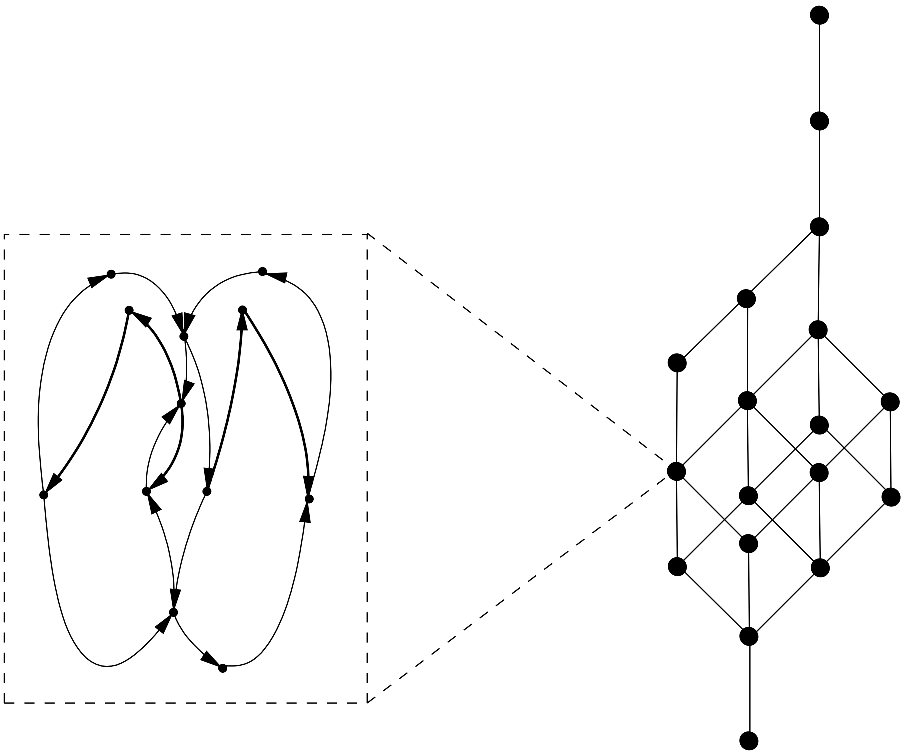
Partial Orders on Orientations via Cycle Flips



Diplomarbeit
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Die selbständige und eigenhändige Anfertigung versichere ich an Eides Statt.

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Introduction

The subject of this thesis are partial orders on orientations and reorientations of directed structures. The papers of Felsner [7] and Propp [12] prompt this.

Felsner constructs a distributive lattice on those orientations of a planar graph, that have fixed outdegree on every vertex (α-orientations). The α-orientations of a graph generalize f-factors, spanning trees, Eulerian orientations and Schnyder woods.

Propp presents a method to generate a distributive lattice on those orientations of a (not necessarily planar) graph, that have the number of forward arcs in cycles as invariant (c-orientations).

The motivation of the thesis is based upon the question, whether or with which obstructions one can generalize Felsner's and Propp's results to oriented matroids. It turns out that the generalization is possible, but yields a theory which is not as nice as in [7, 12]. Therefore we focus on special matroid classes and particular variants.

By identifying the orientations of an undirected graph with the reorientations of a directed graph, we can bring together and generalize the structures investigated by Felsner and Propp. We transfer the invariants of the considered reorientation classes to the terminology of oriented matroids and show that they are dual in this sense.

Furthermore we reformulate the generating methods (flip flop sequences) for the distributive lattices in [7, 12] and show that they are essentially the same. Flip flop sequences can be applied not only to directed graphs but to arbitrary sign matrices. As oriented matroids can be displayed as sign matrices, we are indeed heading towards the desired generalization. Sign matrices still can be organized in directed graphs. In most cases these graphs fail by large to be the graphs of distributive lattices. Actually we show that every connected, loop-free, directed graph arises from a sign matrix, this way.

For positive, we find sufficient conditions on sign matrices, to generate digraphs that are Hasse diagrams of distributive lattices. In addition we obtain a natural embedding into the higher dimensional lattice of integers. As a corollary we obtain the distributivity of the lattices in [7] and [12] together with an embedding into some $\mathbb{Z}^d$. This result is based upon the existence of a 2-basis of the cycle space and the cut space, respectively.

Then we put together what we have learned about flip flop sequences with a suitable generalization of α-reorientations to oriented matroids. This enables us to construct sign matrices that produce posets on the α-reorientations of general oriented matroids. We prove that a stronger analogy to the graph case in terms of cycle bases does only hold for regular oriented matroids. So we specialize our analysis to this class.

Seymour's decomposition theorem for regular matroids [13] leads to the investigation of the three splitters:

- The matroid $\mathbf{R10}$ has a finite number of reorientations. We find generating sets for posets on the respective α-reorientations by computer enumeration.
- We investigate the structure of α-reorientations of **graphic oriented matroids**. From this we derive the main theorems of [7] and [12] as corollaries. Moreover we give a precise description of the partial order on the Eulerian orientations of the square torus grid.
- The α-reorientations of **cographic oriented matroids** coincide dually with the c-orientations in [12]. Therefore they carry the structure of a distributive lattice, as well.

Our last result is that every distributive lattice comes from the flip flop sequences of the α-reorientations of a cographic oriented matroid. We describe the relations between cographic oriented matroids that generate the same distributive lattice.

Chapter 1

Essentials

As already explained in the introduction, the aim of this thesis is to generalize results obtained for planar graphs by Felsner in [7] and dual results for general graphs by Propp in [12]. We will introduce the fundamental terms and initial assumptions, that underlie the whole thesis. Then we briefly present the results of [7, 12]. They will be phrased in a language that is suitable for the generalization we are aiming for. We assume the reader to be familiar with basic concepts of graphs, digraphs and partially ordered sets and refer to [6] and [5] for good introductions, respectively. At the end of the chapter we will point out the aspects we attempt to generalize and give a short preview of the following chapters.

Given an undirected graph $G = (V, E)$ we call a directed graph $D = (V, A)$ an **orientation of** G if G is the underlying undirected graph $\underline{D}$ of D. For a directed graph denote by δ^+ the function that maps every vertex to the number of outgoing arcs, i.e $\delta(v)$ is the **outdegree** of $v \in V$.

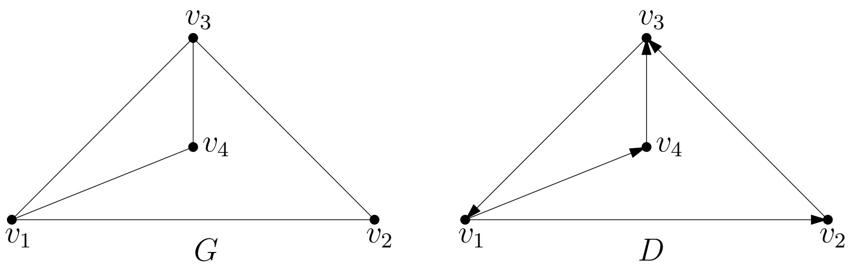

Figure 1.1: The digraph D is a $(2, 1, 1, 1)$-orientation of the undirected graph G.

Let $\alpha \in \mathbb{Z}^V$. We define the set of orientations

$$\alpha\text{-}or(G) := \{D = (V, A) \mid \underline{D} = G \textit{ and } \delta^+ \equiv \alpha\}$$

as the set of α**-orientations of** G.

In order to generalize the ideas in [7, 12], it turns out that considering reorientations instead of orientations is convenient.

So given a directed graph $D = (V, A)$ we can define the set of **α-reorientations of D** as

$$\textit{reor}_\alpha(D) := \{D' = (V, A') \mid \underline{D} = \underline{D'} \textit{ and } \delta^+(D) \equiv \delta^+(D')\}.$$

Note that in the definition of $\textit{reor}_\alpha(D)$ the letter α does not stand for a vector anymore. As the information about the outdegree is already represented by D, the α stands for the invariance of the outdegree among this set of reorientations.

The orientations of an undirected graph coincide with the reorientations of one of its orientations. Thus one clearly has

$$\textit{reor}_\alpha(D) = \alpha\textit{-or}(G) \textit{ if and only if } D \in \alpha\textit{-or}(G).$$

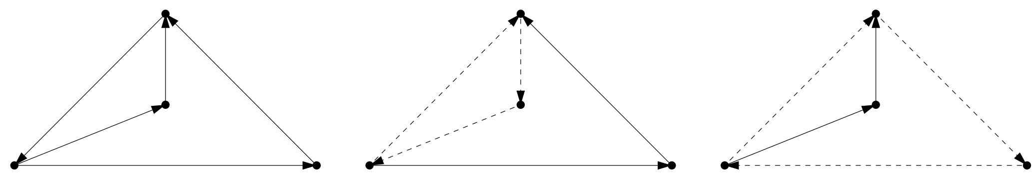

Figure 1.2: The digraphs D, D' and D'' are α-reorientations of D and $(2, 1, 1, 1)$-orientations of G. Dashed lines stand for the reoriented arc sets with respect to D.

In [7] the question whether for given G and α the set α-$\textit{or}(G)$ is empty is translated to the construction of a maximal flow in another graph. One can obtain an α-orientation or a certificate for the non-existence in polynomial time. In the sequel we will always assume to have an α-orientation at hand. This is we can investigate $\textit{reor}_\alpha(D)$ instead of α-$\textit{or}(G)$.

Before we begin to investigate it is useful to introduce some concepts, that we will frequently refer to.

We start with signed sets. We call a set X a **signed set** if it comes with an ordered 2-partition $X^+ \sqcup X^- = X$. We call X^+ the **positive elements** of X and X^- the **negative elements**. A signed set is called **positively directed** or **negatively directed** if the respective other part of it is empty. Denote by $-X$ the signed set given by $(-X)^+ := X^-$ and $(-X)^- = X^+$. We write $\underline{X}$ for the underlying unsigned set of X. These two operations can be applied elementwise to set of signed sets and will be denoted the same. Moreover let A^+ denote the set of positive signed sets of a set A of signed sets.

In the way one associates a $(0, 1)$-vector as incidence vector to a usual subset of some ground set, one can display a signed set as a $(1, -1, 0)$-vector - its **signed incidence**

vector. We will often go back and forth between the concepts of (signed) sets and (signed) incidence vectors. Sets of (signed) sets will also be understood as matrices, whose rows correspond to the (signed) incidence vectors of the element sets. In the signed case we refer to such a matrix as **sign matrix**.

One important sign matrix in the context of digraphs is the **incidence matrix of a directed graph** $D = (V, A)$. We denote it by *Inc*(D). It has a row for every vertex and a column for every arc of D, such that the column labelled by the arc $a = (u, v) \in A$ is $e_u - e_v$. Here e_v stands for the vector that has a 1 in the entry that corresponds to the vertex v and zeros elsewhere.

For a directed graph $D = (V, A)$ we call a signed arc set $E \subseteq A$ a **Eulerian** of D if E is in the kernel of *Inc*(D). Denote by $D[E]$ the subgraph of D induced by the arcs of E. If E is a Eulerian one has that the undirected induced graph $\underline{D[E]}$ has even degree on all its vertices. Conversely every arc set with this property can be signed to become a Eulerian of D. We call a Eulerian **directed** if its signed incidence vector is non-negative. If E is a directed Eulerian, then $D[E]$ is a **Eulerian orientation** of $\underline{D[E]}$, i.e. every vertex in $D[E]$ has the same in- and outdegree. Conversely every unsigned arc set E, which induces a Eulerian orientation $D[E]$ can be signed to become a directed Eulerian of D.

Denote by $\mathcal{E}(D)$ the set of Eulerians of D and by $\mathcal{C}(D)$ the set of cycles - the minimal elements of $\mathcal{E}(D)\backslash\{\emptyset\}$ with respect to inclusion of the underlying set. Every Eulerian is the disjoint union of cycles.

We already switched the view from α-orientations to α-reorientations. Now, instead of looking at *reor*$_\alpha(D)$ as a set of directed graphs, we look at a reorientation D' of D as the arc set of D which has to be reoriented in order to obtain D'. Due to this identification, we can make a first observation.

Proposition 1. *reor*$_\alpha(D) \cong \underline{\mathcal{E}^+(D)}$.

Proof. Let $D' \in$ *reor*$_\alpha(D)$ and $E \subseteq A$ the set of arcs that is reoriented from D to D'. Changing the orientation on E does not change the outdegree if and only if at every vertex of D the numbers of incoming and outgoing arcs in E coincide. This is $D[E]$ is a Eulerian orientation of $\underline{D[E]}$, i.e. E can be signed to become a directed Eulerian, i.e. $E \in \underline{\mathcal{E}^+(D)}$. □

With Proposition 1 one easily sees, that the digraphs in Figure 1.2 are all the α-reorientations of D. The dashed arc sets are exactly those that can be signed to become directed Eulerians of D.

We will take advantage of Proposition 1 by working with $\underline{\mathcal{E}^+(D)} \cong \mathcal{E}^+(D)$ as a subset of the digraphs cycle space instead of α-reorientations as a set of directed graph. The **cycle space** of D is defined as the integral row space $span_{\mathbb{Z}}(\mathcal{C}(D))$ or equivalently as the integral kernel of the incidence matrix of D. As one has

$$\mathcal{E}(D) = \ker_{\{-1,0,1\}}(\mathit{Inc}(D)) = span_{\mathbb{Z}}(\mathcal{C}(D)) \cap \{-1,0,1\}^{|A|}$$

we can consequently identify

$$\mathcal{E}^+(D) = \ker_{\{1,0\}}(\mathit{Inc}(D)) = span_{\mathbb{Z}}(\mathcal{C}(D)) \cap \{1,0\}^{|A|}.$$

Now, we can regard $\mathit{reor}_\alpha(D)$ as the set of $(1,0)$-vectors in the cycle space. This has some advantages.

First, it is a standard result, that $span_{\mathbb{Z}}(\mathcal{C}(D))$ is a direct product of the cycle spaces of the 2-connected components of D, so by Proposition 1 this holds for the α-orientations, too. We can analyse them component by component and restrict ourselves to the 2-connected components (blocks) of D.

Second, Proposition 1 gives an interpretation of the following important notion. For a directed $D = (V, A)$ we call an arc $a \in A$ **rigid** if it has the same orientation in all the members of $\mathit{reor}_\alpha(D)$. To us the arcs of interest are those which are not rigid. We can delete the rigid arcs and obtain some D'. This new graph has a different outdegree at some vertices. But, the sets of $(0,1)$-vectors $\underline{\mathcal{E}^+(D)}$ and $\underline{\mathcal{E}^+(D')}$ can be identified. So the α-reorientations of D' and the α-reorientations of D coincide.

Now, the non-rigid arcs, are those contained in some subgraph induced by a directed Eulerian. As Eulerians are disjoint sums of cycles, every such arc is contained in directed cycle. As every arc of a directed graph is either contained in a directed cycle or in a directed cut, the set of rigid arcs is the set of arcs that are contained directed cuts. So after deleting the rigid arcs, we end up with a graph that has no directed cuts. In other words: we can restrict our attention to strongly connected components of D.

Hence, by Proposition 1, we can focus on the class of 2-connected, strongly connected directed graphs.

All we have done until now worked on general directed graphs, without any additional properties. To present Felsner's results, we take a look at a planar digraph D given with a crossing-free embedding in the plane (planar map). We want to find a partial order on $\mathit{reor}_\alpha(D)$. Therefore we will assign directions to those α-reorientations, that can be performed at members of $\mathit{reor}_\alpha(D)$, by reorienting facial cycles.

We orient all the faces of the embedding counter clockwisely (ccw). This leads to a choice of orientation for the **facial cycles of** D - the boundaries of the faces of the embedding. The ccw-orientation uniquely partitions every facial cycle into forward arcs and backward arcs, by walking around the corresponding face in ccw-direction. We call the set of facial cycles $\mathcal{F}$ and the facial cycle, which is induced by the unbounded face, **the forbidden facial cycle** denoted by $\mathbf{X}$.

Taking forward and backward arcs as positive and negative elements, respectively, terms as positively and negatively directed specialize from general signed sets to facial cycles.

To establish an order on *reor*$_\alpha(D)$ we introduce special α-reorientations of D. Reversing the orientation on the arc set of a positively directed facial cycle is called a **flip**. Reversing the orientation on the arc set of a negatively directed Eulerian is called a **flop**. As in both cases the reoriented arc set lies in $\underline{\mathcal{E}^+(D)}$ one cannot leave *reor*$_\alpha(D)$ by flipping and flopping facial cycles. The flip and the flop will be understood as going up and going down, respectively, in *reor*$_\alpha(D)$. Felsner's Theorem says that indeed every arc set in $\underline{\mathcal{E}^+(D)}$ can be reoriented by a sequence of flips an flops of facial cycles. To capture the consecutive application of flips and flops and the resulting partial order, we define the following.

Let D be planar ccw embedded digraph with facial cycles $\mathcal{F}$ and forbidden facial cycle $\mathbf{X}$. A **flip flop sequence based at** $\mathcal{F}\backslash\{\mathbf{X}\}$ is a sequence $(F_{s(1)}, \ldots, F_{s(k)})$ of elements of $\mathcal{F}\backslash\{\mathbf{X}\}$ such that for every $1 \leq i \leq k$ after reorienting $F_{s(1)}, \ldots, F_{s(i-1)}$ the next cycle $F_{s(i)}$ can be flipped or flopped in the resulting digraph.

We can now establish a partial order on D's α-reorientations the following way. Let $D', D'' \in$ *reor*$_\alpha(D)$. Define $D' \leq_{\mathrm{ff}} D''$ if there is a flip sequence of bounded facial cycles of D' that transforms D' to D''.

In [7] Felsner proves:

Theorem 1. *Let $D = (V, A)$ be a strongly connected directed graph, given with a crossing free planar embedding. Denote by $\mathbf{X}$ the facial cycle induced by the unbounded face. Then*

- *Every α-reorientation of D can be obtained from D by a flip flop sequence based at $\mathcal{F}\backslash\{\mathbf{X}\}$.*
- *Moreover the resulting partial order $($reor$_\alpha(D), \leq_{\mathrm{ff}})$ is a distributive lattice.*

Proof. The proof of the first part of Theorem 1 works by induction on the area that is enclosed by the directed Eulerian E, corresponding to the desired α-reorientation of D. If the area of E is minimal it is a facial cycle, hence can be flipped or flopped. If the area enclosed by E is not minimal, by strong connectivity, E contains a directed path which cuts the area of E into two disjoint parts, enclosed by Eulerians E' and E''. One of both is directed and can be reoriented by the induction hypothesis. Afterwards the other one is directed and can be reoriented by the same argument. So by induction E can be reoriented by a flip flop sequence of bounded facial cycles.
We will not restate the proof of the second part of Theorem 1, as carried out in [7]. It works with an order isomorphism to a set of potential functions. We will prove this statement differently in Theorem

3.6.7. Anyways potential functions will come up again in Chapter 4. □

The distributive lattice on the α-reorientations of our example from Figure 1.2 is a 3-path with minimum D and maximum D'' as displayed in Figure 1.3.

