M \subseteq Q$.

A *frame(locale)* L is a complete lattice such that

$$a \bigwedge(\bigvee M) = \bigvee\{a \bigwedge m \mid m \in M\},$$

for any $a \in L,\ M \subseteq L$.

Definition 2 (Cf. [5, 9])

Given a quantale Q, a left quantale module is a pair $(A, *)$, where A is a $\bigvee$-lattice and $Q \times A \xrightarrow{*}$ such that:

1. $q * (\bigvee S) = \bigvee_{s \in S}(q * s)$ for every $q \in Q,\ S \subseteq A$;
2. $(\bigvee T) * a = \bigvee_{t \in T}(t * a)$ for every $a \in A,\ T \subseteq Q$;
3. $q_1 * (q_2 * a) = (q_1 q_2) * a$ for every $a \in A,\ q_1, q_2 \in Q$.

Definition 3 (Cf. [9]) Given a commutative quantale Q, a quantale algebra is a quatnale module $(A, *)$ such that:

1. $(A, \leq, \otimes)$ is a quantale;
2. $q * (a \otimes b) = (q * a) \otimes b$ for every $a, b \in A,\ q \in Q$.

$Pos_{\leq}$

Objects: posemigroups;

Morphisms: order-preserving submultiplicative mappings;

$\varepsilon_{\leq}$: a morphism $f : S \to T$ in $Pos_{\leq}$ belongs to $\varepsilon_{\leq}$ if it satisfies:

$$f(a_1) \cdots f(a_n) \leq f(a) \Longrightarrow a_1 \cdots a_n \leq a,$$

for each $a_1, \cdots, a_n, a \in S$.

Theorem 8 (2014, Zhang, Laan)

$\varepsilon_{\leq}$-injective objects in the category $Pos_{\leq}$ are exactly quantales.

Example: quantale $(\mathscr{P}(S), \cdot, \subseteq)$

S: a posemigroup,
$\mathscr{P}(S)$: the set of all downsets of S,

$$X \cdot Y = (XY) \downarrow = \{x \in S \mid s \leq xy \text{ for some } x \in X,\ y \in Y\},$$

for $X, Y \in \mathscr{P}(S)$.

Definition 4: Nucleus

a *nucleus* on a quantale Q is a submultiplicative closure operator on Q.

the nucleus cl:

For any down-set I of a posemigroup S we define its closure by

$$\mathrm{cl}(I) := \{x \in S \mid aIc \subseteq b\downarrow \text{ implies } axc \leq b \text{ for all } a, b, c \in S\}.$$

Then $\mathrm{cl} : \mathscr{P}(S) \to \mathscr{P}(S)$ is a nucleus.

Construction: $\mathscr{Q}(S)$

$$\mathscr{Q}(S) := \{I \in \mathscr{P}(S) \mid I = \mathrm{cl}(I)\}$$

define a multiplication $\circ$ on $\mathscr{Q}(S)$ by

$$I \circ J := \mathrm{cl}(I \cdot J). \tag{1}$$

Theorem 9 (2014 Zhang, Laan)[Cf. Theorem 5.8 in [6]]

Let S be a posemigroup such that $\mathrm{cl}(s\downarrow) = s\downarrow$ for every $s \in S$. Then $\mathscr{Q}(S)$ is an $\varepsilon_{\leq}$-injective hull of S in $Pos_{\leq}$.

In this work, S is always a *pomonoid*.

S-posets

A poset $(A, \leq)$ together with a mapping $A \times S \to A$ (under which a pair (a, s) maps to an element of A denoted by as) is called a *right S-poset*, denoted by A_S, if for any $a, b \in A,\ s, t \in S$,

1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leq b,\ s \leq t$ imply that $as \leq bt$.

A left S-poset can be defined similarly.

S: a pomonoid

Fakhruddin S.M., *On the category of S-posets*. Acta Sci. Math., 1988

Result 1:

Each injective S-poset is **"complete"** if it is a complete lattice and satisfying the distributions of arbitrary joins with S-actions.

Result 2:

Let G be a pogroup. Then a comlete G-poset is injective.

Definition 5 (2015, Zhang, Laan)

We call a right S-poset A_S a *right S-quantale* if

1. the poset A is a complete lattice;
2. $(\bigvee M)s = \bigvee\{ms \mid m \in M\}$ for each subset M of A and each $s \in S$.

