n-tuple semigroups

Anatolii Zhuchok

Luhansk Taras Shevchenko National University
Starobilsk, Ukraine
E-mail: zhuchok.av@gmail.com

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1. Introduction

The notion of an n-tuple algebra of associative type was introduced in [1] in connection with an attempt to obtain an analogue of the Chevalley construction for modular Lie algebras of Cartan type. This notion is based on the notion of an n-tuple semigroup. Recall that a nonempty set G is called an n-tuple semigroup [1], if it is endowed with n binary operations, denoted by $\boxed{1}, \boxed{2}, ..., \boxed{n}$, which satisfy the following axioms: $(x \boxed{r} y) \boxed{s} z = x \boxed{r} (y \boxed{s} z)$ for any $x, y, z \in G$ and $r, s \in \{1, 2, ..., n\}$. The class of all n-tuple semigroups is rather wide and contains, in particular, the class of all semigroups, the class of all commutative trioids (see, for example, [2, 3]) and the class of all commutative dimonoids (see, for example, [4, 5]).

2-tuple semigroups, causing the greatest interest from the point of view of applications, occupy a special place among *n*-tuple semigroups. So, 2-tuple semigroups are closely connected with the notion of an interassociative semigroup (see, for example, [6, 7]). Moreover, 2-tuple semigroups, satisfying some additional identities, form so-called restrictive bisemigroups, considered earlier in the works of B. M. Schein (see, for example, [8, 9]). Restrictive bisemigroups have applications in the theory of binary relations, as well allow effectively to study differentiable manifolds.

It is known, for example, that a differentiable manifold is characterized by a restrictive bisemigroup of coordinate systems [10]. Sets with two binary associative operations $\boxed{1}$ and $\boxed{2}$, satisfying the axiom $(x\boxed{1}y)\boxed{2}z = x\boxed{1}(y\boxed{2}z)$, were considered in [11] and in this paper the free object of rank 1 in the corresponding variety was constructed. At the same time, it should be noted that in [9] a pair of indicated associative operations $\boxed{1}$ and $\boxed{2}$ is called an associative pair, and algebras with two associative operations, forming an associative pair, have the name of a bisemigroup. Associative pairs of operations were considered by A. Sade [12]. In [1], with the help of the construction of an n-tuple semigroup some properties of n-tuple algebras of associative type are studied.

This work is devoted to the study of n-tuple semigroups. Here we present the free n-tuple semigroup of an arbitrary rank and, as a consequence, characterize one-generated free n-tuple semigroups. Moreover, we give examples of n-tuple semigroups, establish the independence of axioms of an n-tuple semigroup, as well show that the semigroups of the constructed free n-tuple semigroup are isomorphic, and its automorphism group is isomorphic to the symmetric group. We also construct the free product of arbitrary n-tuple semigroups and study its structure.

2. Examples of n-tuple semigroups and the independence of axioms

In this section, we give examples of n-tuple semigroups and establish the independence of axioms of an n-tuple semigroup.

1) Obviously, every semigroup is an 1-tuple semigroup.

2) Every semigroup $(S, \cdot)$ can be considered as an n-tuple semigroup T, if assume $T = (S, \underbrace{\cdot, \cdot, \ldots, \cdot}_{n})$.

3) Recall that a nonempty set T equipped with three binary associative operations $\dashv$, $\vdash$ and $\perp$, satisfying the following axioms:
$(x \dashv y) \dashv z = x \dashv (y \vdash z)\,(T1)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)\,(T2)$,
$(x \dashv y) \vdash z = x \vdash (y \vdash z)\,(T3)$, $(x \dashv y) \dashv z = x \dashv (y \perp z)\,(T4)$,
$(x \perp y) \dashv z = x \perp (y \dashv z)\,(T5)$, $(x \dashv y) \perp z = x \perp (y \vdash z)\,(T6)$,
$(x \vdash y) \perp z = x \vdash (y \perp z)\,(T7)$, $(x \perp y) \vdash z = x \vdash (y \vdash z)\,(T8)$

for all $x, y, z \in T$, is called a trioid. Trioids were introduced by J.-L. Loday and M.O. Ronco [2] in the context of algebraic topology. A trioid is called commutative [3], if its three operations are commutative.