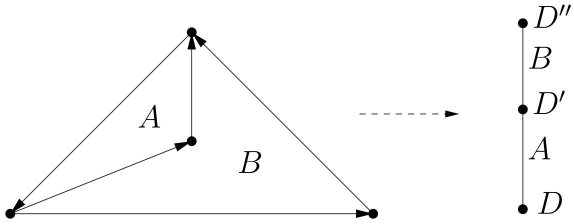

Figure 1.3: Distributive lattice on the α-reorientations of D. The generating facial cycles A and B induce an arc labeling.

A bigger example of a strongly connected planar digraph together with the distributive lattice on its α-reorientations is depicted on the cover of this thesis.

We will now have a look at the results in [12]. The objects investigated there are dual to the settings treated until now. This duality is in terms of oriented matroids, see Chapter 3.5.

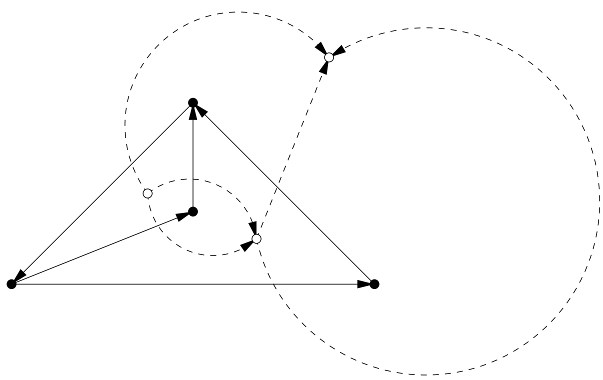

Figure 1.4: Primal and dual (dashed) planar digraphs.

In the case of oriented matroids that come from planar digraphs matroid duality coincides with usual duality of planar digraphs. Duality of planar digraphs is duality of the underlying graphs together with duality on the arc orientations. We map a forward arc of a ccw facial cycle to an outgoing arc of the corresponding dual vertex, as depicted in Figure 1.4. There are possibly several ways to embed the planar dual of a graph. We just pick one of them. We regard the consecutive application of duality as switching between the two underlying graphs, while changing the arc orientations according to the above rules.

Note that in contrast to duality of undirected graphs, the directed version is no involution anymore.

By duality of planar digraphs, the objects and properties treated until now translate as follows:

strongly connected $\leftrightsquigarrow$ acyclic
directed Eulerian $\leftrightsquigarrow$ directed cut
facial cycle $\leftrightsquigarrow$ vertex cut
α-reorientation $\leftrightsquigarrow$ c-reorientation

There are two concepts in the above table that still shall be defined:

We understand a **cut** as an arc set $A(X, \overline{X})$ induced by a 2-partiton $(X, \overline{X})$ of the vertex set. The cut consists of all the arcs that are incident to X and $\overline{X}$. We sometimes refer to X and $\overline{X}$ as the **sides of the cut**. A cut is **directed** if either all its arcs point from X to $\overline{X}$ or from $\overline{X}$ to X. A cut is called a **vertex cut** if X consists of a single vertex. The set of cuts of a digraph, seen as sign vectors, is integrally spanned by the vertex cuts. The set of non-empty inclusion minimal cuts is denoted as $\mathcal{C}^*$.

To define c-reorientations let $D = (V, A)$ be a directed graph with some fixed basis $\mathbf{C}$ of cycles of its cycle space. Let $c_D \in \mathbb{Z}^{\mathbf{C}}$ count the number of forward arcs in the cycles of $\mathbf{C}$. We call

$$\textit{reor}_c(D) := \{D' = (V, A') \mid \underline{D'} = \underline{D} \textit{ and } c_{D'} \equiv c_D\}$$

the set of **c-reorientations of** D.

Flip flop sequences of vertex cuts are defined analogously to flip flop sequences of facial cycles.

So dualizing the statement of Theorem 1 according to the above rules we obtain the following:

Theorem 2. *Let $D = (V, A)$ be an acyclic directed graph given with a planar embedding. Let $v \in V$ be an arbitrary fixed vertex. Then*

- *Every c-reorientation of D can be obtained from D by a flip flop sequence of vertex cuts different from the vertex cut induced by v.*
- *Moreover $(\textit{reor}_c(D), \leq_{\text{ff}})$ is a distributive lattice.*

Here the vertex cut of v corresponds to the unbounded face $\mathbf{X}$ of an embedding of the primal graph.

Actually the more general theorem proved in [12] says that in the dual setting planarity is no longer needed.

Theorem 3. *Let $D = (V, A)$ be an acyclic directed graph. And $v \in V$ an arbitrary fixed vertex. Then*

- *Every c-reorientation of D can be obtained from D by a flip flop sequence of vertex cuts without using the vertex cut of v.*
- *And moreover $(reor_c(D), \leq_{\mathrm{ff}})$ is a distributive lattice.*

Proof. Both parts of the theorem will be corollaries of later results with different proof ideas, even if some similarities come up in Section 4. □

One might think now that using duality one can show the statement of Theorem 1 for non-planar graphs. This is not the case. The duality between the concepts at concern is purely combinatorial in terms of oriented matroids (see Chapter 3.5), which exactly in the case of planar graphs coincides with topological duality. So application of duality to Theorem 3 gives an analogue of Theorem 1 on cographic oriented matroids.

The main purpose of this work is to generalize the presented concepts, methods and results. We pursue this with respect to three concepts:

- The concept of α-reorientations will be generalized to oriented matroids (Chapter 3)
- The generating method given by flip flop sequences will be generalized to sign matrices, which specialize to oriented matroids (Chapter 2).
- The proof method of Theorem 1, which relies on an induction on the area of Eulerians, will be applied to 2-cell-embeddings of digraphs on arbitrary orientable surfaces (Section 3.6).

These steps together enable us to prove Theorem 1 and Theorem 3 as corollaries of the more general Theorem 2.2.3 and Theorem 3.6.7.

The following preview is a bit more elaborate:

- In Chapter 2 the definition of flip flop sequences will be transfered from signed incidence vectors of cycles or cuts to arbitrary sign vectors. The resulting structures are connected loop-free digraphs instead of distributive lattices. In fact we can prove, that every connected loop-free digraph arises from the flip flops of a set of signed sets.

  As a main feature of the application of flip flops to arbitrary sign matrices a necessary condition to generate a distributive lattice will be proved. The distributivity of the posets in Theorem 1 and Theorem 3 is covered by this result (Corollary 2.2.6).

- In Chapter 3 the concept of α-reorientations will be generalized to the language of oriented matroids.

  1. In Section 3.1 we introduce some basic facts, definitions and notation from oriented and ordinary matroid theory.
  2. In Section 3.2 several problems of a possible transcription of the concept of α-orientations to oriented matroids will be discussed. We establish a generalization of α-reorientations. Some first observations concerning α-reorientations will be provided.
  3. In Section 3.3 we justify why further consideration will be restricted to the class of regular oriented matroids. A necessary condition for a set of circuits to generate all α-reorientations of such matroids will be given. Seymour's decomposition theorem for regular matroids gives rise to certain lines of investigation, treated in the following sections of the thesis.
  4. In Section 3.4 we briefly mention the structure of α-reorientations of the regular splitter $\mathbf{R10}$. By computer enumeration we disprove sufficiency of the necessary condition on a generating system for the α-reorientations, given in Section 3.3.
  5. In Section 3.5 we will make explicit how the set of c-reorientations of a digraph and therefore Theorem 3 dualizes to a theorem about α-reorientations of cographic oriented matroids.
  6. In Section 3.6 the α-reorientations of graphic oriented matroids will be investigated. This problem can be stated in terms of non-planar graphs. Application of the proof method of the first part of Theorem 1 leads to a theorem having the first parts of Theorem 1 and Theorem 3 as corollaries. Some examples, negative results as well as positive results (Section 3.7) will be provided.

- In Chapter 4 we investigate more about α-orientations of cographic oriented matroids. It will be shown that every distributive lattice comes from the α-orientations of a cographic oriented matroid. As several of oriented matroids can produce the same lattice, their structure will be explored.

Chapter 2

Order Structure from Flip Flops

In the present chapter we will generalize the idea of flip flop sequences. In Chapter 1 we have looked at flip flop sequences of sets of cycles or cuts of a digraph. Here we will look at flip flop sequences on a wider set of oriented structures. We consider arbitrary finite multisets of signed sets, displayed as sign matrices.

In Section 2.1 we will define flip flop sequences and flip flop graphs. The latter specializes back to be the Hasse diagram of a distributive lattice in Theorem 1 and Theorem 3. But the class of flip flop graphs is bigger. After some first observations we obtain that every connected loop-free digraph is a flip flop graph.

Section 2.2 will consider flip flop posets and the question how one can force them to have additional properties. We particularly investigate embedding properties into the lattice of integers and distributivity of the flip flop poset. As a corollary we obtain the distributivity of the lattices in Theorem 1 and Theorem 3.

2.1 Directed Graphs from Flip Flop Sequences

We recall some terms that have already been used in Chapter 1. Let B be an $m \times n$ matrix with entries from $\{1, -1, 0\}$. Then B is called a **sign matrix**. Through the entire work we consider sign matrices which have no zero rows. We call a vector whose entries are from $\{1, -1, 0\}$ a **sign vector** or $(1, -1, 0)$**-vector**. A sign vector is said to be **directed** if it is either non-negative (**positively directed** or $(0, 1)$**-vector**) or non-positive (**negatively directed** or $(0, -1)$**-vector**). We will denote the all zeroes and the all ones vector by $\mathbf{0}$ and $\mathbf{1}$, respectively.

For any vector $v \in \mathbb{R}^n$ denote by $sgn(v)$ its signed support vector and by $supp(v)$ its unsigned support vector. Furthermore let $diag(v)$ denote the $n \times n$ sign matrix with the support of v on its diagonal. Denoting by I the n dimensional identity matrix, the **(column) reorientation of** B **with respect to** v denoted B^v is defined as the matrix $B(I - 2diag(v))$. This means that B^v is obtained from B by switching signs in all the columns whose index corresponds to a non-zero entry of v. Actually we will refer to column reorientations by saying just reorientations, when no confusion is ahead. We refer more specificly to row reorientations, when they are about to come up.

If r is a positively directed row of B, the operation $B \rightsquigarrow B^r$ is called a **flip**. The inverse operation (if r is a negatively directed row of B) is called a **flop**. A consecutive application of flips and flops can be understood as $(\ldots((B^{r_{s_1}})^{r_{s_2}})\ldots)^{r_{s_k}}$, where for each $1 \leq i \leq k$ the s_ith row of $(\ldots((B^{r_{s_1}})^{r_{s_2}})\ldots)^{r_{s_{i-1}}}$ is directed. Actually this notation can be changed to a more convenient one as $B^{s_1,\ldots,s_k}$ or just as B^s with $s = (s_1, \ldots, s_k)$ and leads to the following definition.

For an $m \times n$ sign matrix B without zero rows, a sequence of length k of row indices $s : [k] \mapsto [m]$ is called **flip flop sequence based at B** if for each $1 \leq i \leq k$ the s_ith row of $B^{s_1,\ldots,s_{i-1}}$ is directed.

This definition clearly generalizes flip flop sequences of facial cycles as defined in Chapter 1. The difference is that we broadened the set of objects we are applying flip flop sequences to. The next lemma and proposition together give an equivalent way of describing flip flop sequences. It enables us to recognize flip flop sequences based at B without reorienting B.

Lemma 2.1.1. *Let B be a sign matrix and $s = (s_1, \ldots, s_k)$ any sequence of row indices of B. Furthermore let $binsum(s) := (supp(r_{s_1}) + \ldots + supp(r_{s_k})) \mod 2$ denote the* ***binary sum of s****. Then $B^s = B^{binsum(s)}$ holds.*

Proof. We can prove this componentwise. We use, that for a sign vector application of the support function is the same as applying the absolute value function componentwise. For integers $x_1, \ldots, x_l$ we have

$$(|x_1| + \ldots + |x_l|) \mod 2 = (|x_1 + \ldots + x_l|) \mod 2.$$

So the vector $binsum(s)$ carries exactly the information whether a column of B has been multiplied with -1 an odd or an even number of times to get the resulting B^s. This totally describes B^s. □

Proposition 2.1.2. *Let B be an $m \times n$ sign matrix and $s : [k] \mapsto [m]$ a sequence of length k of row indices of B. Then s is a flip flop sequence based at B if and only if there exists a function $\sigma : [k] \mapsto \{+, -\}$ such that for each $1 \leq i \leq k$ the sum of rows of B given by $\Sigma_{j=1}^{i} \sigma(j) r_{s_j}$ is a $(0, 1)$-vector.*
In this case σ is uniquely determined by s and B. Moreover $B^s = B^{\Sigma_{j=1}^{k} \sigma(j) r_{s_j}}$.

Proof. The function σ reflects whether a flip(+) or a flop(-) is performed. We proceed by induction on k.
For the case $k = 1$ the equivalence is obvious by the definition of flip flop sequence based at B.
If $k > 1$ we must be prove that r'_{s_k} is a positively (negatively) directed row of $B^{s_1,\dots,s_{k-1}}$ if and only if there is a $\sigma(k) \in \{+1, -1\}$ such that $(\Sigma_{j=1}^{k-1}\sigma(j)r_{s_j}) + \sigma(k)r_{s_k})$ is a $(0,1)$-vector. Because of symmetry we will only prove the equivalence of r'_{s_k} being a positively directed and $(\Sigma_{j=1}^{k-1}\sigma(j)r_{s_j}) + r_{s_k})$ being a $(0,1)$-vector.
So r'_{s_k} is a positively directed row of $B^{s_1,\dots,s_{k-1}}$ if and only if all the columns where r_{s_k} has a negative sign have been reoriented an odd number of times and those where r_{s_k} has a positive sign have been reoriented an even number of times. This, by induction hypothesis, is equivalent to $(\Sigma_{j=1}^{k-1}\sigma(j)r_{s_j})$ having positive entries where r_{s_k} has negative entries, and 0-entries where r_{s_k} has positive entries. This is the same as saying $(\Sigma_{j=1}^{k-1}\sigma(j)r_{s_j}) + r_{s_k}$ is a $(0,1)$-vector.
As $(\Sigma_{j=1}^{k-1}\sigma(j)r_{s_j})$ and r_{s_k} both are $(0,1)$-vectors, the equation could equally have been stated over $\mathbb{F}_2$. With the induction hypothesis Lemma 2.1.1 gives $B^{(\Sigma_{j=1}^{k-1}\sigma(j)r_{s_j})+\sigma(k)r_{s_k}} = B^{binsum(s)} = B^s$.
Obviously if one of the two choices for $\sigma(k)$ gives a $(0,1)$-vector, as zero rows are forbidden, the other does not, so σ is unique. □

Proposition 2.1.2 leads to the following observations and definitions:

A possible view at flip flop sequences is to put $\sigma(1)r_{s_j}, \dots \sigma(k)r_{s_k}$ in ordered this way as rows into a new matrix. Then s is a flip flop sequences with associated function σ if and only if the resulting matrix is a column-alternating sign matrix, where in every column the first non-zero entry is positive. We will make no explicit use of this view, but sometimes it might be convenient to have it at hand.

Let s be a flip flop sequence based at B of length k, then the **sign sum of s** is $sgnsum(s) := (\Sigma_{j=1}^{k}\sigma(j)r_{s_j})$, where $\sigma : [k] \mapsto \{+,-\}$ is the unique function associated to s, according to Proposition 2.1.2.

For a sign matrix B denote by $\mathrm{FF(B)}$ the set of all flip flop sequences based at B. We can now define **the flip flop span** of B as $\mathrm{ff(B)}:= \{sgnsum(s) \mid s \in \mathrm{FF(B)}\} \subseteq \{0,1\}^{\mathrm{n}}$. Every vector in $\mathrm{ff(B)}$ stands for a reorientation of B.

Recall that Theorem 1 states that $\mathrm{ff}(\mathcal{F}\backslash\{\mathbf{X}\}) = \textit{reor}_\alpha(\mathrm{D})$ for a planar strongly connected digraph. Moreover we have that

$$\textit{reor}_\alpha(D) \cong \underline{\mathcal{E}(D)^+} \cong span_{\mathbb{Z}}(\mathcal{C}) \cap \{0,1\}^{|A|} = span_{\mathbb{Z}}(\mathcal{F}\backslash\{\mathbf{X}\}) \cap \{0,1\}^{|A|}.$$

Switching back to sign matrices this generalizes to the question, how $\mathrm{ff(B)}$ does look like with respect to $span_{\mathbb{Z}}(B) \cap \{0,1\}^n$?

Define for a flip flop sequence s of length k the **integral support of** s by $z(s) := \Sigma_{i=1}^{k}(\sigma(i)e_{s_i})$. Obviously $sgnsum(s) = z(s)^T B$, so $\mathrm{ff(B)}$ is a subset of the set of all $(0,1)$-vectors in the **(integral) row space of** B, i.e. $\mathrm{ff(B)} \subseteq \mathrm{span}_{\mathbb{Z}}(\mathrm{B}) \cap \{0,1\}^{\mathrm{n}}$.

This inclusion can be strict. Consider for instance:

$$B := \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The flip flop span of B does not contain the element $(010) \in span_{\mathbb{Z}}(B) \cap \{0,1\}^3$. The elements of $\mathrm{ff(B)}$ are depicted in Figure 2.1 as the vertex labels of the flip flop graph of B.

With view at the second part of Theorem 1, we want to establish a directed structure on the flip flop span. The **flip flop graph of** B is defined as $D_{\mathrm{ff}}(B) = (V, A)$, where the vertices V correspond to the elements of $\mathrm{ff(B)}$. They can be regarded as labelled by $\mathrm{ff(B)}$. Whenever it is clear that we talk about a flip flop graph we will not always distinguish between the vertices and the corresponding elements of $\mathrm{ff(B)}$. So as a first example of this abuse, we define the arc set $A(D_{\mathrm{ff}}(B))$ of $D_{\mathrm{ff}}(B)$ by

$$(v, w) \in A(D_{\mathrm{ff}}(B)) :\Leftrightarrow w - v \textit{ is a row of } B.$$

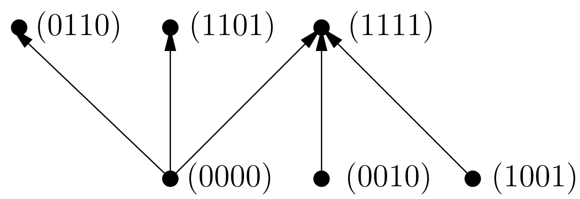

Figure 2.1: Flip flop graph of the matrix B. Vertices are labelled with the corresponding elements of the flip flop span.