Let A_S and B_S be S-posets. We say that a mapping $f : A \to B$ is S-**submultiplicative** if $f(a)s \leq f(as)$ for any $a \in A,\ s \in S$.

$Pos_S^{\leq}$

We denote by $Pos_S^{\leq}$ the category where objects are right S-posets and morphisms are S-submultiplicative order-preserving mappings.

$\varepsilon_{\leq}$

Let $\varepsilon_{\leq}$ be the class of morphisms $e : A_S \to B_S$ in the category $Pos_S^{\leq}$ which satisfy the following condition:

$$e(a)s \leq e(a') \Longrightarrow as \leq a',\ \forall\ a, a' \in A,\ s \in S.$$

Theorem 10 (2015, Zhang, Laan)

Let A_S be an S-poset. Then A_S is $\varepsilon_{\leq}$-injective in $Pos_S^{\leq}$ if and only if A_S is a right S-quantale.

An S-quantale

Let A_S be an S-poset, and let $\mathscr{P}(A)$ be the set of all down-sets of the poset A. Define a right S-action $\cdot$ on $\mathscr{P}(A)$ by

$$D \cdot s = (Ds) \downarrow = \{x \in A \mid x \leq ds \text{ for some } d \in D\},$$

for any $s \in S$, $D \in \mathscr{P}(A)$. Then $(\mathscr{P}(A), \cdot, \subseteq)$ is a right S-quantale.

A nucleus

For any down-set D of an S-poset A_S we define its closure by

$$\mathrm{cl}(D) := \{x \in A \mid Ds \subseteq a\downarrow \text{ implies } xs \leq a \text{ for all } a \in A,\ s \in S\}.$$

Then cl is a nucleus on $\mathscr{P}(A)_S$.

Another S-quantale

For any S-poset A_S, we put

$$\mathscr{Q}(A) := \mathscr{P}(A)_{\mathrm{cl}} = \{D \in \mathscr{P}(A) \mid \mathrm{cl}(D) = D\}$$

and define a right S-action $\circ$ on $\mathscr{Q}(A)$ by

$$D \circ s := \mathrm{cl}(D \cdot s),$$

for any $s \in S$.

Theorem 11 (2015 Zhang, Laan)

For every S-poset A_S, $\mathscr{Q}(A)_S$ is the $\varepsilon_{\leq}$-injective hull of A_S in the category $Pos_S^{\leq}$.

Definition 6

Let Ω be a type. An *ordered Ω-algebra* is a triplet $\mathscr{A} = (A, \Omega_A, \leq_A)$ comprising a poset $(A, \leq_A)$ and a set Ω_A of operations on A (for every k-ary operation symbol $\omega \in \Omega_k$ there is a k-ary operation $\omega_A \in \Omega_A$ on A) such that all the operations ω_A are monotone mappings, where monotonicity of ω_A ($\omega \in \Omega_k$) means that

$$a_1 \leq_A a_1' \wedge \ldots \wedge a_k \leq_A a_k' \Longrightarrow \omega_A(a_1, \ldots, a_k) \leq_A \omega_A(a_1', \ldots, a_k')$$

for all $a_1, \ldots, a_k, a_1', \ldots, a_k' \in A$.

An ordered Ω-algebra $\mathscr{A} = (A, \Omega_A, \leq_A)$ is called a **sup-algebra** if the poset $(A, \leq)$ is a complete lattice and

$$\omega_A(a_1, \ldots, a_{i-1}, \bigvee M, a_{i+1}, \ldots, a_n)$$

$$= \bigvee \{\omega_A(a_1, \ldots, a_{i-1}, m, a_{i+1}, \ldots, a_n) \mid m \in M\}$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1, \ldots, n\}$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$, and $M \subseteq A$.

Sup-algebras are a generalization of quantale-like structures.

Definition 7

Let $\mathscr{A}, \mathscr{B}$ be ordered Ω-algebras of type Ω. A monotone mapping $f : A \to B$ is called a *subhomomorphism* if

$$\omega_B(f(a_1), \ldots, f(a_n)) \leq f(\omega_A(a_1, \ldots, a_n))$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in A$, and

$$\omega_B \leq f(\omega_A)$$

for every $\omega \in \Omega_0$. All ordered Ω-algebras together with their subhomomorphisms form a category which we denote by $\mathbf{OALg}_{\Omega}^{\leq}$.