Proposition

([13], A. Zhuchok) Every commutative trioid is a 3-tuple semigroup.

Examples of commutative trioids can be found in [3].

4) Recall that a dimonoid is a nonempty set equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1) - (T3)$. Dimonoids were introduced by J.-L. Loday [4] during constructing the universal enveloping algebra for a Leibniz algebra. A dimonoid is commutative [5], if both its operations are commutative.

Proposition

([13], A. Zhuchok) Every commutative dimonoid is a 2-tuple semigroup.

Examples of commutative dimonoids can be found in [5, 14].
5) Let $(D, \vdash)$ be an arbitrary semigroup. Consider a semigroup $(D, \dashv)$ defined on the same set. Recall that $(D, \dashv)$ is called an interassociativity of $(D, \vdash)$ (see., e.g., [6, 7]), if the axioms $(T2)$ and $(x \dashv y) \vdash z = x \dashv (y \vdash z)$ are satisfied. Thus, in any 2-tuple semigroup $(D, \dashv, \vdash)$ a semigroup $(D, \dashv)$ is an interassociativity of a semigroup $(D, \vdash)$, and conversely, if a semigroup $(D, \dashv)$ is an interassociativity of a semigroup $(D, \vdash)$, then $(D, \dashv, \vdash)$ is a 2-tuple semigroup.

Descriptions of all interassociativities of a monogenic semigroup and of the free commutative semigroup were represented in [6] and [7, 15], respectively. Methods of constructing interasociativities of a semigroup were developed in [16].

6) Consider connections between 2-tuple semigroups and restrictive bisemigroups.

Let B an arbitrary nonempty set and $\dashv$, $\vdash$ binary operations on B. An ordered triple $(B, \dashv, \vdash)$ is called a restrictive bisemigroup (see, e.g., [8, 9]), if operations $\dashv$, $\vdash$ are associative, idempotent and

$$x \dashv y \dashv z = y \dashv x \dashv z,$$

$$x \vdash y \vdash z = x \vdash z \vdash y,$$

$$(x \dashv y) \vdash z = x \dashv (y \vdash z)$$

for all $x, y, z \in B$.

Thus, every 2-tuple semigroup $(G, \boxed{1}, \boxed{2})$ is a restrictive bisemigroup, if $(G, \boxed{1})$ is a right normal band, and $(G, \boxed{2})$ is a left normal band. And conversely, every restrictive bisemigroup, satisfying the axiom $(T2)$, is a 2-tuple semigroup.
Note that restrictive bisemigroups have relationships with rectangular dimonoids [17].
7) Let G be an arbitrary n-tuple semigroup with operations $\boxed{1}, \boxed{2}, ..., \boxed{\mathrm{n}}$ and $a_1, a_2, ..., a_n \in G$. Define new operations $\boxed{1}_{a_1}, \boxed{2}_{a_2}, ..., \boxed{\mathrm{n}}_{a_n}$ on G by

$$x \boxed{i}_{a_i} y = x \boxed{i} a_i \boxed{i} y$$

for all $x, y \in G$, $i \in \{1, 2, ..., n\}$. These operations are well-defined as the operations $\boxed{i}$, $i \in \{1, 2, ..., n\}$, are associative.

Proposition

([13], A. Zhuchok) $(G, \boxed{1}_{a_1}, \boxed{2}_{a_2}, ..., \boxed{n}_{a_n})$ *is an n-tuple semigroup.*

The n-tuple semigroup $(G, \boxed{1}_{a_1}, \boxed{2}_{a_2}, ..., \boxed{n}_{a_n})$ is called a variant of G, or, alternatively, a sandwich n-tuple semigroup of the algebra G with respect to sandwich elements $a_1, a_2, ..., a_n$, or an n-tuple semigroup with deformed multiplications. The operations $\boxed{1}_{a_1}$, $\boxed{2}_{a_2}$, ..., $\boxed{n}_{a_n}$ are called sandwich operations.
For an n-tuple semigroup the following theorem takes place.