The definition of the arc set $A(D_{\mathrm{ff}}(B))$ gives rise to a coloring $c : A(D_{\mathrm{ff}}(B)) \to [m]$ of the arcs by the row numbers of B. Several arcs can have the same color. Looking at Proposition 2.1.2 one can describe every flip flop sequence s based at B as a walk in $D_{\mathrm{ff}}(B)$. Its initial vertex is the zero vertex and its end vertex is the one labelled with $sgnsum(s)$. The ith arc in the walk is colored with a row that equals $sgnsum(s_1, \ldots, s_i) - sgnsum(s_1, \ldots, s_{i-1})$. The arc is forward with respect to the walk if the associated σ-function has $\sigma(i) = +$ and backward otherwise.

We call a flip flop sequence s **directed** if σ is constant. If σ is constant and positive we call s a **flip sequence**. In the negative case, we call it a **flop sequence**.

Lets now look back to the introduction and convince ourselves that the we have reached a generalization of the concepts presented there.

Take B as the sign matrix whose rows are the signed incidence vectors of the ccw-oriented bounded facial cycles of a strongly connected planar digraph. Theorem 1 says that $\mathrm{ff(B)} = \textit{reor}_\alpha(\mathrm{D})$ and that the transitive closure of $D_{\mathrm{ff}}(B)$ is a distributive lattice.

With B like that by Proposition 1 we have $span_{\mathbb{Z}}(B) \cap \{0,1\}^n = \textit{reor}_\alpha(D)$. So in order to generalize Theorem 1 we must investigate two things:

- Under which conditions is the transitive closure of $D_{\mathrm{ff}}(B)$ a distributive lattice?
- How can we guarantee that $\mathrm{ff(B)} = \mathrm{span}_{\mathbb{Z}}(\mathrm{B}) \cap \{0,1\}^{\mathrm{n}}$?

An analogous reformulation of Theorem 3 in terms of flip flop span and flip flop graph is also possible. Just take the signed incidence vectors of all but one vertex cut as the rows of B.

Nevertheless, the flip flop structure defined in the present section is more general as the following theorem illustrates. We switch the point of view and ask whether a given graph is the flip flop graph of a sign matrix.

Theorem 2.1.3. *For every loop-free connected directed graph $D = (V, A)$, there is a sign matrix B such that $D = D_{\mathrm{ff}}(B)$.*

Proof. Let $D = (V, A)$ be a connected directed graph without loops. We pick an arbitrary $v \in V(D)$ and construct a matrix B such that $D =_{\mathrm{ff}} (B)$ and v corresponds to the zero vector in $\mathrm{ff(B)}$, i.e. the trivial reorientation B of B. We will prove that one solution of this problem is given by $B := \textit{Inc}(D)^T(2diag(e_v) - I)$ - the transpose of D's incidence matrix, where all the columns corresponding to vertices different from v have been multiplied by -1.

We prove this result twice. The first proof is more descriptive. The second one is more formal and includes the stronger statement that $\mathrm{ff(B)} = \mathrm{span}_{\mathbf{Z}}(\mathrm{B}) \cap \{0,1\}^{\mathrm{n}}$.

First proof:
Let $B := \textit{Inc}(D)^T(2diag(e_v) - I)$. Denote by b_a the row of B that corresponds to an arc $a \in A(D)$. The matrix B is oriented in such a way that the positively directed rows are exactly those b_a with $a = (v, \cdot)$. The negatively directed rows are exactly those b_a with $a = (\cdot, v)$. The remaining rows b_a with $a = (u, w)$ equal $e_w - e_u$, thus cannot be flipflopped.
Now, flipping b_a with $a = (v, w)$ leads to the matrix B^{b_a}. In B^{b_a} the vertex w has exactly the particular role, that v played in B.
We identify B with v and B^{b_a} with w. The arc of $D_{\mathrm{ff}}(B)$ coming from the flip of b_a can

be identified with a. Flopping a row leads to the analogue situation.
So standing now at w we can flop back to v or reach any other neighbor of w by flip flopping the corresponding row.
By connectivity of D this way every vertex of D can be represented in $D_{\text{ff}}(B)$.

To convince ourselves that none of the vertices of D is represented multiply. We show that cycles in D correspond to cycles in $D_{\text{ff}}(B)$.
If X is the signed incidence vector of a cycle in D we have *Inc*$(D)X = \mathbf{0}$. But as only rows of *Inc*(D) are reoriented to obtain B, the sign sum of the rows of B that correspond to the elements of X gives $\mathbf{0}$ as well. $\square$

Second proof:
Again, let $B :=$ *Inc*$(D)^T(2diag(e_v) - I)$. In order to prove $V(D) = \text{ff(B)}$ we must find a labeling l of $V(D)$ that is a bijection of $V(D)$ and ff(B). So for any $w \in V(D)$ define $l(w) := (e_v + e_w) \bmod 2$. This mapping is obviously injective. It remains to show, that its image is exactly the flip flop span of B.
We start with proving $l(V(D)) \subseteq \text{ff(B)}$:
Take $w \in V(D)$ and let $d(w)$ denote its distance from v. We proceed by induction on $d(w)$.
If $d(w) = 0$ then $w = v$ and $l(v) = \mathbf{0} \in \text{ff(B)}$.
If $d(w) = 1$ the vector $l(w)$ is a directed row of B, so $l(w) \in \text{ff(B)}$.
If $d(w) > 1$ look at a vertex w' which precedes w in a shortest path from v to w. By induction hypothesis $l(w') = e_v + e_{w'} \in \text{ff(B)}$. We have to check the two (not necessarily disjoint) cases $(w', w) \in A(D)$ and $(w, w') \in A(D)$. Assume $(w', w) \in A(D)$ then by the definition of B the vector $e_w - e_{w'}$ is a row of B. So we have

$$l(w) = e_v + e_w = l(w') + e_w - e_{w'} \in \text{ff(B)}.$$

The case $(w, w') \in A(D)$ is analogue.
Now, in order to prove $l(V(D)) \supseteq \text{ff(B)}$ we prove the stronger

$$l(V(D)) \supseteq span_{\mathbb{Z}}(B) \cap \{0,1\}^n.$$

So let $\Sigma\lambda_i b_i$ be any (integral) linear combination of rows of B such that the sum is a $(0,1)$-vector. Write the numbers λ_i like flows on the arcs a_i associated to the rows b_i. For every $w \in V$ denote by $x(w) := \Sigma_{a_i=(.,w)}\lambda_i - \Sigma_{a_i=(w,.)}\lambda_i$ **the excess of** w. For a flow on a directed graph one has $\Sigma_{w\in V} x(w) = 0$. In our case, the fact that $\Sigma\lambda_i b_i$ is a $(0,1)$-vector translates to the following two conditions in D:

- for every vertex $w \neq v$ we have $x(w) \in \{0, 1\}$.
- for v we have $x(v) \in \{0, -1\}$.

So by $\Sigma_{w\in V} x(w) = 0$ we have that $\Sigma\lambda_i b_i$ is either zero or equals $e_v + e_w$ for some $w \in V$. This is $\Sigma\lambda_i b_i \in l(V(D))$.
It remains to show $A(D) = A(D_{\mathrm{ff}}(B))$. We omit the proof as it is a forward application of the given vertex labeling l to the arc defining property

$$(v, w) \in A(D_{\mathrm{ff}}(B)) :\Leftrightarrow w - v \textit{ is a row of } B.$$

□

We have seen that every connected loop-free digraph is a flip flop graph. But one such graph D can be the flip flop graph of several matrices B. How can we distinguish different ways to flip flop generate the same graph?

Look at the arc coloring $c : A(D_{\mathrm{ff}}(B)) \to [m]$ of the arcs $D_{\mathrm{ff}}(B)$ by the row numbers of B. If two arcs have a vertex in common, they have different colors. The coloring partitions the arc set of $D_{\mathrm{ff}}(B)$ into a set of matchings. One could say, that two sign matrices are "essentially the same" if they induce the same matching partition. It is an open question, how the different matching partitions that arise from the flip flop structure could be characterized.

The matching partition induced by the matrix in Theorem 2.1.3 is the trivial one, i.e. every arc is a matching. Any matching partition consists of at least maximum degree many matchings. Together we obtain that the minimal number of rows of a sign matrix B, that generates a given graph D, lies between its maximal degree and its number of arcs.

2.2 Posets from Flip Flop Sequences

Now we turn back to our original aim, that is to find matrices or conditions on matrices that generate "nice" flip flop graphs. Particularly we want to investigate flip flop posets, which can be defined as the transitive closure of acyclic flip flop graphs..

A first step is the following.

Let B be an $m \times n$ sign matrix and c a sign vector of length m. The **(row) reorientation of B with respect to c** is the matrix $B_c := (I - 2diag(c))B$, which differs from B only by multiplying the set of rows where c has a non-zero entry with -1.

Proposition 2.2.1. *For every $m \times n$ sign matrix B without 0-rows, there is a $(0, 1)$-vector c of length m, such that $D_{\mathrm{ff}}(B_c)$ is acyclic and* $\mathrm{ff(B) = ff(B_c)}$.

Proof. We can use the much stronger fact, that every loop-free oriented matroid has an acyclic orientation, see [1]. As B has no 0-rows, applying this to the row space of B gives us, that there is a row reorientation B_c, such that there are no positive linear dependences among the rows of B_c. This means in particular that we will not have flip flop sequences s with $sgnsum(s) = 0$, whose associated σ function is constant. In other words, row reorienting B, we obtain an acyclic $D_{\text{ff}}(B_c)$, which differs from $D_{\text{ff}}(B)$ only by orientation of arcs, thus has the same flip flop span as B. $\square$

Proposition 2.2.1 particularly tells us that a B with linearly independent rows generates an acyclic flip flop graph.

If $D_{\text{ff}}(B)$ is acyclic we call the transitive closure of $D_{\text{ff}}(B)$ the **flip flop poset of B** denoted by $P_{\text{ff}}(B)$.

Posets can be order embedded into some $\mathbb{Z}^d$ with the dominance order. We now want to investigate under which conditions the flip flop structure leads to an order embedding of $P_{\text{ff}}(B)$ into $\mathbb{Z}^m$. Denote by FF the set of flip flop sequences based at B. Mapping a flip flop sequence s to its integral support $z(s)$ with respect to B, we obtain a mapping of FF into $\mathbb{Z}^m$. We will now investigate if the integral supports of FF somehow geometrically represent $P_{\text{ff}}(B)$.

One can ask whether the partial order $P_{\text{ff}}(B)$ has anything to do with the dominance order $\leq_{dom}$ on $z(\text{FF}) \subseteq \mathbb{Z}^{\text{m}}$. A positive awnser to this question would be that the map $sgnsum \circ z^{-1} : z(\text{FF}) \to sgnsum(S)$ is an order preserving bijection, i.e. that for any $s, s' \in \text{FF}$ the following two conditions are satisfied:

(i) $sgnsum(s) = sgnsum(s) \Leftrightarrow z(s) = z(s')$

(ii) $sgnsum(s) <_{\text{ff}} sgnsum(s') \Leftrightarrow z(s) <_{dom} z(s')$

In this case we say that $P_{\text{ff}}(B)$ is **integral**.

It is easy to see that every $P_{\text{ff}}(B)$ satisfies the "$\Leftarrow$" direction of (i) and that (i) implies the "$\Rightarrow$" direction of (ii).

The rest is not generally satisfied as illustrated by the following examples:

In Figure 2.2 (a) we see the Hasse diagram of

$$P_{\text{ff}}(\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}),$$

which satisfies (ii) but not (i) at the top element.

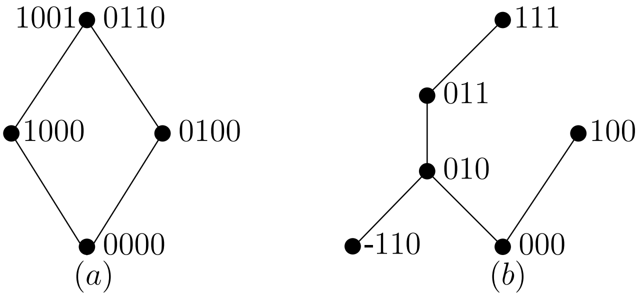

Figure 2.2: Examples of non-integral flip flop posets. (Multiple) vertex labels stand for integral supports z of corresponding flip flop sequences. Figure (a) satisfies (ii) but not (i). Figure (b) satisfies (i) but not (ii)

Figure 2.2 (b) is the Hasse diagram of

$$P_{\text{ff}}(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}).$$

It satisfies (i) but not (ii), which can be seen by the incomparability of the two vertices labelled with (111) and (100), respectively.

Integrality of a flip flop poset $P_{\text{ff}}(B)$ means that the function $(sgnsum \circ z^{-1})^{-1}$ exists and is an order embedding of $P_{\text{ff}}(B)$ into $\mathbb{Z}^m$. The **dimension of a poset** P is the minmal d such that P can be embedded into $\mathbb{Z}^d$. So if a P is ismorphic to some integral $P_{\text{ff}}(B)$ its dimension is bounded from above by the row number m of B.

A property of the embedding given by $(sgnsum \circ z^{-1})^{-1}$, which suggests that its dimension is generally bigger than the dimension of the poset, is the following:

Proposition 2.2.2. *If $P_{\text{ff}}(B)$ is an integral flip flop poset the elements of the embedding into $\mathbb{Z}^m$ via $(sgnsum \circ z^{-1})^{-1}$ are convex independent, i.e. are the vertices of a polytope.*

Proof. The proof works by elementary arguments of polytope theory. As we do not make any use of the result, the proof will be omitted. □

Next we want to characterize those $P_{\text{ff}}(B)$ that are integral and distributive lattices. In order to do so we prove a slightly more general theorem, that needs one more term to be introduced.

Given a directed graph $D = (V, A)$ with an arc coloring $c : A \to [k]$ we define the **colored incidence vector** of a signed arc set X as

$$c(X) = (| X^+ \cap c^{-1}(i) | - | X^- \cap c^{-1}(i) |)_{1 \leq i \leq k}.$$

As usual, we will see forward arcs as positive elements of an arc set and backward arcs as negative elements.

Before stating the theorem note the following: Color the arcs of a flip flop graph with the row numbers of the rows they correspond to. The colored incidence vector of a path corresponds to the integral support vector of the flip flop sequence, that is represented by the path.

Theorem 2.2.3. *Let P be a poset. There is an order isomorphism ϕ from P to a distributive lattice $L \subseteq \mathbb{Z}^k$ such that $p \prec q \Leftrightarrow \phi(q) - \phi(p) = e_i$ for some $i \in [k]$*

if and only if

P is the transitive closure of an acyclic directed graph $D = (V, A)$ that admits an arc coloring $c : A \to [k]$, which satisfies the following conditions. For every $u \in V$:

$\bigvee_1$: $(u, v), (u, w) \in A \Rightarrow c(u, v) \neq c(u, w)$

$\bigvee_2$: $(u, v), (u, w) \in A \Rightarrow \exists (v, x), (w, x) \in A : c(u, v) = c(w, x)$ *and* $c(u, w) = c(v, x)$

$\bigwedge_1$: $(v, u), (w, u) \in A \Rightarrow c(v, u) \neq c(w, u)$

$\bigwedge_2$: $(v, u), (w, u) \in A \Rightarrow \exists (x, v), (x, w) \in A : c(v, u) = c(x, w)$ *and* $c(w, u) = c(x, v)$

In this case the coloring and the embedding can be chosen such that

$$\phi(q) - \phi(p) = e_i \Leftrightarrow c(p, q) = i.$$

Proof. "$\Rightarrow$":
Let P be a distributive lattice embedded into $(\mathbb{Z}^k, \leq_{dom})$ via ϕ such that

$$p \prec q \Leftrightarrow \phi(q) - \phi(p) = e_i \text{ for some } i \in [k].$$

Define D to be the Hasse diagram of P. The arc (p, q) in D corresponds to the relation $p \prec q$ in P. Define the arc coloring c such that $(p, q) \Leftrightarrow \phi(q) - \phi(p) = e_{c(p,q)}$.
Since D has no parallel arcs $\bigvee_1$ and $\bigwedge_1$ are clearly satisfied.
To see $\bigvee_2$ take two arcs $(u, v), (u, w) \in A$. We have

$$\phi(v) - \phi(u) = e_i \neq e_j = \phi(w) - \phi(u).$$

Take $x := v \vee w$ so in the dominance order $\phi(x)$ is the componentwise maximum $max(\phi(v), \phi(w))$. This is, $\phi(x) - \phi(v) = e_j$ and $\phi(x) - \phi(w) = e_i$. So in terms of c we have $c(u, v) = c(w, x) = i$ and $c(u, w) = c(v, x) = j$.
Property $\bigwedge_2$ follows analogously.

"$\Leftarrow$":
Let $D = (V, A)$ be a connected acyclic digraph that admits an arc coloring $c : A \to [k]$, which satisfies $\bigvee_1$, $\bigvee_2$, $\bigwedge_1$, and $\bigwedge_2$. And let P be the transitive closure of D. The idea of how to embed P into $(\mathbb{Z}^k, \leq_{dom})$ is to define $\phi(v) := 0$ for some $v \in V$. Now, given any other $w \in V$ and a (v, w)-path Q in D we define $\phi(w) := c(Q)$, as the colored incidence vector of Q.
A priori it is not even clear that ϕ is well-defined.

The proof consists of three parts:

1. We introduce the switch operation, that consists of the iterated application of $\bigvee_1$ and $\bigvee_2$. Given $u, v, w \in V$, a directed (u, v)-path Q and a directed (u, w)-path R in D it constructs an element $x \in V$, a directed (v, x)-path S and a directed (w, x)-path T in D.
   Moreover we prove that $c(Q) + c(S) = c(R) + c(T) = c(Q) \vee_{dom} c(R)$.
   So this construction is a generalization of $\bigvee_2$ in the sense that, assuming ϕ to give an order embedding of P, x corresponds to the join of v and u.
   An analogue construction can be done for the iterated application of $\bigwedge_1$ and $\bigwedge_2$.

2. Using the switch operation we show, that ϕ is a well-defined injective function.

3. Using $1.$ and $2.$ we show that ϕ is order-preserving.

These three together give that P is isomorphic via ϕ to a subposet of $(\mathbb{Z}^k, \leq_{dom})$, which is closed with respect do $\vee_{dom}$ and $\wedge_{dom}$. This implies that P is distributive.

We start with proving $1.$
So take $u, v, w \in V$, a directed (u, v)-path Q and a directed (u, w)-path R in D. Applying $\bigvee_2$ iteratively from u on, until $\bigvee_2$ cannot be applied anymore, one obtains a grid as depicted in Figure 2.3. Parallel arcs have the same colour.

Whenever $\bigvee_2$ has been applied to two differently colored arcs of D the resulting two arcs are also arcs of D.
But when we encounter a situation, where $\bigvee_2$ has been applied to two arcs with the same colour, property $\bigvee_1$ tells us that the equally colored arcs, must indeed be the same. So all the arcs that are parallelly above these two equally colored arcs do not exist in D.
We repair all these situations in some order that respects the dominance order of the 2-dimensional grid, seeing u as the minimum. In Figure 2.4 the corresponding vertices are drawn bigger.
So standing at a vertex which is left by two arcs with same colour, we colour the arcs that resulted from the wrongly applied $\bigvee_2$ with 0. Moreover all the parallel arcs above these arcs will be colored with 0 as well. We obtain a picture exemplified by Figure 2.4.

