Injective hulls for $S$-posets

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1. Introduction: Results of injective hulls on posets


**Theorem 1 (Banaschewski B., Bruns G.)** For a partially ordered set $P$, T.A.E.:

1. $P$ is complete;
2. $P$ is strictly injective;
3. $P$ is a retract of every extension;
4. $P$ has no essential extensions.
1. Introduction: Results of injective hulls on posets

**Theorem 2** (Banaschewski B., Bruns G.) For a partially ordered set $P$, T.A.E.:

1. $E$ is a MacNeille completion of $P$;
2. $E$ is an essential, strictly injective extension of $P$;
3. $E$ is a strictly injective extension of $P$ not containing any properly smaller such extension of $P$;
4. $E$ is an essential extension of $P$ not containing in any properly larger such extension of $P$. 
1. Introduction: Results of injective hulls on semilattices


Theorem 3 (Burns G., Lakser H.)
A semilattice $S$ is injective iff it is complete and satisfying

$$a \bigwedge \bigvee M = \bigvee (a \bigwedge m \mid m \in M),$$  \hspace{1cm} (1)

for all $a \in S$, $M \subseteq S$. 
1. Introduction: Results of injective hulls on semilattices


\[ I_D(S) = \{ A \subseteq S \mid A = A \downarrow; M \subseteq A \text{ with } \bigvee M \text{ exists and satisfying (1)} \Rightarrow \bigvee M \in A \}. \]

Theorem 4 (Burns G., Lakser H.)

Injective hulls of a semilattice $S$ is $I_D(S)$. 
1. Introduction: Results of injective hulls on certain $S$-systems over a semilattice


**Theorem 5 (Johnson C.S., J.R., McMorris F.R.)**

Injective hulls of an $S$-system $M_S$ is $I_D(M_S)$. 
1. Introduction: backgrounds and motivations


**Theorem 6. (1974 Schein)** The category of semigroups has only trivial injectives.
The category of po-monoids

Partially ordered monoids (po-monoid) with submultiplicative order-preserving mappings, i.e., an order-preserving mapping \( \phi : A \rightarrow B \) of po-monoids satisfying

\[
\phi(1) = 1, \\
\phi(a)\phi(b) \leq \phi(ab),
\]

for all \( a, b \in A \).
1. Introduction: Results of injective hulls on posemigroups

2012, Lambek J., Barr M., Kennison J.F. and Raphael R.,

Theorem 7 (2012 Lambek, Barr, Kennison and Raphael)
A po-monoid is injective iff it is a quantale.
1. Introduction: Quantales and quantale-like structures

Definitions

Definition 1 (Mulvey C.J., 1986)

A quantale is a complete lattice \( Q \) equipped with an associative binary operation satisfying

\[
a(\bigvee M) = \bigvee \{am \mid m \in M\}, \quad (\bigvee M)a = \bigvee \{ma \mid m \in M\}
\]

for any \( a \in Q, \ M \subseteq Q \).

A frame(locale) \( L \) is a complete lattice such that

\[
a \wedge (\bigvee M) = \bigvee \{a \wedge m \mid m \in M\},
\]

for any \( a \in L, \ M \subseteq L \).
1. Introduction: Quantale modules

Definitions

Definition 2 (Cf. [5, 9])

Given a quantale $Q$, a left quantale module is a pair $(A, \ast)$, where $A$ is a $\bigvee$-lattice and $Q \times A \rightarrow$ such that:

1. $q \ast (\bigvee S) = \bigvee_{s \in S}(q \ast s)$ for every $q \in Q$, $S \subseteq A$;
2. $(\bigvee T) \ast a = \bigvee_{t \in T}(t \ast a)$ for every $a \in A$, $T \subseteq Q$;
3. $q_1 \ast (q_2 \ast a) = (q_1 q_2) \ast a$ for every $a \in A$, $q_1, q_2 \in Q$. 

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Definition 3 (Cf. [9]) Given a commutative quantale $Q$, a quantale algebra is a quantale module $(A, \ast)$ such that:

1. $(A, \leq, \otimes)$ is a quantale;
2. $q \ast (a \otimes b) = (q \ast a) \otimes b$ for every $a, b \in A, q \in Q$. 