We will study injectivity in classes of ordered Ω-algebras with respect to two classes of order-embeddings. The first of them is the class $\mathscr{M}_1$ of order-embeddings that are homomorphisms. The other class has a more complicated definition.

Let us denote by T_Ω^n the set of all n-ary Ω-terms. Let $\mathscr{M}_2$ be the class of mappings $h : \mathscr{A} \to \mathscr{B}$ between ordered Ω-algebras that satisfy the following conditions:

1. h is monotone,
2. $\omega_B(h(a_1), \ldots, h(a_n)) \leq h(\omega_A(a_1, \ldots, a_n))$ for every $n \in N$, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in A$,
3. $h(\omega_A) = \omega_B$ for all $\omega \in \Omega_0$,
4. for all $n \in \mathbb{N}$, $t \in T_\Omega^n$, $a_1, \ldots, a_n, a \in A$,

$$t_B(h(a_1), \ldots, h(a_n)) \leq h(a) \Longrightarrow t_A(a_1, \ldots, a_n) \leq a.$$

Lemma 2

For a class of ordered Ω-algebras, $\mathscr{M}_1 \subseteq \mathscr{M}_2$.

Theorem 12 (2016, Zhang, Laan)

Every sup-algebra $\mathscr{Q} = (Q, \Omega_Q, \leq_Q)$ of type Ω is $\mathscr{M}_2$-injective and therefore, $\mathscr{M}_1$-injective in the category $\mathbf{OALg}_\Omega^{\leq}$.

Proof.
Let $h\colon A \to B$ be a morphism in $\mathscr{M}_2$ and let $f\colon A \to Q$ be a morphism in $\mathbf{OALg}_{\Omega}^{\leq}$. Define a mapping $g\colon B \to Q$ by

$$g(b) = \bigvee \{ t(f(a_1), \ldots, f(a_m)) \mid \\ m \in \mathbb{N}, a_1, \ldots, a_m \in A,\ t \in T_{\Omega}^{m},\ t(h(a_1), \ldots, h(a_m)) \leq b \},$$

for any $b \in B$. We write the last join shortly as

$$\bigvee_{t(h(a_1),\ldots,h(a_m)) \leq b} t(f(a_1), \ldots, f(a_m)).$$

$$
\begin{aligned}
&\omega_Q(g(b_1),\ldots,g(b_n))\\
&= \omega_Q\Big(\bigvee\big\{t_1(f(a_{11}),\ldots,f(a_{1m_1}))\;\big|\;t_1(h(a_{11}),\ldots,h(a_{1m_1}))\le b_1\big\},\ldots,\\
&\qquad\qquad \bigvee\big\{t_n(f(a_{n1}),\ldots,f(a_{nm_n}))\mid t_n(h(a_{n1}),\ldots,h(a_{nm_n}))\le b_n\big\}\Big)\\
&= \bigvee\big\{\omega_Q\left(t_1(f(a_{11}),\ldots,f(a_{1m_1})),\ldots,t_n(f(a_{n1}),\ldots,f(a_{nm_n}))\right)\;\big|\\
&\qquad\qquad t_1(h(a_{11}),\ldots,h(a_{1m_1}))\le b_1,\ldots,t_n(h(a_{n1}),\ldots,h(a_{nm_n}))\le b_n\big\}\\
&\le \bigvee\big\{t(f(a_1),\ldots,f(a_m))\;\big|\;t(h(a_1),\ldots,h(a_m))\le\omega_B(b_1,\ldots,b_n)\big\}\\
&= g(\omega_B(b_1,\ldots,b_n)),
\end{aligned}
$$

Theorem 6(2016, Zhang, Laan[12])

For an ordered Ω-algebra $\mathscr{A} = (A, \Omega_A, \leq_A)$, the following assertions are equivalent:

1. $\mathscr{A}$ is $\mathscr{M}_2$-injective in $\mathbf{OALg}_{\Omega}^{\leq}$,
2. $\mathscr{A}$ is $\mathscr{M}_1$-injective in $\mathbf{OALg}_{\Omega}^{\leq}$,
3. $\mathscr{A}$ is a sup-algebra.