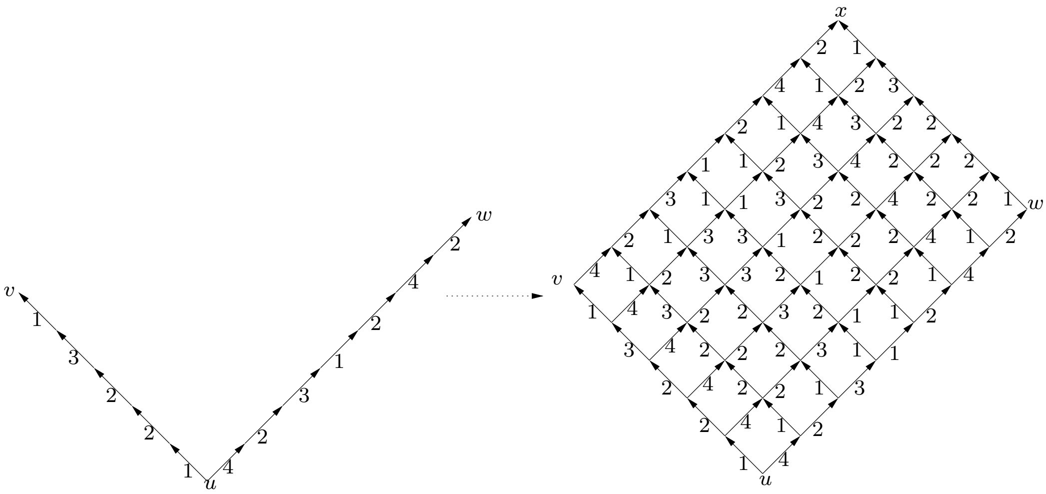

Figure 2.3: Formal completition of the paths Q and R to a square grid. The arcs are labeled with their colour.

We can now contract all the arcs which are colored with 0 and delete resulting parallel arcs. We call the obtained graph D'. We want to prove that the directed (v, x)-path S and the directed (w, x)-path T of D' are also in D.

So first we prove that the graph D' can mapped homomorphically to a subgraph of D. We show that every vertex of D' is either the top vertex of a **legal square** in the sense of $\bigvee_{1,2}$ and $\bigwedge_{1,2}$ or is in Q or R.

Before any 0-arc is contracted this condition is satisfied by all the vertices but those **bad vertices** which lie above one of the vertices, where we started a parallel 0-coloring. We call the square below a bad vertex an **illegal square**. See Figure 2.5.

We show that starting with the formal grid in Figure 2.3 we can contract 0-arc by 0-arc, such that whenever we destroy a legal square below some vertex, this situation can be repaired by contracting another 0-arc. At the end every vertex the top vertex of a legal square in Q or R.

Every 0-arc is contained in one or two squares. So if we contract a 0-arc we harm at most two legal squares. But if a legal square is harmed by contracting a 0-arc it must contain another parallel 0-arc because it was legal. We can contract this other 0-arc in order to repair the situation. We then obtain an equally colored double arc. Delete one of both. If the remaining one is 0-arc it is clearly again contained in at most two squares. The remaining vertices of the former square, were top vertices of legal squares before. So they still are or they can be repaired.

If we harm one of the legal squares, we just leave it like that.

So after contracting all the 0-arcs we have not obtained new bad vertices. Moreover the

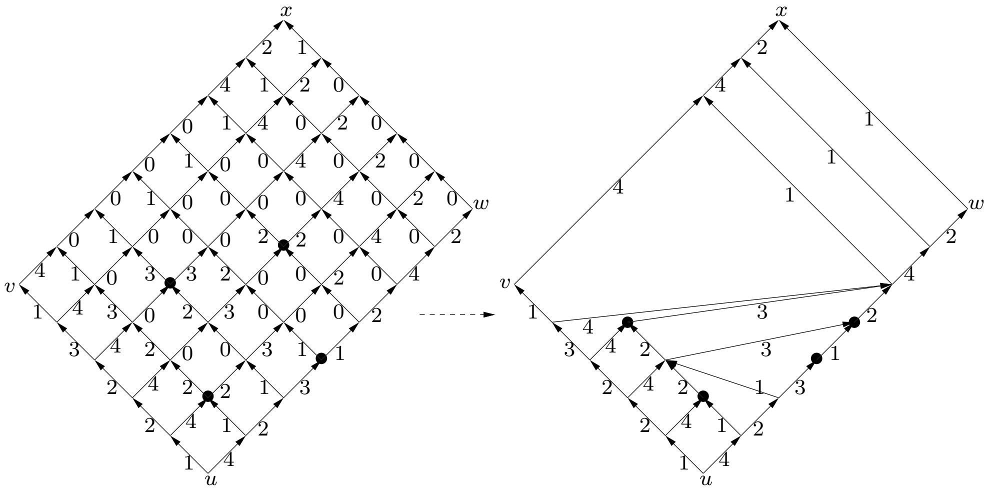

Figure 2.4: Arcs with 0-colour result from wrongly applied $\bigvee_2$ and can be contracted. Resulting parallel arcs are deleted.

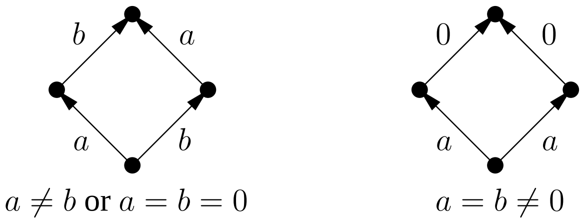

Figure 2.5: Legal and illegal squares in the proof of 1.

contraction of the 0-arcs in an illegal square identifies the bad vertex on its top with the ones on the sides, which are not bad.
So every vertex in D' lies in Q or R or in a legal square. This means that all the vertices and arcs of D' result from application of $\bigvee_2$ to differently colored arcs of D (starting with Q and R). Thus the vertices of D' are vertices of D and if two are connected by an arc, so they are in D. So D' can be mapped homomorphically to a subgraph of D.

One vertex of D can possibly occur several times in D', i.e the homomorphism is possibly not injective. But by acyclicity of D such different representatives of a vertex of D in D' cannot lie on a common directed path in D'. So S and T are indeed directed paths in D.

We still want to prove how the colored incidence vectors of the new paths S and T look like.
Observe that the points where we repaired the grid by coloring arcs with 0 corresponds to arcs of a maximum matching that identifies equally colored arcs of Q and R. The number

of arcs in such a matching that correspond to a fixed colour i is $min(c(Q)(i), c(R)(i))$. As T and S differ from Q and R respectively only by contracting the 0-colored arcs, we obtain $c(T) = c(Q) - (c(Q) \wedge_{dom} c(R))$ and $c(S) = c(R) - (c(Q) \wedge_{dom} c(R))$. This implies $c(Q) + c(S) = c(R) + c(T) = c(Q) \vee_{dom} c(R)$.
It should be clear that interchanging $\bigvee$s and $\bigwedge$s everywhere an analogue construction can be performed if $u, v, w \in V$, a directed (v, u)-path Q and a directed (w, u)-path R are given.
During the rest of the proof we call the concatenated path $(T, -S)$ that results from our construction applied to Q and R, the **switch** of the path $(Q, -R)$. By the above construction $(T, -S)$ has less or equal arcs than $(Q, -R)$ and $c(T, -S) = c(Q, -R)$.

Now we prove 2.
We show, that ϕ is a well-defined function.
By the definition of ϕ, we must show that for $u, v \in V$ all the (u, v)-paths have the same colored incidence vector. Or, equivalently, that every circular walk C of D has $c(C) = 0$. Suppose there is a cycle C with $c(C) \neq 0$. By successively replacing parts of C with their switches we can obtain a new cycle C', that is bipolarly oriented, has less or equal arcs than C and has the same colored incidence vector as C.
It could indeed happen that C' is only a circular walk, i.e. uses arcs several times, but then it can be decomposed into smaller cycles some of them having colored incidence vector different from 0.

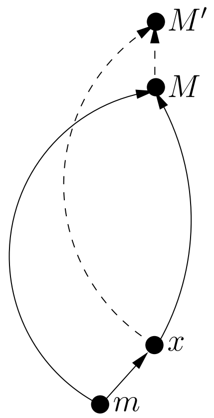

Figure 2.6: The bipolarly oriented cycles C and C' in the proof of 2.

So among the smallest bipolarly oriented cycles in D that are counterexamples to our claim, take C to be one with maximal sink M with respect to P. Applying a switch to the (M, m, x)-path leads to a new (x, M', M)-path, depicted as a dashed path in Figure 2.6. Both have the same colored incidence vector. So gluing the (x, M', M)-path to the part of C, which is not in the (M, m, x)-path, leads to a new cycle C', with $c(C) = c(C')$ and $\mid C' \mid \leq \mid C \mid$. (By minimality C' cannot take arcs several times as argued above).
As $M \leq_P M'$ either M was not maximal or $\mid C \mid$ not minimal. This is a contradiction.

To see that ϕ is injective, again we proceed by contradiction. Applying switches we can reduce every more general counterexample to the following case: There are two directed paths Q, R with the same initial vertex, different end vertices and $c(Q) = c(R)$. Applying a switch to this situation we obtain that the resulting paths T and S have length 0, so the end vertices were the same as well. This is a contradiction.

Now we prove 3.
We show that ϕ is an order embedding. Clearly $v <_P w \Rightarrow \phi(v) <_{dom} \phi(w)$, as there is a directed (v, w)-path in P. On the other hand as any $v, w \in V$ are connected by some shortest path, consecutive application of switches to parts of this path allows us to construct a directed (u, v)-path Q and a directed (u, w)-path R. Switching this gives the element x together with directed (v, x)-path S and a (w, x)-path T. Moreover 1. gives that $\phi(x) = \phi(v) \vee_{dom} \phi(w)$, so given $\phi(v) \leq_{dom} \phi(w)$ we have that the path T is empty and S is a directed (v, w)-path. So $v \leq_P w$.

We have shown until now that, P is order-isomorphic to a subposet of $(\mathbb{Z}^k, \leq_{dom})$. By 1. the poset $\phi(P)$ is closed with respect to taking joins and meets. So $\phi(P)$ is a sublattice of $(\mathbb{Z}^k, \leq_{dom})$. This implies that $\phi(P) \cong P$ is a distributive lattice as well. $\square$

In a flip flop graph the row numbers of the generating matrix give a natural arc coloring. We can apply Theorem 2.2.3 to the so given arc colorings. As a corollary we obtain the desired characterization of integral distributive flip flop posets.

Corollary 2.2.4. *Let B be a sign matrix, such that $D_{\text{ff}}(B)$ is acyclic. Then $P_{\text{ff}}(B)$ is integral and a distributive lattice if and only if whenever two different rows r_i, r_j of B can be flipped at the same time, then after flipping one of them the other can still be flipped. The analogue must hold for flops.*

Proof. "$\Rightarrow$":
Let $P_{\text{ff}}(B)$ be integral and a distributive lattice. Then $x + e_i, x + e_j \in z(\text{FF})$ implies $(x + e_i) \vee_{dom} (x + e_j) = x + e_i + e_j \in z(\text{FF})$. This means, that after flipping r_j the other row r_j can be flipped. Obviously they can be flipped the other way around as well.

The analogue holds for the flop case.

"$\Leftarrow$":
Let $D_{\text{ff}}(B)$ be an acyclic flip flop graph, such that whenever two different rows r_i, r_j of B can be flipped at the same time, then after flipping one of them the other one can still be flipped. Every arc a of $D_{\text{ff}}(B)$ is corresponds to a row r_i of B. Define the colour $c(a) := i$ to be the corresponding row number. Obviously no two arcs that are outgoing arcs of the same vertex can be colored the same. So we have property $\bigvee_1$ of Theorem 2.2.3. Furthermore the fact that rows that are flippable at the same time comute, translates

to $\bigvee_2$ of Theorem 2.2.3. Properties $\bigwedge_1$ and $\bigwedge_2$ follow analogously.
Mapping the 0-vertex of $D_{\text{ff}}(B)$ via ϕ to $\mathbf{0} \in \mathbb{Z}^m$ we can apply Theorem 2.2.3 and obtain an embedding with respect to the coloring c. This is, every vertex is mapped to its integral support. □

The condition of Corollary 2.2.4 does not sound very easy to read off the matrix B. The following corollary gives simple sufficient conditions on the generating matrix.

Corollary 2.2.5. *Let B be a sign matrix, such that $D_{\text{ff}}(B)$ is acyclic. If any two rows y, z of B have either disjoint supports or at least one entry i, where they are signed differently, i.e. $y(i)z(i) = -1$, then $P_{\text{ff}}(B)$ is integral and a distributive lattice.*

Proof. Let B be a sign matrix, such that $D_{\text{ff}}(B)$ is acyclic, $x \in \text{ff(B)}$ and y, z rows of B. As if the supports of y and z are not disjoint, there is an i such that $y(i)z(i) = -1$, we have $\{x(i) + y(i), x(i) + z(i)\} \cap \{-1, 2\} \neq \emptyset$ in this case. This is, $\{x + y, x + z\}$ cannot be a subset of ff(B). So if y and z can be flipped their supports are disjoint. This is, $x + y, x + z \in \text{ff(B)}$ implies $x + y + z \in \text{ff(B)}$.
The same argument works for the ”minus”-case.
We can use Corollary 2.2.4, to obtain that $P_{\text{ff}}(B)$ is integral and a distributive lattice. □

In order to derive Theorem 1 and Theorem 3, we introduce an important class of sign matrices that satisfy the conditions of Corollary 2.2.5. We call a sign matrix B a 2-**basis** (of its row space) if every column contains at most one $+1$-entry and at most one -1-entry and the rows of B are linearly independent. Obviously any 2-basis fulfills the requirements of Corollary 2.2.5, thus has an integral, distributive flip flop poset.

Any independent set of coherently oriented facial cycles of a digraph which is 2-cell embedded into an orientable surface gives a 2-basis. Furthermore every independent set of directed vertex cuts of a digraph forms a 2-basis.

Therefore Corollary 2.2.5 implies the second parts of Theorem 1 and of Theorem 3:

Corollary 2.2.6. *The flip flop poset of the bounded facial cycles of a planar digraph and the flip flop poset of an independent set of vertex cuts of any digraph are integral and distributive.*

As it is conversely easy to extend any 2-basis by one row in order to obtain the incidence matrix of a directed graph, one can easily prove that an oriented matroid is cographic if and only if its circuit space is spanned by a 2-basis of signed circuits, see [15].

Chapter 3

Oriented Matroids

In the present chapter we will discuss the generalization of α-orientations and α-reorientations from (directed) graphs to (oriented) matroids. After introducing the basic terminology and fundamental lemmas, we explain some problems that come up when transcribing the graphical concepts to matroids. First, we obtain that α-orientations are not really suitable to be generalized and explain the problems. We manage to avoid these problems by defining the α-reorientations of an oriented matroid in a suitable way. Second, we will justify the restriction to regular oriented matroids. The α-reorientations of more general oriented matroids are not closed with respect to flip flops of circuits. As the class of regular oriented matroids can be decomposed via certain operations into three splitters, we then investigate these splitters. One of the splitters is the matroid $\mathbf{R10}$, whose α-reorientations can be enumerated. Another splitter is the class of cographic oriented matroids, whose α-reorientations can be described by dualization from Theorem 3. The last splitter consists of the class of graphic oriented matroids, which will be investigated in the last part of this chapter.

3.1 Basics

For a real introduction to ordinary matroids [15] and [11] are standard references. Here we make some use of [14], too. For an introduction to oriented matroids we refer to [2]. In the present section we list up some basic terms very briefly. Not much more than notational explanation is provided. We will use the notation for signed sets as given in Chapter 1. Again we will neither distinguish between signed sets and sign vectors nor between sets of signed sets and sign matrices.

An **oriented matroid** is a pair $\mathcal{M} = (E, \mathcal{C})$ of a finite set E and a set of signed subsets $\mathcal{C}$ of E called **circuits**, that satisfy the following axioms:

1. $\emptyset \notin \mathcal{C}$
2. $\mathcal{C} = -\mathcal{C}$
3. $X, Y \in \mathcal{C}$ and $\underline{X} \subseteq \underline{Y} \Rightarrow X = \pm Y$
4. $X, Y \in \mathcal{C}$ and $e \in X^+ \cap Y^-$ then there is some $Z \in \mathcal{C}$ such that $Z^+ \subseteq X^+ \cup Y^+ \backslash \{e\}$ and $Z^- \subseteq X^- \cup Y^- \backslash \{e\}$.

Leaving the signing of the sets away, the resulting concept denoted $\mathbf{M} = (E, \mathbf{C})$ is called a (ordinary) matroid. So to every oriented matroid $\mathcal{M}$ we can associate an ordinary matroid $\underline{\mathcal{M}}$, the **underlying matroid of** $\mathcal{M}$.

There are several ways to obtain a new oriented matroid from $\mathcal{M}$.

A **reorientation of** $\mathcal{M}$ is an oriented matroid $\widetilde{\mathcal{M}} = \mathcal{M}^{\widetilde{E}}$ on the same ground set. It is obtained from $\mathcal{M}$ by reversing the signs on a given subset $\widetilde{E} \subseteq E$ in every circuit of $\mathcal{C}$. The new set of circuits can be thought of as $\widetilde{\mathcal{C}} = \mathcal{C}(I - 2diag(\widetilde{E}))$. Thus, as in the digraph case we can identify a reorientation of $\mathcal{M}$ with a subset of E.

The **deletion of** $A \subseteq E$ is an oriented matroid $\mathcal{M} \backslash A$ that can be obtained from an oriented matroid $\mathcal{M} = (E, \mathcal{C})$. Its ground set is $E \backslash A$ and its circuits are those circuits of $\mathcal{M}$ that are disjoint from A.

The **contraction of** $A \subseteq E$ is an oriented matroid $\mathcal{M}/A$ that can be obtained from an oriented matroid $\mathcal{M} = (E, \mathcal{C})$. Its ground set is $E \backslash A$ and its circuits are the support minimal signed sets $Min(\{X \backslash A \mid X \in \mathcal{C}\} \backslash \{\emptyset\})$.

Another oriented matroid that is induced by $\mathcal{M} = (E, \mathcal{C})$ is $\mathcal{M}^* = (E, \mathcal{C}^*)$. Its circuits $Y \in \mathcal{C}^*$ are given by the inclusion minima of non-empty signed sets satisfying

$$\underline{X} \cap \underline{Y} \neq \emptyset \Rightarrow ((X^- \cap Y^-) \cup (X^+ \cap Y^+) \neq \emptyset \textit{ and } (X^- \cap Y^+) \cup (X^+ \cap Y^-) \neq \emptyset),$$

for every $X \in \mathcal{C}$. The oriented matroid $\mathcal{M}^*$ is called the **dual of** $\mathcal{M}$. The circuits of $\mathcal{M}^*$ are called the **cocircuits of** $\mathcal{M}$. The defining property of the cocircuits (besides the minimality) is called **(combinatorial) othogonality**.