1. Introduction: Quatale algebras

Definitions
2. Injective hulls for posemigroups

**Category** $\text{Pos}_\leq$

**Objects:** posemigroups;

**Morphisms:** order-preserving submultiplicative mappings;

$\varepsilon_\leq :$ a morphism $f : S \to T$ in $\text{Pos}_\leq$ belongs to $\varepsilon_\leq$ if it satisfies:

$$f(a_1) \cdots f(a_n) \leq f(a) \implies a_1 \cdots a_n \leq a,$$

for each $a_1, \cdots, a_n, a \in S.$
Theorem 8 (2014, Zhang, Laan)

$\varepsilon_{\leq}$-injective objects in the category $Pos_{\leq}$ are exactly quantales.

Example: quantale $(P(S), \cdot, \subseteq)$

$S$: a posemigroup,
$P(S)$: the set of all downsets of $S$,

$$X \cdot Y = (XY) \downarrow = \{x \in S \mid s \leq xy \text{ for some } x \in X, \ y \in Y\},$$

for $X, Y \in P(S)$. 

Definition 4: Nucleus

A nucleus on a quantale $Q$ is a submultiplicative closure operator on $Q$.

The nucleus $\text{cl}$:

For any down-set $I$ of a posemigroup $S$ we define its closure by

$$\text{cl}(I) := \{ x \in S \mid alc \subseteq b \downarrow \text{ implies } axc \leq b \text{ for all } a, b, c \in S \}.$$ 

Then $\text{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ is a nucleus.


2. Injective hulls for posemigroups

**Constructions**

**Construction: \( \mathcal{Q}(S) \)**

\[
\mathcal{Q}(S) := \{ I \in \mathcal{P}(S) \mid I = \text{cl}(I) \}
\]

define a multiplication \( \circ \) on \( \mathcal{Q}(S) \) by

\[
I \circ J := \text{cl}(I \cdot J).
\] (1)

**Theorem 9 (2014 Zhang, Laan)**[Cf. Theorem 5.8 in [6]]

Let \( S \) be a posemigroup such that \( \text{cl}(s \downarrow) = s \downarrow \) for every \( s \in S \). Then \( \mathcal{Q}(S) \) is an \( \varepsilon_{\leq} \)-injective hull of \( S \) in \( \text{Pos}_{\leq} \).
3. Injective hulls for $S$-posets

Definitions

In this work, $S$ is always a *pomonoid*.

### $S$-posets

A poset $(A, \leq)$ together with a mapping $A \times S \rightarrow A$ (under which a pair $(a, s)$ maps to an element of $A$ denoted by $as$) is called a *right $S$-poset*, denoted by $A_S$, if for any $a, b \in A$, $s, t \in S$,

1. $a(st) = (as)t$,
2. $a1 = a$,
3. $a \leq b$, $s \leq t$ imply that $as \leq bt$.

A left $S$-poset can be defined similarly.
3. Injective hulls for $S$-posets

**$S$: a pomonoid**


**Result 1:**

Each injective $S$-poset is **“complete”** if it is a complete lattice and satisfying the distributions of arbitrary joins with $S$-actions.

**Result 2:**

Let $G$ be a pogroup. Then a complete $G$-poset is injective.
3. Injective hulls for $S$-posets

**Definitions**

**Definition 5** (2015, Zhang, Laan)

We call a right $S$-poset $A_S$ a *right $S$-quantale* if

1. the poset $A$ is a complete lattice;
2. $(\bigvee M)s = \bigvee \{ms \mid m \in M\}$ for each subset $M$ of $A$ and each $s \in S$. 

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Let $A_S$ and $B_S$ be $S$-posets. We say that a mapping $f : A \to B$ is **$S$-submultiplicative** if $f(a)s \leq f(as)$ for any $a \in A$, $s \in S$.

**Definitions**

\[ Pos_S^{\leq} \]

We denote by $Pos_S^{\leq}$ the category where objects are right $S$-posets and morphisms are $S$-submultiplicative order-preserving mappings.