4. Injective hulls for ordered algebras Constructions of injective hulls in $\mathbf{OALg}_{\Omega}^{\leq}$

Step 1

Let $\mathscr{A} = (A, \Omega_A, \leq_A)$ be an ordered Ω-algebra, and let $\mathscr{P}(A)$ be the set of all down-sets of the poset $(A, \leq_A)$. For any $n \in \mathbb{N}, \omega \in \Omega_n$ and $D_1, \ldots, D_n \in \mathscr{P}(A)$ we denote

$$\omega_A(D_1, \ldots, D_n) = \{\omega_A(d_1, \ldots, d_n) \mid d_1 \in D_1, \ldots, d_n \in D_n\}$$

and define an operation $\omega_{\mathscr{P}(A)}$ on $\mathscr{P}(A)$ by

$$\omega_{\mathscr{P}(A)}(D_1, \ldots, D_n) := \omega_A(D_1, \ldots, D_n) \downarrow .$$

For each $\omega \in \Omega_0$ we put

$$\omega_{\mathscr{P}(A)} := \omega_A \downarrow .$$

Then that $(\mathscr{P}(A), \Omega_{\mathscr{P}(A)}, \subseteq)$ is a sup-algebra.

4. Injective hulls for ordered algebras Constructions of injective hulls in $\mathbf{OALg}_{\Omega}^{\leq}$

Let $\mathscr{A} = (A, \Omega_A, \leq_A)$ be an ordered Ω-algebra. We denote by P_A^1 the set of all unary polynomial functions on $\mathscr{A}$. Recall that a *unary polynomial function* on $\mathscr{A}$ is a mapping $p : A \to A$ such that $p = t_{\mathscr{A}}(a_1, \ldots, a_{n-1}, \_)$ for some $n \in \mathbb{N}$ and $a_1, \ldots, a_{n-1} \in A$. Clearly, every unary polynomial function is monotone.

Step 2

For any $D \in \mathscr{P}(A)$ denote

$$cl(D) := \{u \in A \mid p(D) \subseteq a \downarrow \Rightarrow p(u) \leq a \text{ for all } a \in A, p \in P_A^1\},$$

where $p(D) = \{p(d) \mid d \in D\}$.

4. Injective hulls for ordered algebras Constructions of injective hulls in $\mathbf{OALg}_{\Omega}^{\leq}$

Step 3

For any ordered Ω-algebra $\mathscr{A} = (A, \Omega_A, \leq_A)$, we denote

$$\mathscr{Q}(A) := \mathscr{P}(A)_{cl} = \{D \in \mathscr{P}(A) \mid cl(D) = D\}.$$

Theorem 13 (2016, Zhang, Laan)

Let $\mathscr{A} = (A, \Omega_A, \leq_A)$ be an ordered Ω-algebra such that $cl(a\downarrow) = a\downarrow$ for every $a \in A$. Then $\mathscr{Q}(A)$ is the $\mathscr{M}_2$-injective hull of $\mathscr{A}$ in the category $\mathbf{OALg}_{\Omega}^{\leq}$.

5. Work to be continued

2017, Xia C.C., Zhao B., Han S.W., A Note on injective hulls of posemigropus. Theory Appl. Categ.

Let $(S, \cdot)$ be a posemigroup, $D \subseteq S$. Define

$$D^\star \triangleq D^{ul} \cap D^L \cap D^R \cap D^T,$$

where

$$D^{ul} \triangleq \{x \in A \mid (\forall b \in A)\ D \subseteq b{\downarrow} \Longrightarrow x \leq b\},$$
$$D^L \triangleq \{x \in A \mid (\forall a, b \in A)\ Da \subseteq b{\downarrow} \Longrightarrow xa \leq b\},$$
$$D^R \triangleq \{x \in A \mid (\forall a, b \in A)\ aD \subseteq b{\downarrow} \Longrightarrow ax \leq b\},$$
$$D^T \triangleq \{x \in A \mid (\forall a, b, c \in A)\ aDc \subseteq b{\downarrow} \Longrightarrow axc \leq b\},$$

$\mathscr{S}^\star = \{D \subseteq S \mid D^\star = D\}$ is the $\varepsilon_\leq$-injective hull for a posemigroup S.

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Thank you!