Obviously usual vectorial orthogonality of sign vectors implies their combinatorial orthogonality but not vice versa. We call a sign vector **vectorial** if it is vectorially orthogonal to $\mathcal{C}$ or $\mathcal{C}^*$.

A basic fact about oriented matroid duality is that $(\mathcal{M} \backslash A)^* = \mathcal{M}^*/A$ and conversely $(\mathcal{M}/A)^* = \mathcal{M}^* \backslash A$, (see [2],p123).

The Farkas Lemma (see, [2],p122) says that every element $e \in E$ is either contained in a positive circuit or in a positive cocircuit of $\mathcal{M}$.

An oriented matroid $\mathcal{M}$ is called **totally cyclic** if every element of E is contained in a positive circuit. We call $\mathcal{M}$ **acyclic** if $\mathcal{M}^*$ is totally cyclic.

In [1] it is shown that every $\mathcal{M}$ without one-element-circuits has an acyclic reorientation. Moreover the minimal subsets of E that can be reoriented to give an acyclic reorientation are exactly those sets, which are minimal with respect to intersecting every positive circuit.

The **composition** $Y := X_1 \circ \ldots \circ X_k$ of signed sets $X_1, \ldots, X_k$ is defined as $Y(e) := X_{min\{i \in [k] | X_i(e) \neq 0\}}(e)$ if possible and 0 else, for every $e \in E$. This binary operation is associative non-commutative and has the empty set as neutral element. The set $\mathcal{V}$ of **vectors** of $\mathcal{M}$ is the set of signed sets that result from compositions of circuits endowed by the empty set. The set $\mathcal{V}^*$ of **covectors** is defined analogously in terms of cocircuits. A composition $X_1, \ldots, X_k$ is called **conformal** if $X_i(e)X_j(e) \geq 0$ for all $i, j \in [k]$ and entries $e \in E$. Every vector $X \in \mathcal{V}$ is even a conformal composition of circuits, see [2], p141. If we have $X_1(e)X_2(e) \geq 0$ and $X_1 \leq X_2$ for two sign vectors X_1, X_2 and every $e \in E$, we say that X_1 is conformingly contained in X_2. This is the **conformal inclusion** of signed sets.

Every oriented matroid is uniquely determined by any of the set systems $\mathcal{C}$, $\mathcal{C}^*$, $\mathcal{V}$, or $\mathcal{V}^*$. Each of them can be described by an axiomatization similar to the one we gave for $\mathcal{C}$.

An important class of oriented matroids are **vectorial matroids**. Vectorial matroids arise the following way. Given a vector subspace V of $\mathbb{R}^n$, the support minimal vectors in $sgn(V \backslash \{0\})$ form the set of circuits of an oriented matroid on the ground set $[n]$. A standard way to represent these matroids is to represent V as the real kernel of a $m \times n$ matrix B. Clearly Gauss row operations on B do not change $\mathcal{M}$, so B can assumed to be of the form $[I \mid A]$. The matrix A is then called a **representation matrix** for $\mathcal{M}$. The rank of B defines $rank(\mathcal{M})$, **the rank of** $\mathcal{M}$. Moreover the dual matroid of the matroid induced by V comes from the orthogonal complement of V.

Special cases of vectorial matroids are **graphic** and **cographic** matroids given by the linear dependencies of the incidence matrix of a directed graph (the cycle space) and the orthogonal vectorspace (the cut or cocycle space), respectively. We denote the graphic oriented matroid induced by the directed graph D as $\mathcal{M}(D)$. The circuits of $\mathcal{M}(D)$ are exactly the signed incidence vectors of the cycles of D. The cocircuits are the inclusion minimal signed incidence vectors of the cuts of D. A cographic oriented matroid is the dual matroid of a graphic matroid, i.e. $\mathcal{M}^*(D)$. Total cyclicity and acyclicity generalize strong connectivity and acyclicity of digraphs. Deletion and contraction of arcs correspond to the analogue operations on the induced oriented matroid.

Ordinary vectorial matroids can come from (representation) matrices over any field $\mathbb{F}$. If a matroid can be represented over every field it is called **regular**.

Another important class of ordinary matroids are the **uniform matroids** denoted $\mathbf{U}_{m,n}$. They consist of the ground set $[n]$ and the set of circuits $\binom{[n]}{m+1}$.

3.2 Formal Transcription and First Observations

In the present section we will discuss the suitability of the concepts of α-orientations and α-reorientations for generalization from (directed) graphs to (oriented) matroids. We display a number of problems that come up when attempting a natural transcription. Later we present a possible way to avoid these problems.

How can we generalize α-orientations? Our guiding idea is that vertex cuts generalize to cocircuits. As the vertex cuts of a graph span all the cut space, prescribing the outdegree on the vertex cuts fixes the number of positive elements for all the cuts of an orientation of the graph. So the outdegree could be generalized to the number of positive entries of cocircuits.

We look at the case of α-orientations. So we have an ordinary matroid $\mathbf{M}$, which we want to orient. This means that we search an orientend matroid $\mathcal{M}$ with $\underline{\mathcal{M}} = \mathbf{M}$ and more properties with respect to the outdegree. At this step we already encounter the first big difference between graphs and oriented matroids. Not every ordinary matroid is **orientable**, i.e. is underlying matroid of an oriented matroid, e.g. the Fano matroid $\mathbf{F}(7)$ (see [3]). So there are ordinary matroids which have no α-orientations at all (independent of the α). We will not really consider this problem and always think of orientable matroids.

So let $\mathbf{M}$ be an orientable ordinary matroid on a ground set E with a set of cocircuits $\mathbf{C}^*$. In the graph case the α vector was counting positive elements of the vertex cuts, which form a basis of the cut space of the directed graph. The first problem when attempting to generalize this notion to oriented matroids is the following. When orienting an orientable matroid every cocircuit in $\mathbf{C}^*$ will be represented by two cocircuits $X, -X \in \mathcal{C}^*$. In contrast to the graph case, there is no canonical choice, no way to distinguish X and $-X$. We have no analogue to indegree and outdegree. So something like α-orientations cannot really be defined.

A bigger set of orientations that is suitable to generalize, is the set of those orientations that fix the absolute value of the difference of in- and outdegree. Here in- and outdegree are treated symmetrically. In digraphs this concept coincides with α-orientations exactly in the case of Eulerian orientations, i.e $\alpha \equiv \frac{deg}{2}$. So we could restrict the generalization to **Eulerian orientations of an oriented matroid**, which are defined by $\langle X, \mathbf{1} \rangle = 0$ for every $X \in \mathcal{C}^*$.

But there is another problem, which exactly in the case of Eulerian orientations cannot be solved. Given a directed graph D and the graph $-D$, obtained from D by reversing the orientation of all the arcs, then the graphic oriented matroids $\mathcal{M}(D)$ and $\mathcal{M}(-D)$ induced by D and $-D$ respectively are considered the same in oriented matroid theory. They have the same set of circuits. So the oriented matroid cannot distinguish between the two orientations D and $-D$. But in the analysis of α-orientations of graphs, both orientations are considered to be different.

Identifying the orientations D and $-D$ would be one idea. But regarding posets on the α-orientations, this identification would only work as long as $\textit{reor}_\alpha(D) \cap \textit{reor}_\alpha(-D) = \emptyset$. This condition is violated if and only if we have a poset on the Eulerian orientations of a graph. In this case D and $-D$ occur as comparable elements in the poset and after identifying them, one obtains cycles.

The next difficulty is that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are oriented matroids with the same underlying matroid then $\mathcal{M}_1$ is not necessarily a reorientation of $\mathcal{M}_2$, e.g. the matroid $\mathbf{U}_{3,6}$ has orientations which cannot be obtained from each other by reorientation, see [10]. So the set of orientations of a matroid needs not to be connected by reorientations.

In [4] there are even examples of different Eulerian orientations of a rank 3 matroid that cannot be obtained one from another by reorientation. This is quite a big difference to the graph case because there, our results were strongly connected to the fact that we could investigate reorientations of a directed graph instead of orientations of an undirected graph. Also the flip flop structure developed in Chapter 2 clearly reflects reorientation classes of sign matrices.

So in order to escape from all these problems we restrict further investigation to the set of α**-reorientations of an oriented matroid** $\mathcal{M}$ defined as:

$$\textit{reor}_\alpha(\mathcal{M}) := \{\widetilde{\mathcal{M}} \mid \widetilde{\mathcal{M}} \textit{ is a reorientation of } \mathcal{M} \textit{ and we have } \mathcal{C}^*\mathbf{1} = \widetilde{\mathcal{C}}^*\mathbf{1}\}.$$

What are the advantages of this definition? Here from the beginning on we consider an oriented matroid and do not have the problem of orientability anymore. Moreover we only consider reorientations and do not include orientations of the underlying matroid, that are no reorientations. We identify the reorientation $\widetilde{\mathcal{M}}$ with a subset $\widetilde{E} \subseteq E$. As we want to have a correspondence of reorientations and subsets of E we break with the convention $\mathcal{M} = -\mathcal{M}$, i.e. we consider $\mathcal{M}$ and the reorientation of all its elements as different reorientations. The equation of sign marices $\mathcal{C}^*\mathbf{1} = \widetilde{\mathcal{C}}^*\mathbf{1}$ says that corresponding cocircuits of the reorientation have the same number of positive entries. This generalizes the idea of the invariance of the outdegree. By considering sign matrices instead of sets of signed cocircuits we can take $\widetilde{\mathcal{C}}^*$ as $\mathcal{C}^*(I - 2diag(\widetilde{E}))$. So the sets $\mathcal{C}^*$ and $\widetilde{\mathcal{C}}^*$ are given with a fixed order, which guarantees that indeed the sign numbers of corresponding cocircuits are compared.

Having in mind that for digraphs $\textit{reor}_\alpha(D) \cong \mathcal{E}^+(D) = \ker_{\{0,1\}} \textit{Inc}(D) = \ker_{\{0,1\}} \mathcal{C}^*$, the definition of $\textit{reor}_\alpha(M)$ allows us to prove an analogue to the essential Proposition 1:

Proposition 3.2.1. *For an oriented matroid $\mathcal{M}$ we have $\textit{reor}_\alpha(\mathcal{M}) \cong \ker_{\{0,1\}} \mathcal{C}^*$.*

Proof. Let $\widetilde{E} \subseteq E$ and denote by $\mathcal{M}^{\widetilde{E}}$ the matroid obtained by reorienting the elements of $\widetilde{E}$, then
$\mathcal{M}^{\widetilde{E}} \in \textit{reor}_\alpha(\mathcal{M})$
$\Leftrightarrow \mathcal{C}^*\mathbf{1} = (\mathcal{C}^*(I - 2diag(\widetilde{E}))\mathbf{1})$
$\Leftrightarrow \mathcal{C}^*\widetilde{E} = 0$
$\widetilde{E} \in \ker_{\{0,1\}} \mathcal{C}^*$

□

So in analogy to the bounded facial cycles $\mathcal{F}\backslash\{\mathbf{X}\} \subset \ker_{\{1,-1,0\}} \textit{Inc}(D) = \mathcal{E}(D)$ of a planar graph D, which by Theorem 1 flip flop generate all $\ker_{\{1,0\}} \textit{Inc}(D) = \textit{reor}_\alpha(D)$, we would like to find a small subset $B \subset \ker_{\{-1,0,1\}} \mathcal{C}^* =: \mathcal{E}(\mathcal{M})$ such that seeing B as a sign matrix $\mathrm{ff(B)} = \ker_{\{0,1\}} \mathcal{C}^* = \textit{reor}_\alpha(\mathcal{M})$. The set $\mathcal{E}(\mathcal{M})$ of signed sets generalizes the set of Eulerians of a digraph.

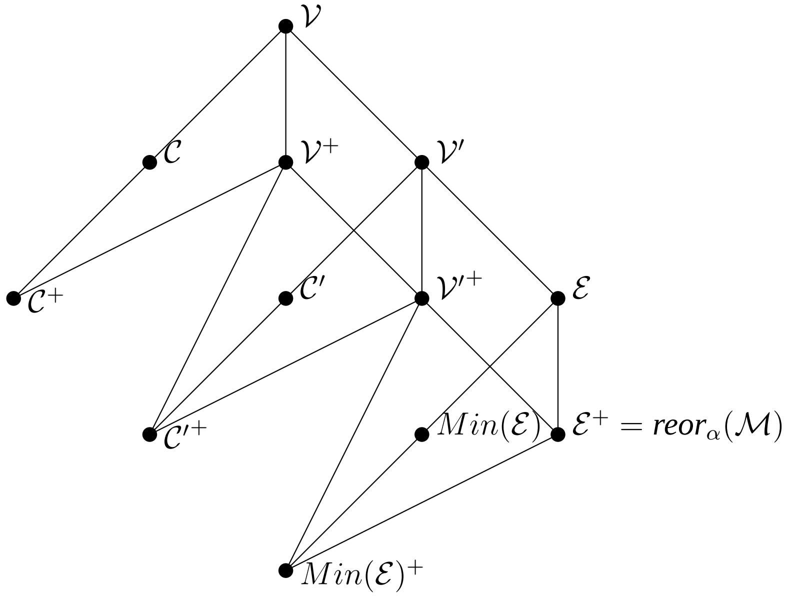

Figure 3.1: Inclusions of sets of signed sets occurring in the analysis of α-reorientations.

The sets $\textit{reor}_\alpha(M)$ and $\mathcal{E}(\mathcal{M})$ as well as the flip flop operations reflect linear algebra structures that are related to the oriented matroid $\mathcal{M}$. General oriented matroids do not necessarily have such a direct link to linear algebra. In order to interpolate between the oriented matroid $\mathcal{M}$ and the linear properties at our concern we can associate to $\mathcal{M}$ the vectorial oriented matroid $\mathcal{M}'$. It is induced by the linear dependences of the columns of $\mathcal{C}^*$. So all the sign vectors in $\mathcal{C}'$ are combinatorially orthogonal to $\mathcal{C}^*$. This means that every circuit of $\mathcal{M}'$ is in the set of vectors $\mathcal{V}$ of $\mathcal{M}$. We visualize the inclusion relations of the set systems given by $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{E}(\mathcal{M})$ in Figure 3.1.

The next proposition gives us a hint where to look for sets $B \subset \mathcal{E}(\mathcal{M})$ that satisfy $\mathrm{ff}(\mathrm{B}) = \textit{reor}_\alpha(\mathcal{M})$.

Proposition 3.2.2. *Denote by $Min(\mathcal{E}(\mathcal{M}))$ the minima of $\mathcal{E}(\mathcal{M})$ with respect to conformal inclusion. Then $\mathrm{ff}(\mathrm{Min}(\mathcal{E}(\mathcal{M}))) = \textit{reor}_\alpha(\mathcal{M})$.*

Proof. As $\mathcal{E}(\mathcal{M})$ is closed with respect to taking sign sums we have

$$\mathrm{ff}(\mathrm{Min}(\mathcal{E}(\mathcal{M}))) \subseteq \mathcal{E}(\mathcal{M})^+ = \textit{reor}_\alpha(\mathcal{M}).$$

Suppose on the other hand there is a non-minmal element X in some $B \subseteq \mathcal{E}(\mathcal{M})$. This means there is a $Y \in \mathcal{E}(\mathcal{M})$ that is conformingly contained in X. We define

$$B' := B \backslash \{X\} \cup \{Y, X - Y\}.$$

Obviously $X - Y \in \mathcal{E}(\mathcal{M})$ and $X - Y$ is conformingly contained in X. Moreover whenever X can be flipped, first Y and then $X - Y$ can be flipped. The flopping situation works analogously. So $\mathrm{ff}(\mathrm{B}) \subseteq \mathrm{ff}(\mathrm{B}')$ and starting with $B = \mathcal{E}(\mathcal{M})$ we obtain the result. $\square$

Recall that, having a B with $\mathrm{ff}(\mathrm{B}) = \textit{reor}_\alpha(\mathcal{M})$ by Proposition 2.2.1 we can row reorient B, such that $D_{\mathrm{ff}}(B_c)$ is acyclic and the flip flop span is not changed. So we have flip flop posets $P_{\mathrm{ff}}(B_c)$ on $\textit{reor}_\alpha(\mathcal{M})$ for every oriented matroid $\mathcal{M}$. This result should not be overestimated as B_c can still be fairly big, and nothing more specific can be said about properties of $P_{\mathrm{ff}}(B_c)$. For instance $B = Min(\mathcal{E})^+$ would be a possibility, which specializing back to digraphs is not nearly as nice as $B = \mathcal{F} \backslash \{\mathbf{X}\}$.

Proceeding along the lines of the introduction, we call an element $e \in E$ **rigid with respect to** $\mathcal{M}$ if its orientation will not be changed among all reorientations that appear in $\textit{reor}_\alpha(\mathcal{M})$. Again as in the graph case, we are not interested in rigid elements of an oriented matroid and can throw them out. After that we can say the following about $\mathcal{M}$.

Proposition 3.2.3. *If $\mathcal{M}$ has no rigid elements it is totally cyclic.*

Proof. If $\mathcal{M}$ has no rigid elements, by Proposition 3.2.1 for every element $e \in E$ there is a vector $v \in \ker_{\{0,1\}} \mathcal{C}^*$ such that $v_e = 1$. But $\mathcal{C}^* v = 0$ and $v = sgn(v)$ imply $v \in \mathcal{V}^+$, see Figure 3.1. So every element $e \in E$ is contained in a positive vector of $\mathcal{M}$. But since every vector is a conformal composition of circuits, positive vectors are compositions of positive circuits. Thus every element $e \in E$ is contained in a positive circuit, which is the definition of totally cyclic. $\square$

The other direction of the proposition does not hold in general for oriented matroids, e.g. any totally cyclic orientation of $\mathbf{U}_{2,4}$ consists only of rigid elements. But as in the graph case total cyclicity is maintained by α-reorientations. If after reorientation there were a new directed cocircuit, the number of positive entries would have been changed.

Analogously to the digraph case, we can restrict our attention to the totally cyclic components of $\mathcal{M}$ in the study of *reor*$_\alpha(\mathcal{M})$. As matroid connectivity generalizes graph 2-connectivity the restriction to 2-connected graphs corresponds to the natural restriction to connected matroids.

3.3 Regular Oriented Matroids

In the present chapter we want to investigate a stronger analogy from digraphs to oriented matroids. It turns out, that it holds for regular oriented matroids only.