**Definitions**

\[ \mathcal{E}_S^{\leq} \]

Let $\mathcal{E}_S^{\leq}$ be the class of morphisms $e : A_S \to B_S$ in the category $Pos_S^{\leq}$ which satisfy the following condition:

\[ e(a)s \leq e(a') \implies as \leq a', \quad \forall \ a, a' \in A, \ s \in S. \]
3. Injective hulls for $S$-posets

**Theorem 10 (2015, Zhang, Laan)**

Let $A_S$ be an $S$-poset. Then $A_S$ is $\varepsilon_\leq$-injective in $\text{Pos}_S^{\leq}$ if and only if $A_S$ is a right $S$-quantale.

**An $S$-quantale**

Let $A_S$ be an $S$-poset, and let $\mathcal{P}(A)$ be the set of all down-sets of the poset $A$. Define a right $S$-action $\cdot$ on $\mathcal{P}(A)$ by

$$D \cdot s = (Ds) \downarrow = \{x \in A \mid x \leq ds \text{ for some } d \in D\},$$

for any $s \in S$, $D \in \mathcal{P}(A)$. Then $(\mathcal{P}(A), \cdot, \subseteq)$ is a right $S$-quantale.
A nucleus

For any down-set $D$ of an $S$-poset $A_S$ we define its closure by

$$\text{cl}(D) := \{ x \in A \mid D_s \subseteq a \downarrow \text{ implies } xs \leq a \text{ for all } a \in A, \ s \in S \}.$$  

Then $\text{cl}$ is a nucleus on $\mathcal{P}(A)_S$. 
Another $S$-quantale

For any $S$-poset $A_S$, we put

$$\mathcal{Q}(A) := \mathcal{P}(A)_{\text{cl}} = \{ D \in \mathcal{P}(A) \mid \text{cl}(D) = D \}$$

and define a right $S$-action $\circ$ on $\mathcal{Q}(A)$ by

$$D \circ s := \text{cl}(D \cdot s),$$

for any $s \in S$. 
Theorem 11 (2015 Zhang, Laan)

For every $S$-poset $A_S$, $Q(A)_S$ is the $\varepsilon_\leq$-injective hull of $A_S$ in the category $\text{Pos}_S^\wedge$. 
Definition 6

Let $\Omega$ be a type. An ordered $\Omega$-algebra is a triplet $A = (A, \Omega_A, \leq_A)$ comprising a poset $(A, \leq_A)$ and a set $\Omega_A$ of operations on $A$ (for every $k$-ary operation symbol $\omega \in \Omega_k$ there is a $k$-ary operation $\omega_A \in \Omega_A$ on $A$) such that all the operations $\omega_A$ are monotone mappings, where monotonicity of $\omega_A$ ($\omega \in \Omega_k$) means that

$$a_1 \leq_A a'_1 \land \ldots \land a_k \leq_A a'_k \implies \omega_A(a_1, \ldots, a_k) \leq_A \omega_A(a'_1, \ldots, a'_k)$$

for all $a_1, \ldots, a_k, a'_1, \ldots, a'_k \in A$. 
An ordered $\Omega$-algebra $\mathcal{A} = (A, \Omega_A, \leq_A)$ is called a **sup-algebra** if the poset $(A, \leq)$ is a complete lattice and

$$\omega_A(a_1, \ldots, a_{i-1}, \bigvee M, a_{i+1}, \ldots, a_n)$$

$$= \bigvee \{\omega_A(a_1, \ldots, a_{i-1}, m, a_{i+1}, \ldots, a_n) \mid m \in M\}$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1, \ldots, n\}$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$, and $M \subseteq A$.

Sup-algebras are a generalization of quantale-like structures.
Definition 7

Let $A$, $B$ be ordered $\Omega$-algebras of type $\Omega$. A monotone mapping $f : A \to B$ is called a *subhomomorphism* if

$$\omega_B(f(a_1), \ldots, f(a_n)) \leq f(\omega_A(a_1, \ldots, a_n))$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in A$, and

$$\omega_B \leq f(\omega_A)$$

for every $\omega \in \Omega_0$. All ordered $\Omega$-algebras together with their subhomomorphisms form a category which we denote by $\mathbf{OALg}^{\leq}_\Omega$. 
4. Injective hulls for ordered algebras

We will study injectivity in classes of ordered \( \Omega \)-algebras with respect to two classes of order-embeddings. The first of them is the class \( \mathcal{M}_1 \) of order-embeddings that are homomorphisms. The other class has a more complicated definition.