Theorem

([13], A. Zhuchok) A system of all axioms of an n-tuple semigroup $(n > 1)$ *is independence.*

3. Free n-tuple semigroups

In this section, we construct the free n-tuple semigroup of an arbitrary rank and consider separately one-generated free n-tuple semigroups. Moreover, we show that the semigroups of the free n-tuple semigroup are isomorphic, and its automorphism group is isomorphic to the symmetric group.
As usual, $\mathbb{N}$ denotes the set of all positive integers.

Lemma

([13], A. Zhuchok) In an n-tuple semigroup $(G, \boxed{1}, \boxed{2}, ..., \boxed{n})$ *for any* $m > 1, m \in \mathbb{N}$, *and for any* $x_i \in G$, $1 \leq i \leq m+1$, *and* $*_j \in \{\boxed{1}, \boxed{2}, ..., \boxed{n}\}$, $1 \leq j \leq m$, *any parenthesizing of*

$$x_1 *_1 x_2 *_2 \cdots *_m x_{m+1}$$

gives the same element from G.

An n-tuple semigroup, which is free in the variety of n-tuple semigroups, is called a free n-tuple semigroup.
The main result of this section is the following theorem.

Theorem

([13], A. Zhuchok) Any free n-tuple semigroup is a subdirect product of the free semigroup on some set and of the variant of some n-tuple semigroup defined on the free monoid of rank n.

Let X be an arbitrary nonempty set and ω an arbitrary word in the alphabet X. The length of ω will be denoted by l_ω. By definition, the length of the empty word is equal to 0. Fix $n \in \mathbb{N}$ and let $Y = \{y_1, y_2, ..., y_n\}$ be an arbitrary set consisting of n elements.

Let further $F[X]$ be the free semigroup on X, $F^{\theta}[Y]$ the free monoid on Y and $\theta \in F^{\theta}[Y]$ the empty word. Define n binary operations $\boxed{1}, \boxed{2}, ..., \boxed{\text{n}}$ on $XY_n = \{(w, u) \in F[X] \times F^{\theta}[Y] \,|\, l_w - l_u = 1\}$ by

$$(w_1, u_1)\boxed{i}(w_2, u_2) = (w_1 w_2, u_1 \cdot_{y_i} u_2)$$

for all $(w_1, u_1), (w_2, u_2) \in XY_n$ and $i \in \{1, 2, ..., n\}$, where $\cdot_{y_i}$ is the sandwich-operation on $F^{\theta}[Y]$. The algebra $(XY_n, \boxed{1}, \boxed{2}, ..., \boxed{\text{n}})$ is denoted by $F_n TS(X)$. By the last theorem (see also [13]), $F_n TS(X)$ is the free n-tuple semigroup.

From the last theorem we obtain a corollary which describes the free n-tuple semigroup of rank 1.

Corollary

([13], A. Zhuchok) $(F^{\theta}[Y], \cdot_{y_1}, \cdot_{y_2}, ..., \cdot_{y_n})$ *is the free n-tuple semigroup of rank* 1.

The following lemma establishes relationships between semigroups of the free n-tuple semigroup $F_nTS(X)$.

Lemma

([13], A. Zhuchok) For any $i, j \in \{1, 2, ..., n\}$ the semigroups $(XY_n, \boxed{i})$ and $(XY_n, \boxed{j})$ are isomorphic.

Denote by $\Im[X]$ the symmetric group on a set X and by $Aut\ G^{'}$ the automorphism group of an n-tuple semigroup $G^{'}$.

Lemma

([13], A. Zhuchok) $Aut\ F_nTS(X) \cong \Im[X]$.

We finish this section with the description of one congruence on $F_nTS(X)$ which helps to obtain the free semigroup from the free n-tuple semigroup.

Let μ be an arbitrary fixed congruence on the free semigroup $F[X]$. Define a relation $\widetilde{\mu}$ on $F_nTS(X)$ by

$$(w_1, u_1)\widetilde{\mu}(w_2, u_2) \Leftrightarrow w_1 \mu w_2$$

for all $(w_1, u_1), (w_2, u_2) \in F_nTS(X)$.