Theorem 1 states that every planar digraph D has a set of cycles, such that its flip flop span covers exactly the α-reorientations of D. In particular every set of cycles of D flip flop spans a subset of *reor*$_\alpha(D)$. So for the set of all cycles $\mathcal{C}$ we have $\mathrm{ff}(\mathcal{C}) =$ *reor*$_\alpha(\mathrm{D})$. In the present section we will investigate under which conditions the equation $\mathrm{ff}(\mathcal{C}) =$ *reor*$_\alpha(\mathcal{M})$ holds for general oriented matroids. This analogy is desirable, because it would enable us to continue investigating cycle/circuit spaces. That this requirement is plausibly quite restrictive can already be read off Figure 3.1, where $\mathcal{C}$ and *reor*$_\alpha(\mathcal{M})$ are far apart.

We start with a case where $\mathrm{ff}(\mathcal{C}) =$ *reor*$_\alpha(\mathcal{M})$ does not hold at all. Take a Eulerian orientation $\mathcal{M}$ of $\mathbf{U}_{3,6}$. The matrix B displays a "representative half" of $\mathcal{C}^*$. i.e. $B \cup -B = \mathcal{C}^*$.

$$B = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix},$$

It is easy to see that

$$\textit{reor}_\alpha(M) = \{(0,0,0,0,0,0), (1,1,1,1,1,1)\}$$

and

$$\{(0,0,0,0,0,0)\} \subsetneq \mathrm{ff}(\mathcal{C}) \subseteq \{\mathrm{x} \in \{0,1\}^6 \mid \langle \mathrm{x}, \mathbf{1} \rangle \in \{0,4\}\}.$$

So for this particular $\mathcal{M}$ we have neither $\text{ff}(\mathcal{C}) \subseteq$ *reor*$_\alpha(\mathcal{M})$ nor $\text{ff}(\mathcal{C}) \supseteq$ *reor*$_\alpha(\mathcal{M})$.

How do positive examples look like? In the sequel we will prove that regular oriented matroids are the only oriented matroids which satisfy $\text{ff}(\mathcal{C}) =$ *reor*$_\alpha(\mathcal{M})$ for all their totally cyclic reorientations.

An oriented matroid $\mathcal{M}$ is said to be **regular** if its underlying ordinary matroid $\underline{\mathcal{M}}$ is regular. We will need different characterizations of regular oriented matroids.

Theorem 3.3.1. *Let $\mathcal{M}$ be an oriented matroid, then the following statements are equivalent:*

(i) $\mathcal{M}$ is regular.

(ii) $\mathcal{M}$ has a totally unimodular representation matrix.

(iii) $\mathcal{M}$ is vectorial, such that every signed circuit of $\mathcal{M}$ is - seen as a sign vector - an element of the corresponding vector space. Moreover these vectors are spanning.

(iv) $\mathcal{M}$ is a 1-, 2- *or* 3-*sum of graphic and cographic oriented matroids and* $R10$.

Proof. For $(ii) \Leftrightarrow (iii)$, see [14], chap5.

For $(i) \Leftrightarrow (iii)$, see [15], p175.

For $(i) \Leftrightarrow (iv)$, see [13].

□

The first result of the present section will be an extension of this characterization, with respect to totally cyclic regular oriented matroids. We will prove that for an oriented matroid $\mathcal{M}$ regularity is equivalent to $\text{ff}(\mathcal{C}) =$ *reor*$_\alpha(\mathcal{M})$ for all its totally cyclic reorientations. So we almost characterize the oriented matroids, that generalize the $\text{ff}(\mathcal{C}) =$ *reor*$_\alpha(\mathrm{D})$ of graphs.

For this we need the following Lemma:

Lemma 3.3.2. *An oriented matroid $\mathcal{M}$ is regular if and only if $\mathcal{C} \perp \mathcal{C}^*$ as sets of integral vectors.*

Proof. As regularity of $\mathcal{M}$ is equivalent to regularity of $\mathcal{M}^*$ (see [14], chap5), both come from orthogonal vector spaces that, by Theorem 3.3.1(iii), contain the signed incidence vectors of their circuits. This is equivalent to $\mathcal{C} \perp \mathcal{C}^*$ as sets of vectors. □

Now we are ready to prove

Theorem 3.3.3. *Let $\mathcal{M} = (E, \mathcal{C})$ be an oriented matroid without one-element-circuits. We have $\text{ff}(\mathcal{C}) =$ reor$_\alpha(\mathcal{M})$ for all its totally cyclic reorientations $\Leftrightarrow$ $\mathcal{M}$ is regular.*

Proof. ”$\Leftarrow$”
Let $\mathcal{M}$ be regular. This property is invariant under reorientation. So by Lemma 3.3.2 we have for every reorientation of $\mathcal{M}$, that $\mathcal{C} \subseteq \ker \mathcal{C}^*$. In particular $\mathcal{C}^+ \subseteq \ker \mathcal{C}^*$, which means that we cannot leave the set of α-reorientations by flips of circuits.
Equality is obtained by the following. Every element $v \in \ker_{\{1,0\}} \mathcal{C}^* \subseteq \mathcal{V}^+$ is a vectorial vector of $\mathcal{M}$. But as in a regular matroid every vectorial vector can be written as a sum of circuits conforming to it (see [14], chap1.2), v must be the disjoint union of directed circuits of $\mathcal{M}$. Thus, flipping these directed circuits successively, one obtains the reorientation given by v.
”$\Rightarrow$”
Suppose $\mathrm{ff}(\mathcal{C}) =$ *reor*$_\alpha(\mathcal{M})$ for all totally cyclic reorientations of $\mathcal{M}$. As $\mathcal{M}$ has no one-element-cocircuits, by [1] there $\mathcal{M}$ has totally cyclic reorientations. So let $\mathcal{M}$ be a totally cyclic oriented matroid with the above property. We have that $\mathcal{C}^+(\mathcal{M}^v) \subseteq \ker_{\{0,1\}}(\mathcal{C}^*)^v$ for every $(0,1)$-vector v that stands for a totally cyclic reorientation. For a $(0,1)$-vector x we can transform

$$x \in \ker_{\{0,1\}}(\mathcal{C}^*)^v \Leftrightarrow \mathcal{C}^*(I - 2diag(v))x = 0 \Leftrightarrow x^v \in \ker_{\{1,-1,0\}} \mathcal{C}^*.$$

So particularly every circuit of $\mathcal{M}$, that appears positively directed in a totally cyclic reorientation is in $\ker_{\{1,-1,0\}} \mathcal{C}^*$.
We show, that every circuit appears positively directed in some totally cyclic reorientation of $\mathcal{M}$. So let $X \in \mathcal{C}(\mathcal{M})$ with $1 \leq |X^-| \leq |X^+|$. First we reorient $\mathcal{M}$ on X^-, so X is positively directed in the actual orientation, say $\widetilde{\mathcal{M}}$. But $\widetilde{\mathcal{M}}$ does not need to be totally cyclic. If it is not, there are positively directed cocircuits in $\widetilde{\mathcal{M}}$. We have that $\widetilde{\mathcal{M}}$ has no one-element-cocircuits. As shown in [1], in order to obtain a totally cyclic reorientation of $\widetilde{\mathcal{M}}$ it is enough to reorient a set that intersects each positively directed cocircuit of $\widetilde{\mathcal{M}}$. As X is positively directed by Farkas Lemma the positively directed cocircuits are disjoint to X. So we obtain a totally cyclic reorientation of $\mathcal{M}$ that has X as a positively directed circuit.
Together we have shown, that $\mathcal{C} \subseteq \ker \mathcal{C}^*$ so by Lemma 3.3.2 we have that $\mathcal{M}$ is regular.
□

We conjecture the stronger statement

”for totally cyclic $\mathcal{M}$ regularity is equivalent to $\mathrm{ff}(\mathcal{C}) =$ reor$_\alpha(\mathcal{M})$”,

which would really be the characterization of those oriented matroids that generalize the digraph property $\mathrm{ff}(\mathcal{C}) =$ *reor*$_\alpha(\mathrm{D})$.

An analogue proof to the one of Theorem 3.3.3 is still failing. The ”$\Leftarrow$”-direction is no problem. The first part of the ”$\Rightarrow$”-direction would give that every circuit that is ever flipped among $\mathrm{ff}(\mathcal{C})$ is also in $\ker \mathcal{C}^*$. But it still cannot be shown that every circuit of such oriented matroid is flippable, or a linear combination of flippable circuits.

Now that we have a reason to restrict further investigation to the class of regular oriented matroids we can think of, what else are nice advantages of dealing with regular matroids only.

Theorem 3.3.1(ii) gives a connection to totally unimodular matrices. Every linear program induced by such a matrix and integral cost vectors is integral, i.e has an optimal solution that is integral, one obtains a polynomial method to check whether a fixed regular matroid can be reoriented in order to have some given α-value on its circuits. This is a weak analogue to a planar graph result of [7]. Felsner proves that given a planar undirected graph G and an integral vector α, one can obtain an α-orientation of G or a certificate for the non-existence in polynomial time.

In order to find suitable of circuits whose flip flop span consists of all the α-reorientations of a regular matroid, the following gives a necessary condition for such sets.

Proposition 3.3.4. *Let $\mathcal{M}$ be a regular, totally cyclic, oriented matroid and $B \subseteq \mathcal{C}$, then* $\mathrm{ff(B)} = \textit{reor}_\alpha(\mathcal{M}) \Rightarrow \mathrm{span}_\mathbb{Z}(\mathrm{B}) = \mathrm{span}_\mathbb{Z}(\mathcal{C})$.

Proof. As $B \subseteq \mathcal{C}$ the inclusion $span_\mathbb{Z}(B) \subseteq span_\mathbb{Z}(\mathcal{C})$ trivially holds.

To see "$\supseteq$" observe the following:
As $\ker_{\{0,1\}} \mathcal{C}^* = \textit{reor}_\alpha(\mathcal{M}) = \mathrm{ff(B)}$ we have $span_\mathbb{Z}(B) = span_\mathbb{Z}(\ker_{\{0,1\}} \mathcal{C}^*)$. On the other hand Lemma 3.3.2 implies $span_\mathbb{Z}(\mathcal{C}) \subseteq span_\mathbb{Z}(\ker_{\{1,-1,0\}} \mathcal{C}^*)$. Therefore it is enough to show that $span_\mathbb{Z}(\ker_{\{0,1\}} \mathcal{C}^*) \supseteq \ker_\mathbb{Z} \mathcal{C}^*$.
Take $v \in \ker_\mathbb{Z} \mathcal{C}^*$ and let m_v denote the number of negative entries of v. We proceed by induction on m_v.
Let $m_v = 0$
By regularity we have that v is a sum of circuits conforming to v (see [14], chap1.2). These circuits are positively directed, i.e. they are in $\ker_{\{0,1\}} \mathcal{C}^*$ and v is an integral combination of them.
Take now $m_v > 0$. So there is some $e \in E$ such that $v_e < 0$. By total cyclicity of $\mathcal{M}$, there is some $c \in \mathcal{C}^+$ with $c_e = 1$. So we can find a $\lambda \in \mathbb{Z}_{>0}$ such that $m_{v+\lambda c} < m_v$. By induction hypothesis we have $(v + \lambda c) \in span_\mathbb{Z}(\ker_{\{0,1\}} \mathcal{C}^*)$ which implies $v \in span_\mathbb{Z}(\ker_{\{0,1\}} \mathcal{C}^*)$. $\square$

In the proof of Proposition 3.3.4 we have particularly shown that in a totally cyclic regular oriented matroid $\mathcal{M}$ there are bases $B \subseteq \mathcal{C}^+$ for the circuit space $span_\mathbb{Z}(\mathcal{C})$. On the other hand, if is not totally cyclic, there are obviously no circuit bases consisting of positively directed circuits. So a regular oriented matroid is totally cyclic if and only if its circuit space is spanned by a set of positively directed circuits. This generalizes the corresponding result for directed graphs [9].

As for a regular oriented matroid $\mathcal{M}$ the integral dimension of $span_\mathbb{Z}(\mathcal{C})$ is known to be $|E| - \textit{rank}(\mathcal{M})$, Proposition 3.3.4 gives an "easy to check" lower bound for the cardinality of a flip flop generating set.

Recalling the discussion in Section 3.2, the following is an advantage of regular oriented matroids. The set of orientations of a regular matroid is connected by reorientations (see, [10]), so considering α-reorientations instead of α-orientations is no restricition.

Another feature of regular matroids is Seymour's famous decomposition theorem (Theorem 3.3.1(iv)). It tells us that every regular matroid $\mathcal{M}$ is graphic, cographic, R10 or a 1-,2- or 3-sum of such matroids.

Now on the one hand we can try to analyze the flip flop structure of any of the three splitters. On the other hand we can look, whether a "nice" flip flop structure of $\mathcal{M}_1$ and $\mathcal{M}_2$, somehow will be preserved by taking i-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$, for $i \in \{1, 2, 3\}$.

First we take a look at 1-sums. The 1**-sum of two sign matrices** B_1 and B_2 is defined as

$$B_1 \oplus_1 B_2 := \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

For regular oriented matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ with representation matrices B_1 and B_2 the 1**-sum** $\mathcal{M}_1 \oplus \mathcal{M}_2$ is the regular oriented matroid with representation matrix $B_1 \oplus_1 B_2$.

Define the product $D_1 \times D_2$ of two directed graphs D_1 and D_2 as the directed graph with vertex set $V(D_1) \times V(D_2)$ and $((u_1, u_2), (v_1, v_2)) \in A(D_1 \times D_2) :\Leftrightarrow$

$$\textit{for one } \{i, j\} = \{1, 2\} \textit{ we have } (u_i, v_i) \in A(D_i) \textit{ and } u_j = v_j.$$

Then we have $D_{\text{ff}}(B_1 \oplus_1 B_2) = D_{\text{ff}}(B_1) \times D_{\text{ff}}(B_2)$.

One easily obtains $\text{ff}(\text{B}_1) = \textit{reor}_\alpha(\mathcal{M}_1)$ and $\text{ff}(\text{B}_2) = \textit{reor}_\alpha(\mathcal{M}_2)$ if and only if $\text{ff}(\text{B}_1 \oplus_1 \text{B}_2) = \textit{reor}_\alpha(\mathcal{M}_1 \oplus_1 \mathcal{M}_2)$. Moreover the 1-sum preserves order structures induced by the transitive closures of $D_{\text{ff}}(B_1)$ and $D_{\text{ff}}(B_2)$.

The properties of 2-*sums* and 3-*sums* seem to be more tricky, so we turn to the analysis of the splitters. This will be the subject of the following sections.

The splitter **R10** is a 10-element oriented matroid and all its reorientations. The different totally cyclic α-reorientation classes can be enumerated with a computer. Some results will be described in Section 3.4.

Dualizing Theorem 3 we get a distributive lattice on the α-orientations of any cographic oriented matroid. This will be made explicit in Section 3.5.

For the class of graphic matroids we present positive and negative results in the last two sections of this chapter.

3.4 α-Reorientations of R10

We say $\mathbf{R10}$ to any oriented matroid given by the linear dependencies of any column reorientation $E' \subseteq [10]$ of

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

Computer enumeration gives that there are 51 different α-reorientation classes of $\mathbf{R10}$ without rigid elements.

The circuit space of any such reoriented $\mathbf{R10}$ has dimension 5. Nevertheless some of the α-reorientation classes can only be flip flop generated by a 6 element set of signed circuits. This shows that the necessary condition given by Proposition 3.3.4 is not sufficient for the regular oriented matroids $\mathbf{R10}$.

3.5 α-Reorientations of Cographic Oriented Matroids and c-Reorientations of Directed Graphs

An oriented matroid $\mathcal{M}$ is called **graphic**, if there is a directed graph $D = (V, A)$ such that the circuits of $\mathcal{M}$ and the the inclusion minimal cuts of D coincide, i.e. $\mathcal{C}^*(D) = \mathcal{C}^*(\mathcal{M})$. We denote such matroids by $\mathcal{M}^*(D)$.

We have already mentioned a couple of times that the c-reorientations of an acyclic digraph D correspond dually to the α-reorientations of the totally cyclic cographic oriented matroid induced by D. This correspondence also conserves Theorem 3 and leads to a dual statement in terms of α-reorientations of totally cyclic cographic oriented matroids.

In the present section we will prove this formally.

First we recall the definition of c-reorientations. Let D be an acyclic digraph and $\mathbf{C}$ be any basis of cycles of its cycle space $span_{\mathbb{Z}}(\mathcal{C})$. Define $c_D \in \mathbb{Z}^{\mathbf{C}}$ to be the vector that counts the positive entries of the elements of $\mathbf{C}$. Now we define the c-reorientations of D as *reor*$_c(D) := \{D' = (V, A') \mid \underline{D'} = \underline{D} \text{ and } c_{D'} \equiv c_D\}$.

It is an essential observation that the set *reor*$_c(D)$ does not depend on the choice of $\mathbf{C}$. Analogously to Proposition 1, identifying the reorientations of D with the arc sets that are reoriented one obtains *reor*$_c(D) = \ker_{\{0,1\}} \mathcal{C}$.

Any set B of all but one vertex cut of D integrally spans the cut space $span_{\mathbb{Z}}(\mathcal{C}^*)$ of D. So reading Theorem 3 again with these terms filled in and replacing D by the graphic oriented matroid $\mathcal{M}$ induced by D, it sounds as follows.

Theorem 3.5.1. *Let $\mathcal{M}(D)$ be the acyclic graphic matroid induced by an acyclic directed graph D. There is a spanning set $B \subseteq \mathcal{C}^*$ such that*

- $\mathrm{ff(B)} =$ ***reor***$_{\mathrm{c}}(\mathrm{D}) = \ker_{\{0,1\}} \mathcal{C}$
- *Moreover $P_{\mathrm{ff}}(B)$ is a distributive lattice.*

This is a statement about the graphic oriented matroid $\mathcal{M}$ associated to D. Oriented matroid duality consists of interchanging the roles of $\mathcal{C}$ and $\mathcal{C}^*$ and therefore acyclicity and total cyclicity are switched as well. So we can now translate the theorem to the dual statement. We denote by $\mathcal{M}$ the cographic oriented matroid $\mathcal{M}^*(D)$ associated to D.

Theorem 3.5.2. *Let $\mathcal{M}$ be a totally cyclic cographic matroid. There is a spanning set $B \subseteq \mathcal{C}$ such that*

- $\mathrm{ff(B)} = \ker_{\{0,1\}} \mathcal{C}^*$ *which by definition is reor*$_\alpha(\mathcal{M})$
- *Moreover $P_{\mathrm{ff}}(B)$ is a distributive lattice.*

We have seen now that the set of α-reorientations of cographic oriented matroids is already understood. This is a corollary of Theorem 3, which is proven by Propp in [12]. Later on we will give a different proof for Theorem 3.5.2, which comes as Corollary 3.6.8 in the analysis of α-reorientations of graphic oriented matroids.

3.6 α-Reorientations of Graphic Oriented Matroids

An oriented matroid $\mathcal{M}$ is called **graphic**, if there is a directed graph $D = (V, A)$ such that the circuits of $\mathcal{M}$ and the the cycles of D coincide, i.e. $\mathcal{C}(D) = \mathcal{C}(\mathcal{M})$. We denote such matroids by $\mathcal{M}(D)$.