Let us denote by \( T^n_\Omega \) the set of all \( n \)-ary \( \Omega \)-terms. Let \( \mathcal{M}_2 \) be the class of mappings \( h : \mathcal{A} \rightarrow \mathcal{B} \) between ordered \( \Omega \)-algebras that satisfy the following conditions:

1. \( h \) is monotone,
2. \( \omega_B(h(a_1), \ldots, h(a_n)) \leq h(\omega_A(a_1, \ldots, a_n)) \) for every \( n \in \mathbb{N} \), \( \omega \in \Omega_n \), \( a_1, \ldots, a_n \in \mathcal{A} \),
3. \( h(\omega_A) = \omega_B \) for all \( \omega \in \Omega_0 \),
4. for all \( n \in \mathbb{N} \), \( t \in T^n_\Omega \), \( a_1, \ldots, a_n, a \in \mathcal{A} \),

\[
t_B(h(a_1), \ldots, h(a_n)) \leq h(a) \implies t_A(a_1, \ldots, a_n) \leq a.
\]
4. Injective hulls for ordered algebras

**Lemma 2**
For a class of ordered $\Omega$-algebras, $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

**Theorem 12** (2016, Zhang, Laan)
Every sup-algebra $\mathcal{D} = (Q, \Omega_Q, \leq_Q)$ of type $\Omega$ is $\mathcal{M}_2$-injective and therefore, $\mathcal{M}_1$-injective in the category $\text{OALg}_{\Omega}$. 
Proof.
Let $h: A \to B$ be a morphism in $\mathcal{M}_2$ and let $f: A \to Q$ be a morphism in $\text{OALg}_{\Omega}^{\leq}$. Define a mapping $g: B \to Q$ by

$$g(b) = \bigvee \{ t(f(a_1), \ldots, f(a_m)) \mid m \in \mathbb{N}, a_1, \ldots, a_m \in A, \ t \in T^m_\Omega, \ t(h(a_1), \ldots, h(a_m)) \leq b \},$$

for any $b \in B$. We write the last join shortly as

$$\bigvee_{t(h(a_1), \ldots, h(a_m)) \leq b} t(f(a_1), \ldots, f(a_m)).$$
\[
\omega_Q(g(b_1), \ldots, g(b_n))
\]
\[
= \omega_Q \left( \bigvee \left\{ t_1(f(a_{11}), \ldots, f(a_{1m_1})) \mid t_1(h(a_{11}), \ldots, h(a_{1m_1})) \leq b_1 \right\}, \ldots, \\
\bigvee \left\{ t_n(f(a_{n1}), \ldots, f(a_{nm_n})) \mid t_n(h(a_{n1}), \ldots, h(a_{nm_n})) \leq b_n \right\} \right)
\]
\[
= \bigvee \left\{ \omega_Q(t_1(f(a_{11}), \ldots, f(a_{1m_1})), \ldots, t_n(f(a_{n1}), \ldots, f(a_{nm_n}))) \mid \\
\quad t_1(h(a_{11}), \ldots, h(a_{1m_1})) \leq b_1, \ldots, t_n(h(a_{n1}), \ldots, h(a_{nm_n})) \leq b_n \right\}
\]
\[
\leq \bigvee \left\{ t(f(a_1), \ldots, f(a_m)) \mid t(h(a_1), \ldots, h(a_m)) \leq \omega_B(b_1, \ldots, b_n) \right\}
\]
\[
= g(\omega_B(b_1, \ldots, b_n)),
\]
Theorem 6 (2016, Zhang, Laan[12])

For an ordered $\Omega$-algebra $\mathcal{A} = (A, \Omega_A, \leq_A)$, the following assertions are equivalent:

1. $\mathcal{A}$ is $\mathcal{M}_2$-injective in $OALg^{<}_\Omega$;
2. $\mathcal{A}$ is $\mathcal{M}_1$-injective in $OALg^{<}_\Omega$;
3. $\mathcal{A}$ is a sup-algebra.
4. Injective hulls for ordered algebras Constructions of injective hulls in $\text{OALg}_{\Omega}^{\leq}$

**Step 1**

Let $\mathcal{A} = (A, \Omega_A, \leq_A)$ be an ordered $\Omega$-algebra, and let $\mathcal{P}(A)$ be the set of all down-sets of the poset $(A, \leq_A)$. For any $n \in \mathbb{N}, \omega \in \Omega_n$ and $D_1, \ldots, D_n \in \mathcal{P}(A)$ we denote

$$\omega_A(D_1, \ldots, D_n) = \{\omega_A(d_1, \ldots, d_n) | d_1 \in D_1, \ldots, d_n \in D_n\}$$

and define an operation $\omega_{\mathcal{P}(A)}$ on $\mathcal{P}(A)$ by

$$\omega_{\mathcal{P}(A)}(D_1, \ldots, D_n) := \omega_A(D_1, \ldots, D_n) \downarrow.$$  

For each $\omega \in \Omega_0$ we put

$$\omega_{\mathcal{P}(A)} := \omega_A \downarrow.$$

Then that $(\mathcal{P}(A), \Omega_{\mathcal{P}(A)}, \subseteq)$ is a sup-algebra.
Let $\mathcal{A} = (A, \Omega_A, \leq_A)$ be an ordered $\Omega$-algebra. We denote by $P_A^1$ the set of all unary polynomial functions on $\mathcal{A}$. Recall that a \textit{unary polynomial function} on $\mathcal{A}$ is a mapping $p : A \rightarrow A$ such that $p = t_\mathcal{A}(a_1, \ldots, a_{n-1}, -)$ for some $n \in \mathbb{N}$ and $a_1, \ldots, a_{n-1} \in A$. Clearly, every unary polynomial function is monotone.

**Step 2**

For any $D \in \mathcal{P}(A)$ denote

$$cl(D) := \{ u \in A \mid p(D) \subseteq a \Downarrow \Rightarrow p(u) \leq a \text{ for all } a \in A, p \in P_A^1 \},$$

where $p(D) = \{ p(d) \mid d \in D \}$. 
4. Injective hulls for ordered algebras  Constructions of injective hulls in $\text{OALg}_{\leq}^\Omega$

Step 3

For any ordered $\Omega$-algebra $\mathcal{A} = (A, \Omega_A, \leq_A)$, we denote

$$\mathcal{Q}(A) := \mathcal{P}(A)_{cl} = \{D \in \mathcal{P}(A) | cl(D) = D\}.$$ 

Theorem 13 (2016, Zhang, Laan)

Let $\mathcal{A} = (A, \Omega_A, \leq_A)$ be an ordered $\Omega$-algebra such that $cl(a \downarrow) = a \downarrow$ for every $a \in A$. Then $\mathcal{Q}(A)$ is the $\mathcal{M}_2$-injective hull of $\mathcal{A}$ in the category $\text{OALg}_{\leq}^\Omega$. 
Let \((S, \cdot)\) be a posemigroup, \(D \subseteq S\). Define

\[
D^* \triangleq D^{ul} \cap D^L \cap D^R \cap D^T,
\]

where

\[
D^{ul} \triangleq \{x \in A \mid (\forall b \in A) \ D \subseteq b \downarrow \implies x \leq b\},
\]

\[
D^L \triangleq \{x \in A \mid (\forall a, b \in A) \ D a \subseteq b \downarrow \implies xa \leq b\},
\]

\[
D^R \triangleq \{x \in A \mid (\forall a, b \in A) \ a D \subseteq b \downarrow \implies ax \leq b\},
\]

\[
D^T \triangleq \{x \in A \mid (\forall a, b, c \in A) \ a D c \subseteq b \downarrow \implies axc \leq b\},
\]

\[
\mathcal{S}^* = \{D \subseteq S \mid D^* = D\}
\]

is the \(\varepsilon_{\leq}\)-injective hull for a posemigroup \(S\).
References


Thank you!