Lemma

([13], A. Zhuchok) The relation $\widetilde{\mu}$ is a congruence on the free n-tuple semigroup $F_nTS(X)$ and operations of the factor-algebra $F_nTS(X)/\widetilde{\mu}$ coincide.

From the last lemma we obtain the following corollary.

Corollary

([13], A. Zhuchok) If μ is the diagonal of $F[X]$, then $F_nTS(X)/\widetilde{\mu}$ is the free semigroup.

4. Free products of n-tuple semigroups

In this section, we construct the free product of n-tuple semigroups. A principal importance of free products of algebras from a given class $\Re$ is the fact that any homomorphisms of free multipliers into some fixed algebra from $\Re$ can be uniquely extended to a homomorphism of the free product into this algebra. Moreover, every endomorphism of the free product of algebras is uniquely defined by homomorphisms of free multipliers into the free product, and conversely, any homomorphisms of free multipliers into the free product of algebras can be uniquely extended to an endomorphism of the free product.

It is well-known (see, e.g., [18], [19]) that any free algebra is the free product of one-generated free algebras. It is natural to consider the problem of constructing a free product in the variety of n-tuple semigroups.

Let $Fr[T_i]_{i\in I}$ be the free product of arbitrary semigroups $T_i, i \in I$. For every $w \in Fr[T_i]_{i\in I}$ denote by $w^{(0)}$ (respectively, $w^{(1)}$, l_w) the first letter (respectively, last letter, length) of w. Fix $n \in \mathbb{N}$ and let $Y = \{y_1, y_2, \ldots, y_n\}$ be an arbitrary set consisting of n elements. Let further $\{(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n})\}_{i\in I}$ be a family of arbitrary pairwise disjoint n-tuple semigroups, $F^\theta[Y]$ the free monoid on Y and $\theta \in F^\theta[Y]$ the empty word. For every $j \in \{1, 2, \ldots, n\}$ by $\boxed{j^*}$ denote the operation on $Fr[(S_i, \boxed{i_j})]_{i\in I}$. Fix $j \in \{1, 2, \ldots, n\}$ and define n binary operations

$\boxed{1}^{'}, \boxed{2}^{'}, \ldots, \boxed{n}^{'}$ on

$$V = \left\{ (w, u) \in Fr[(S_i, \boxed{i_j})]_{i \in I} \times F^{\theta}[Y] \,|\, l_w - l_u = 1 \right\}$$

by

$$(w_1, u_1) \boxed{r}^{'} (w_2, u_2)$$

$$= \begin{cases} (w_1 w_2, u_1 y_r u_2), w_1^{(1)} \in S_k, w_2^{(0)} \in S_m, k, m \in I, k \neq m, \\ \left(w_1 \boxed{r^*} w_2, u_1 u_2 \right), w_1^{(1)}, w_2^{(0)} \in S_k, k \in I \end{cases}$$

for all $(w_1, u_1), (w_2, u_2) \in V$, $r \in \{1, 2, \ldots, n\}$. The algebra $(V, \boxed{1}^{'}, \boxed{2}^{'}, \ldots, \boxed{n}^{'})$ is denoted by $Fr_n T(S_i)_{i \in I}$. Obviously, constructing the algebra $Fr_n T(S_i)_{i \in I}$ does not depend on the choice j in the definition of V.

Theorem

$Fr_nT(S_i)_{i\in I}$ is the free product of n-tuple semigroups $(S_i, \boxed{i_1}, \boxed{i_2}, \ldots, \boxed{i_n})$, $i \in I$.

Recall that the construction of the free n-tuple semigroup of rank 1 was given in Section 3. Let further $\{\Lambda_i\}_{i\in I}$ be a family of free n-tuple semigroups of rank 1. Using the fact that every free algebra is the free product of one-generated free algebras, from the last theorem we obtain a corollary which gives the free n-tuple semigroup.

Corollary

The free product of n-tuple semigroups $\Lambda_i, i \in I$, is the free n-tuple semigroup of rank $|I|$.

Recall that the free *n*-tuple semigroup of an arbitrary rank was first constructed in [13].
We also introduce the notion of an *n*-band of *n*-tuple semigroups and in terms of this notion describe the structure of $Fr_n T(S_i)_{i \in I}$.

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