So in the present section we return our attention to graphs. The concept of α-reorientations of oriented matroids as defined in Section 3.2 specializes to the α-reorientations of digraphs as described in Chapter 1. Proposition 1 remains valid and we have *reor*$_\alpha(D) \cong \underline{\mathcal{E}^+(D)}$. As the latter equals $span_{\mathbb{Z}}(\mathcal{C}(D) \cap \{0,1\}^{|A|})$ the present section will mainly consist of an analysis of the cycle space of D. Again, we can restrict ourselves to strongly connected, 2-vertex-connected digraphs.

In contrast to Chapter 1, we now have to deal with non-planar graphs. In order to apply some of the proof techniques from [7], we consider these graphs together with a 2-cell embedding into an orientable surface S. Our goal is to understand the (directed) Eulerians of D. The topology of the embedding into S gives us a way to split this investigation into two steps. The cycle space of D - particularly $\mathcal{E}(D)$ - consists of elements that are integral combinations of facial cycles and those that are not. Eulerians of the first type have a lot in common with Eulerians of planar graphs. The corresponding α-reorientations can be ordered in a very nice way as well. Our main result is a necessary and sufficient condition for a directed Eulerian to be reversable by flip flops of facial cycles. Eulerians that are not integral combinations of facial cycles bring more difficulty into the analysis of *reor*$_\alpha(D)$. Homology theory turns out to be a good tool for understanding them.

In the first part of this chapter, our strategy will be to start with flip flops of facial cycles. We will investigate the resulting poset on a subset of *reor*$_\alpha(D)$. As we cannot generate all *reor*$_\alpha(D)$ by facial flip flops, at the end of the chapter we investigate how to extend the set of facial cycles in order to flip flop generate the entire *reor*$_\alpha(D)$.

So let $D = (V, A)$ be a directed graph and S an orientable surface. The pair (D, S) is called a **2-cell embedding of** D **into** S, if the topological graph D can be mapped continously into S, such that two arcs of D intersect exactly in their common vertices and every connected component f of the space $S\backslash D$ is homeomorphic to an open disk. It is a fact from topological graph theory, that every graph has a 2-cell embedding in some orientable surface. The set F of components of $S\backslash D$ is then called the set of **faces** of the embedding. It is important to note that the closure of a face f is not required to be homeomorphic to a closed disk.

This way (D, S) leads to a (non-regular) cell decomposition of S where $V(D)$ are the 0-dimensional cells, $A(D)$ are the 1-dimensional cells and the faces are the 2-dimensional cells. We will consider this cell decomposition together with an orientation of its cells. For the 1-dimensional cells this orientation is given by D. As in the planar case in Chapter 1, we define all the 2-cells to be oriented counterclockwisely (ccw). This can be done coherently because of the orientability of S.

The set of the ccw oriented faces F leads to the set $\mathcal{F}$ of **facial cycles of** (D, S) analogously to the planar case. Since we did orient the 2-cells of the embedding counterclockwisely, we can distinguish forward and backward arcs of the facial cycles. We take the facial cycles of (D, S) to have the forward arcs as positive arcs and the backward arcs as negative.

Because S is orientable every arc of D appears once backward and once forward among the elements of $\mathcal{F}$. If an arc appears forward and backward in the same facial cycle, we consider it not to appear in the signed incidence vector.

In analogy to the unbounded face of a planar embedding of a graph, we will fix an arbitrary facial cycle $\mathbf{X}$ and call it **the forbidden facial cycle**. Then $\mathcal{F}\backslash\{\mathbf{X}\}$ is a set of linearly independent sign vectors. We denote by $\mathcal{E}^0(D,S)$ the set of Eulerians given by $span_{\mathbb{Z}}(\mathcal{F}\backslash\{\mathbf{X}\}) \cap \{1,-1,0\}^{|A|} \subseteq \mathcal{E}(D)$. Together with the ccw orientation we have that $\mathcal{F}\backslash\{\mathbf{X}\}$ is a 2-basis, as defined in Chapter 2.

We have seen in Corollary 2.2.5, that being a 2-basis implies that $P_{\mathrm{ff}}(\mathcal{F}\backslash\{\mathbf{X}\})$ is integral and a distributive lattice. For the sake of investigating $\mathrm{ff}(\mathcal{F}\backslash\{\mathbf{X}\})$ the following lemma about 2-bases will be important:

Lemma 3.6.1. *Let B be a 2-basis and $u \in span_{\mathbb{Z}}(B) \cap \{1,-1,0\}^n$ and $v = supp(u)$. Then we have:*

1. *There is a $w \in span_{\mathbb{Z}_{\geq 0}}(B) \cap \{-1,0,+1\}^n$ with $supp(w) = v$.*
2. *Moreover if v is support minimal, then $w = \pm u$.*
3. *If $w, -w \in span_{\mathbb{Z}_{\geq 0}}(B) \cap \{1,-1,0\}^n$ then $w = -w = \mathbf{0}$.*

Proof. Let $u \in span_{\mathbb{Z}}(B) \cap \{1,-1,0\}^n$ and $v = supp(u)$. As B is a 2-basis every entry $u(i)$ of u comes from at most two rows r_1 and r_2 of B. As the entries of u are in $\{1,-1,0\}$ the coefficients of λ_1 and λ_2 of r_1 and r_2, respectively, must satisfy $|\lambda_1 - \lambda_2| \in \{0,1\}$. So rows of B with differently signed coefficients in the combination of u cannot share an entry. Regard ”sharing an entry” as symmetric relation among the rows of B. Then the set of rows in the combination of u decomposes into connected components of the same sign. Equivalently the coefficient vector of u, say λ, decomposes into a disjoint sum of non-negative and non-positive vectors $(\lambda_i)_{1\leq i\leq k}$. The vectors $(u_i)_{1\leq i\leq k}$ induced by $(\lambda_i)_{1\leq i\leq k}$ are mutually disjoint and conformingly contained in u. Denote by $|\lambda|$ the componentwise absolute value vector of λ. So w defined as $|\lambda|B$ has the same support as u itself. This proves 1..

In particular for each of the subvectors of u_i and w_i induced by some λ_i we have $u_i = \pm w_i$. This proves 2..

Statement 3. follows from the linear independency of the rows of B and the non-negative coefficient vectors, that w and $-w$ are required to have.

□

Taking $B := \mathcal{F}\backslash\{\mathbf{X}\}$ we can look what Lemma 3.6.1 tells us about $\mathcal{E}^0(D,S)$.

- For every $\underline{E} \in \underline{\mathcal{E}^0(D,S)}$ there is a E^0, which is a positive integral combination of facial cycles and $\underline{E} = \underline{E^0}$. We call such E^0 a **0-Eulerian**.
- If E is support minimal among $\mathcal{E}^0(D,S)$ than $E = \pm E^0$ for every 0-Eulerian with the same support as E. Because of 3. we have that E^0 is unique in this case. If E is not support minimal, every decomposition into disjoint support minimal Eulerians leads to a unique representation in terms of 0-Eulerians.

These results are particularly at interest for directed Eulerians as we want to investigate $\underline{(\mathcal{E}^0(D,S))^+}$, because this set coincides with the α-reorientations at our concern.

Later, we will give a necessary and sufficient condition for arc sets in $\underline{(\mathcal{E}^0(D,S))^+}$ to be reversable by facial flip flops. To prove this it will be useful to have a set of signed Eulerians at hand whose supports cover all the sets in $\underline{(\mathcal{E}^0(D,S))^+}$. By Lemma 3.6.1 we have that every element $\underline{(\mathcal{E}^0(D,S))^+}$ has the support of a 0-Eulerian, we call these 0-Eulerians **directed**. Directed 0-Eulerians turn out to be more useful than the vectors in $\mathcal{E}^0(D,S))^+$ to cover $\underline{(\mathcal{E}^0(D,S))^+}$, because they have an easier representation with respect to $\mathcal{F}\backslash\{\mathbf{X}\}$.

The orientation of the facial cycles, automatically leads to the notion of positively and negatively directed 0-Eulerians. We call elements of $span_{\mathbb{Z}\geq 0}(\mathcal{F}\backslash\{\mathbf{X}\}) \cap \{1,0\}^{|A|}$ **positively directed** 0-Eulerians and those of $span_{\mathbb{Z}\geq 0}(\mathcal{F}\backslash\{\mathbf{X}\}) \cap \{-1,0\}^{|A|}$ **negatively directed** 0-Eulerians.

By Lemma 3.6.1 no element of $\underline{(\mathcal{E}^0(D,S))^+}$ is covered by a negatively and a positively directed 0-Eulerian at the same time. Moreover every directed 0-Eulerian is a disjoint union of minimal negatively and positively directed 0-Eulerians. If a positively directed 0-Eulerian can be reoriented by facial flip flops, it can indeed be reversed by flips only. The analogue holds for negatively directed 0-Eulerians.

Different 0-Eulerians can be signings of the same $(0,1)$-vector as exemplified in Figure 3.2.

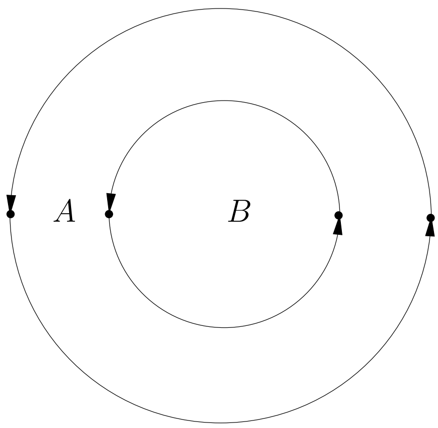

Figure 3.2: The 0-Eulerians $A = (1,1,-1,-1)$ and $A + 2B = (1,1,1,1)$ have the same arc sets but different signings and different interiors.

We have already seen that, when investigating whether $\mathrm{ff}(\mathcal{F}\backslash\{\mathbf{X}\}) = \underline{(\mathcal{E}^0(\mathrm{D},\mathrm{S}))^+}$, we can restrict our investigation to directed 0-Eulerians. By Lemma 3.6.1 every element of $\underline{(\mathcal{E}^0(D,S))^+}$ is a disjoint union of supports of positively and negatively directed 0-Eulerians. It is enough to characterize, when these can be reoriented. On the next pages we develop the necessary theory to pursue this investigation.

First of all the choice of the forbidden face $\mathbf{X}$, leads to a notion of inside and outside for all the 0-Eulerians of our embedding.

Let E be a 0-Eulerian. Define **the interior of** E as the set of facial cycles in $\mathcal{F}\backslash\{\mathbf{X}\}$ that are necessary to combine E. Denote the interior of E as *Int*(E).

The interior can be understood as the support vector of the integral support $z(E)$ of E with respect to $\mathcal{F}\backslash\{\mathbf{X}\}$ as defined in Chapter 2. The inclusion order on the interiors of the 0-Eulerians brings a new aspect of comparability to the set of 0-Eulerians besides the inclusion as sigend sets of arcs. This order will give rise to an induction in the "$\Leftarrow$"-proof of Theorem 3.6.7. As different 0-Eulerians can have the same interior, the next lemma gives an important representative for the 0-Eulerians with fixed interior.

Lemma 3.6.2. *For every 0-Eulerian E, there is a unique 0-Eulerian denoted ∂E, which is arc minimal with respect to conformal inclusion satisfying Int$(\partial E) =$ Int(E). It will be called **the boundary of** E.*

Proof. Let $z(E)$ be the integral support of E with respect to $\mathcal{F}\backslash\{\mathbf{X}\}$ and $supp(z(E))$ its support. Define $\partial E := supp(z(E))(\mathcal{F}\backslash\{\mathbf{X}\})$ as the result of the corresponding combination of facial cycles.
First, it is clear that E and ∂E have the same interior. In particular ∂E is a 0-Eulerian.
To see uniqueness, first observe that $\partial E = \partial E'$ if *Int*$(E) =$ *Int*(E'). So it is enough to prove that ∂E is conformingly contained in E. Now, if a is an arc of ∂E it lies between two facial cycles F_1 and F_2, with $supp(z(E))(F_1) = 1$ and $supp(z(E))(F_2) = 0$. But as E is $(1,-1,0)$-vector, incident facial cycles must have coefficients in $z(E)$ that differ by one. So $z(E)(F_1) = 1$ and $z(E)(F_2) = 0$. This gives, that a is contained in E with the same sign as in ∂E. □

Lemma 3.6.2 in particular implies that the boundary of a directed 0-Eulerian is a directed 0-Eulerian again. It is not true that ∂E is a cycle, it is not even arc inclusion minimal among the 0-Eulerians. Take for instance $\partial A = A$ in Figure 3.2. The picture also exemplifies that two 0-Eulerians that differ only by their signing can have different interior and boundary.

The interior of a 0-Eulerian E leads to a subgraph $D(E)$ of D, which consists of all the arcs that are incident to facial cycles in *Int*(E).

Lemma 3.6.3. *Let E be a 0-Eulerian which is minimal with respect to inclusion of (unsigned) arc sets among the 0-Eulerians of (D,S). Let E' a 0-Eulerian and a subgraph of $D(E)$. Then Int$(E') \subseteq$ Int(E).*
Equality holds if and only if the signed arc sets ∂E and $\partial E'$ are the same.

Proof. By Lemma 3.6.2 $\partial E'$ is conformingly contained in E' and has the same interior, we assume $\partial E' = E'$.
Now suppose *Int*$(E') \not\subseteq$ *Int*(E). Denote the facial cycles in *Int*$(E')\backslash$*Int*(E) by $\{F'_i \mid i \in I\}$ and those in *Int*(E) by $\{F_j \mid j \in J\}$. As E' is a subgraph of $D(E)$, every arc a that is incident to exactly one F'_i must be incident to exactly one F_j as well. So the sum of all the F'_i is a non-empty 0-Eulerian E'', which is contained in E and oppositely signed on common elements. If the arc sets are equal then $-E$ and E are 0-Eulerians. By Lemma 3.6.1 this is not possible. So $\underline{E''}$ is strictly contained in $\underline{E}$, which contradicts arc minimality of E.
By Lemma 3.6.2 we have that *Int*(E') = *Int*(E) is equivalent to $\partial E = \partial E'$ as signed sets. □

In Lemma 3.6.3 the arc minimality of E is necessary. Otherwise the 0-Eulerians A and B of Figure 3.2 in the roles of E and E', respectively, give a counterexample.

Now, we introduce the concept of topological duality, which will bring us closer to oriented matroids again. It will establish us to regard $\mathcal{E}^0(D, S)$ as the set $\mathcal{E}(\mathcal{M}^*(D^\perp))$ of Eulerians of a cographic oriented matroid.

Let $D = (V, A)$ be a digraph that is 2-cell-embedded into an orientable surface S with faces F. **A topological dual of** (D, S) is a digraph $D^\perp = (V^\perp, A^\perp)$, which is 2-cell-embedded into S. Denote by $F^\perp$ the faces of $(D^\perp, S)$

The dual $D^\perp$ is a directed incidence graph of the faces of (D, S). Topological dualization maps

$$\begin{gathered} V \to F^\perp \\ A \to A^\perp \\ F \to V^\perp \end{gathered}$$

We construct $(D^\perp, S)$ by placing a vertex $f^\perp$ of $V^\perp$ inside of every face $f \in F$. Now every arc $a \in A$ lies between two (not necessarily different) elements $f, g \in F$. The corresponding facial cycles contain a oppositely signed. We introduce an arc

$$a^\perp := (f^\perp, g^\perp) :\Leftrightarrow a \textit{ is a forward arc in the facial cycle induced by } f$$

and

$$a^\perp := (g^\perp, f^\perp) :\Leftrightarrow a \textit{ is a forward arc in the facial cycle induced by } g.$$

This way one obtains a 2-cell-embedding $(D^\perp, S)$. We fix this particular $(D^\perp, S)$. Then we take $((D^\perp)^\perp, S) := (-D, S)$. This way topological dualization is a map of degree 4. Fixing the topological dual, every signed arc set X of D is mapped to a signed arc set of $D^\perp$, which we denote by $X^\perp$. As we do not display arcs that appear twice in a facial cycle in its incidence vector, we introduce the dual convention that a loop at a vertex v does not appear in the signed incidence vector of the vertex cut of v. Then the signed incidence vectors of X and $X^\perp$ are the same. This way every subgraph D' of D can be mapped to a subgraph $D'^\top$ of $D^\top$ as well, by mapping arcs of D to arcs of $D^\top$.

The nice way acyclicity and strong connectivity are treated by matroidal duality is not respected by topological duality. For example in Figure 3.6 we see a strongly connected graph on the torus, whose topological dual is strongly connected again, thus not acyclic at all. Nevertheless, the cographic oriented matroid dual $\mathcal{M}^*(D)$ and the graphic oriented matroid of the topological dual $\mathcal{M}(D^\top)$ have some relation, we need to explore.

The following lemmas link $\mathcal{E}^0(D,S)$ via topological duality to matroid theory.

Lemma 3.6.4. *Let (D,S) be a 2-cell embedding and let $E \in \mathcal{E}(D,S)$. Then $E^\perp$ is a cut of $(D^\perp, S)$ if and only if $E \in \mathcal{E}^0(D,S)$.*

Proof. $E \in \mathcal{E}^0(D,S)$
$\Leftrightarrow$ E is a sum of facial cycles of (D,S)
$\Leftrightarrow$ $E^\perp$ is a sum of vertex cuts of $(D^\perp, S)$
$\Leftrightarrow$ $E^\perp$ is a cut of $(D^\perp, S)$

This motivates a definition that includes the remaining cases. We call a signed arc set $P \subseteq A$ a **pseudocut of** (D,S) if $P^\perp \in \mathcal{E}(D^\perp)$. Let $\mathcal{P}(D,S)$ denote the set of pseudocuts of (D,S).

Lemma 3.6.5. *The sets of support minimal signed sets $Min(\mathcal{E}^0(D,S)\backslash\{\emptyset\})$ and $Min(\mathcal{P}(D,S)\backslash\{\emptyset\})$ are the circuits and cocircuits of an oriented matroid.*

Proof. The pair of signed sets $(Min(\mathcal{P}(D,S)\backslash\{\emptyset\}))$, $Min(\mathcal{E}^0(D,S)\backslash\{\emptyset\})$ is just the same as $(Min(\mathcal{P}(D,S)\backslash\{\emptyset\})^\perp$, $Min(\mathcal{E}^0(D,S)\backslash\{\emptyset\})^\perp)$, which by Lemma 3.6.4 and the definition of pseudocuts is the same as $(Min(\mathcal{E}(D^\perp,S)\backslash\{\emptyset\})$, $Min(\mathcal{C}^*(D^\perp)))$, which is nothing else than $(\mathcal{C}(D^\perp), \mathcal{C}^*(D^\perp))$. As the sets of cycles and minimal cuts, this is the pair of circuits and cocircuits respectively of the graphic oriented matroid $\mathcal{M}(D^\perp)$, given by $D^\perp$.
So the oriented matroid with circuits $Min(\mathcal{E}^0(D,S)\backslash\{\emptyset\})$ and cocircuits $Min(\mathcal{P}(D,S)\backslash\{\emptyset\})$ is its dual, namely $\mathcal{M}^*(D^\perp)$, i.e. a cographic oriented matroid. □

We call the oriented matroid given by Lemma 3.6.5 the **0-matroid of** (D,S) and denote it by $\mathcal{M}^0(D,S)$. Now, for a 0-Eulerian $E \in \mathcal{E}^0(D,S)$ and the induced $D(E)$ we define

$$\mathcal{M}^0(D(E),S) := \mathcal{M}^0(D,S)\backslash(A(D)\backslash A(D(E))),$$

the matroid obtained from $\mathcal{M}^0(D,S)$ by deleting the elements, that are not arcs of $D(E)$. The set $\mathcal{E}^0(D(E),S)$ then consists of all the Eulerians in the span of *Int*(E).

Lemma 3.6.6. *Let E be an arc minimal 0-Eulerian in $\mathcal{E}^0(D,S)$ and $\mathcal{M}(D(E)^0,S)$ totally cyclic. If E' is a directed 0-Eulerian in $\mathcal{E}^0(D(E),S)$ then $\mathcal{M}^0(D(E'),S)$ is totally cyclic as well.*

Proof. First by Lemma 3.6.3 we know, that $A(D(E')) \subseteq A(D(E'))$. Total cyclicity of $\mathcal{M}^0(D(E),S) = \mathcal{M}^0(D,S)\backslash(A(D)\backslash A(D(E)))$ is equivalent to the dual $\mathcal{M}(D^\top)/(A(D^\top)\backslash A(D(E)^\top))$ being acyclic. The latter is just the graphic oriented matroid which arises from $D^\top$ by contracting the arcs in $(A(D^\top)\backslash A(D(E)^\top))$. Now, E' is represented by a directed cut $E'^\top$ in $D(E)^\top$. The arcs $A((D(E))^\top)\backslash A(D(E')^\top)$ are those that have to be contracted to obtain $\mathcal{M}^0(D(E'),S)$ after dualizing. It is easy to see that the graph induced by $A(D(E)^\top)\backslash A(D(E')^\top)$ is the one induced by one side of the directed cut $E'^\top$. So by contracting these arcs no directed cycle can be produced. This is, $\mathcal{M}^0(D(E'),S)$ is totally cyclic. $\square$

Analogously to $\mathcal{M}^0(D,S)$ we call the covectors of $\mathcal{M}^0(D(E),S)$ pseudocuts. The problem about directed 0-Eulerians E with a directed pseudocut in $D(E)$ is, that their orientation cannot be reversed by facial flip flops. Such a situation is illustrated in Figure 3.3.

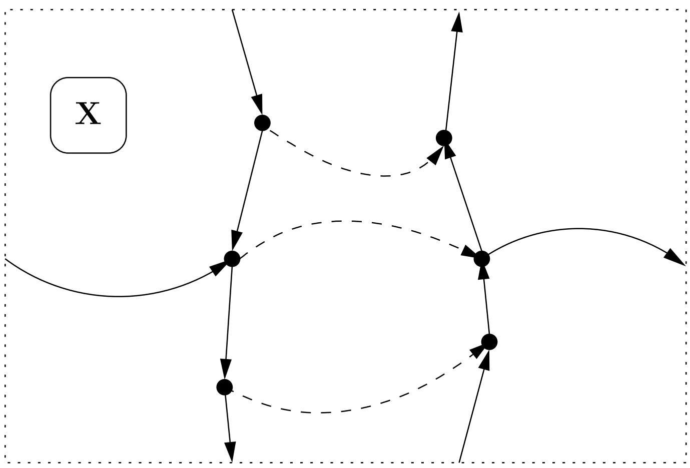

Figure 3.3: A directed 0-Eulerian E (vertical arcs) on the torus with directed pseudocut (dashed arcs) in $D(E)$.

Theorem 3.6.7. *Let D be strongly connected and 2-cell embedded into an orientable surface S, with forbidden facial cycle $\mathbf{X}$. Let $E \in (\mathcal{E}^0(D,S))^+$. Then $\underline{E} \in \mathrm{ff}(\mathcal{F}\backslash\{\mathbf{X}\})$ if and only if there is a 0-Eulerian E^0 with $\underline{E} = \underline{E^0}$ and totally cyclic $\mathcal{M}^0(D(E^0),S)$.*

Proof. ”$\Rightarrow$”:
If $\mathcal{M}^0(D(E^0),S)$ is not totally cyclic, $D(E^0)$ contains a directed pseudocut P. By

Lemma 3.6.5 we have orthogonality, so no directed 0-Eulerian of $D(E^0)$ can ever intersect the directed P. But by definition of the interior every facial cycle in *Int*(E^0) forms part of the integral support of E^0, thus has to be flipped or flopped at least once, in order to reverse the orientation on $\underline{E}$. The facial cycles in *Int*(E^0) that are incident to P can never be directed, thus can not be flipped or flopped, thus $\underline{E}$ cannot be in $\mathrm{ff}(\mathcal{F}\backslash\{\mathbf{X}\})$.

"⇐":
Let $\mathcal{M}^0(D(E^0), S)$ be totally cyclic. We can assume E^0 to be arc minimal in $\mathcal{E}^0(D, S)$, because otherwise we can decompose it into disjoint positively and negatively directed 0-Eulerians. Define **the area of** E^0 as $A(E^0)$ - the height of *Int*(E^0) with respect to the inclusion order on the interiors of 0-Eulerians. We will proceed by induction on $A(E^0)$.
If $A(E^0) = 1$ then E^0 is a facial cycle different from the forbidden one. As $\mathcal{M}^0(D(E^0), S)$ is totally cyclic E^0 is a directed facial cycle, i.e. can be flipped or flopped.
So let $A(E^0) > 1$. Analogously to the proof of Theorem 1 we decompose *Int*(E^0) into *Int*(E') and *Int*(E''), where E', E'' are 0-Eulerians with smaller area, which we are able to flip or flop one after the other.

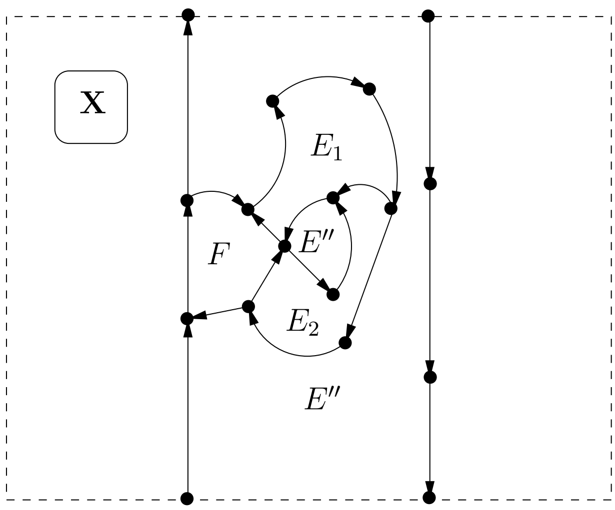

Figure 3.4: Construction of $E' = sgnsum(E_1, E_2, F)$ and E''.

As *Int*(E^0) is not minimal take a facial cycle $F \in$ *Int*(E^0) that is incident to E^0. If the signed arc set $F\backslash E^0$ is not directed, take $a_1 \in (F\backslash E^0)^-$. By total cyclicity we have a minimal directed Eulerian $a_1 \in E_1 \in \mathcal{E}^0(D(E^0), S)$, which by Lemma 3.6.1 can assumed to be a 0-Eulerian. By $E_1 \neq E^0$, arc minimality of E_1, and Lemma 3.6.3 we have $A(E_1) < A(E^0)$. Thus by induction hypothesis and Lemma 3.6.6 the arc set of E_1 can be reversed by facial flip flops.
If E_1 intersects E^0 we take it as E'. Otherwise we reverse its orientation and take $a_2 \in$

$(F \backslash E^0)^-$ in the resulting reorientation. Proceeding this way we obtain a flip flop sequence $(E_1, \ldots, E_k)$ of directed Eulerians, such that $sgnsum(E_1, \ldots, E_k)$ intersects E^0 or turns F into a directed facial cycle. In the second case define $E' := sgnsum(E_1, \ldots, E_k, F)$ otherwise take $E' := sgnsum(E_1, \ldots, E_k)$.
By the usual argumentation we can assume E' to be a directed 0-Eulerian. Moreover we can assume $\partial E' = E'$ by the following two arguments: By "$\Rightarrow$" we know that E' has totally cyclic 0-matroid because it can be flip flopped. By construction we will never take F and all its incident facial cycles in $\int(E^0$, so we have $A(E') < A(E^0)$. As $A(\partial E') = A(E')$ and both have the same 0-matroid, $\partial E'$ can be flipped as well. Moreover it intersect E^0, wherever E' did.
So define the 0-Eulerian E'' as given by *Int*$(E^0) \backslash$*Int*(E'). As E^0 is arc set minmal $E' = \partial E'$, by Lemma 3.6.2, every facial cycle appears at most once in their integral support. Now, E'' is directed oppositely to E' on $E' \cap E''$ and signed as E^0 on $E'' \cap E^0$ and $E' + E'' = E^0$. Moreover $A(E'') < A(E^0)$. So after reversing the orientation on E' we have that E'' is directed, and can thus be reversed, too.
As $E' \cap E''$ has been reoriented twice, we have obtained exactly what we wanted: the reorientation of E by means of facial flip flops.

$\square$

Analyzing the proof of the "$\Leftarrow$"-direction one sees, that if the orientation of E can be reversed, this can be done by facial flip flops of facial cycles in its interior. Moreover one observes, that if E is positively or negatively directed it can be reversed by pure flip or pure flop sequences, respectively.

If one wants to know, whether all the directed 0-Eulerians of an embedding can be reversed this way, by Lemma 3.6.6, it is enough to check the oriented matroids given by the directed 0-Eulerians with inlucion maximal interior.

Take a 0-Eulerians E of (D, S). Theorem 3.6.7 implies the following. If *Int*(E) together with all the faces that induce the facial cycles in *Int*(E) is homeomorphic to a collection of edge disjoint disks then E can always be reversed by facial flip flops. All the pseudocuts in $D(E)$ must be cuts. So a directed pseudocut would be a directed cut and contradict strong connectivity of D. So $\mathcal{M}^0(D(E), S)$ must be totally cyclic and E can be reversed by facial flip flops. The planar case as in [7] is a special case of this situation. Together with Corollary 2.2.4 one obtains Theorem 1.

The nice thing is that Theorem 3.6.7 again together with Corollary 2.2.4 also leads to the main theorem of [12], namely Theorem 3.

Corollary 3.6.8. *Let $D = (V, A)$ be an acyclic directed graph. And $v \in V$ an arbitrary fixed vertex. Then*

- *Every c-reorientation of D can be obtained from D by a flip flop sequence of vertex cuts different from v's vertex cut.*

- *And moreover* $(\textit{reor}_c(D), \leq_{\text{ff}})$ *is a distributive lattice.*

Proof. Embed D into an orientable surface S. Now by Theorem 3.5.2, the c-reorientations of D correspond to the α-reorientations of the cographic oriented matroid induced by D. By Lemma 3.6.5 these are the α-reorientations of $\mathcal{M}^0(D^\perp, S)$. The vertex cut of v corresponds to a facial cycle $\mathbf{X}$ of $(D^\perp, S)$. We have

$$\mathcal{M}^0(D^\perp, S) = \mathcal{M}^0(D^\perp(\mathbf{X}), S).$$

As D is acyclic, $\mathcal{M}^0(D(\mathbf{X}), S)$ is totally cyclic. As we have *Int*$(E) \subseteq$ *Int*$(\mathbf{X})$, for every 0-Eulerian E. Lemma 3.6.6 gives totally cyclic $\mathcal{M}^0(D(E), S)$, for every 0-Eulerian E. So by Theorem 3.6.7 the orientation of every directed 0-Eulerian can be reversed, i.e. every α-reorientation of $\mathcal{M}^0(D^\perp, S)$ can be produced by facial flip flops. Dualizing back, this shows the first part of the corollary.
The second part has already been shown in Chapter 2 as a corollary of Corollary 2.2.4. □

In particular the term *Int*(E) for a 0-Eulerian E dualizes to cuts of acyclic directed graphs. **The interior of a cut** X of such a digraph with forbidden vertex v is the set of vertex cuts induced by the vertices in the side of X, that does not contain v. By Corollary 3.6.8 we know that the orientation on X can be reversed by flipflops of *Int*(X).

After having seen what we can manage by flipping and flopping facial cycles of (D, S), we look now for possible extensions of $\mathcal{F}\backslash\{\mathbf{X}\}$, to hopefully be able to generate all the α-reorientations of D.

If the orientable surface S where D is embedded is different from the sphere not every Eulerian can be combined with $\mathcal{F}\backslash\{\mathbf{X}\}$. Take for instance Figure 3.6 and any straight cycle X_i in it. As the facial cycles do not suffice to generate the cycle space of D, by Proposition 3.3.4, $\text{ff}(\mathcal{F}\backslash\{\mathbf{X}\}) \subseteq \mathcal{E}^0(\mathrm{D}, \mathrm{S}) \subsetneq \textit{reor}_\alpha(\mathrm{D})$. So we can try to extend $\mathcal{F}\backslash\{\mathbf{X}\}$ in such a way, that the resulting set integrally spans the entire cycle space of D.

Here is where some homology comes in. In order to distinguish between topologically different Eulerians of (D, S), we use the concept of the first homology group $H_1(S)$ of the surface S, given by the cell decomposition induced by (D, S). The group $H_1(S)$ can be seen as the quotient space $(\ker_{\mathbb{Z}} \textit{Inc}(D))/(span_{\mathbb{Z}}(\mathcal{F}))$. Every Eulerian E of (D, S) lives in some of the equivalence classes $[E]$, that form $H_1(S)$.

Homology theory tells us that for an orientable surface S of genus $\gamma(S)$ the first homology group is isomorphic to $\mathbb{Z}^{2\gamma(S)}$, see for instance [8]. So every Eulerian arc set E of D gives an element $[E] \in H_1(S)$, which then corresponds to some element of

$(x_1, y_1, \ldots, x_{\gamma(S)}, y_{\gamma(S)}) \in \mathbb{Z}^{2\gamma(S)}$. For an Eulerian E we have $[E] = (0, \ldots, 0)$ if and only if $E \in \mathcal{E}^0(D, S)$

By Proposition 3.3.4, we know that an extension of $\mathcal{F}\backslash\{\mathbf{X}\}$ must span the entire cycle space of D. This requirement can be characterized by the following. As $H_1(S)$ was defined as the quotient of all the cycle space modulo the space spanned by the 0-Eulerians, an extension of $\mathcal{F}$ spans all the cycle space if and only if the corresponding equivalence classes span $H_1(S)$. In other words let $E_1, \ldots, E_k$ be elements of $\mathcal{E}(D)$. The set $\{[E_1], \ldots, [E_k]\}$ integrally generates $H_1(S)$ if and only if $\mathcal{F}\backslash\{\mathbf{X}\} \cup \{E_1, \ldots, E_k\}$ integrally spans the whole cycle space of D.

So given (D, S) the condition $span_{\mathbb{Z}}(\{[E_1], \ldots, [E_k]\}) = \mathbb{Z}^{2\gamma(S)}$ is necessary for $\mathrm{ff}(\mathcal{F}\backslash\{\mathbf{X}\} \cup \{\mathrm{E_1}, \ldots, \mathrm{E_k}\}) = \textit{reor}_\alpha(\mathrm{D})$. This condition is not sufficient.

By Theorem 3.6.7, we know that $\{E_1, \ldots, E_k\}$ also has to be able to repair all those directed 0-Eulerians E which have a directed pseudocut in $D(E)$. This could be rephrased as: "in every α-reorientation D^E that contains a directed pseudocut P, which lies in $D(E')$ for some directed 0-Eulerian E', we can flip flop a Eulerian E'' disjoint from E', that intersects P." By the orthogonality of $\mathcal{P}(D, S)$ and 0-$\mathcal{E}(D, S)$, we know that E'' cannot be a 0-Eulerian. Such a Eulerian E'' is exemplified in Figure 3.3 by the middle dashed arc together with the non-dashed horizontal arcs.

For instance case enumeration shows, that already for the 4×2 hexagonal torus grid as depicted in Figure 3.5, there is no extension of $\mathcal{F}\backslash\{\mathbf{X}\}$, that minimally spans $H_1(S)$ and leads to a flip flop generating set for the α-orientations. As on the other hand the 4×2 hexagonal torus grid is planar, the lower bound for a flip flop generating set given by Proposition 3.3.4 stays tight. Just take the 5 bounded facial cycles of a planar embedding as in Figure 3.5. Theorem 1 implies that all the α-reorientations of the grid can be obtained by flipflops of this set.

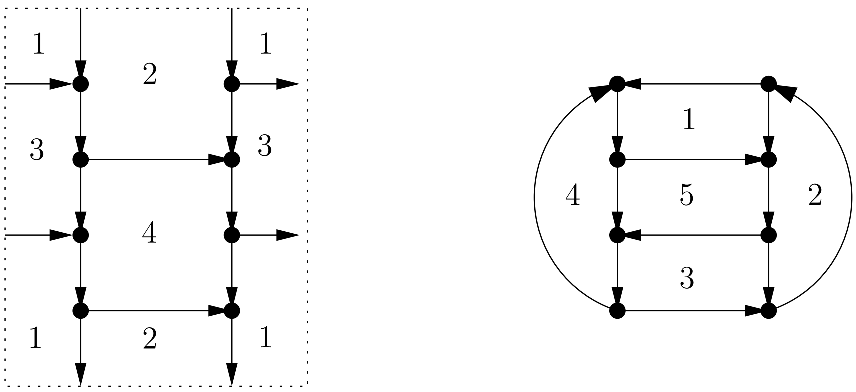

Figure 3.5: The strongly connected 4×2 hexagonal torus grid can be embedded into the torus and into the plane.

The following Section presents a positive non-planar example for Proposition 3.3.4, where a minimal extension of $\mathcal{F}\backslash\{\mathbf{X}\}$ leads to a flip flop generating set, for all the α-orientations.

3.7 Eulerian Orientations of the Torus Square Grid

Eulerian orientations are those α-orientations such that at every vertex the outdegree equals the indegree, i.e $\alpha \equiv \frac{deg}{2}$. In this section we prove that the set of Eulerian orientations of the square grid on the torus carries a poset structure. It will be generated by a minimal extension of $\mathcal{F}\backslash\{\mathbf{X}\}$ to a spanning set of the cycle spaced. The poset consists of distributive lattices given by flipflops of facial cycles, which are related by flips of the two Eulerians X_1 and Y_1.

Let $T_{m,n}$ be the $m \times n$ square grid embedded in the torus. Choose as base point for developing the flip flop poset the Eulerian orientation D of $T_{m,n}$ as depicted in Figure

3.6.

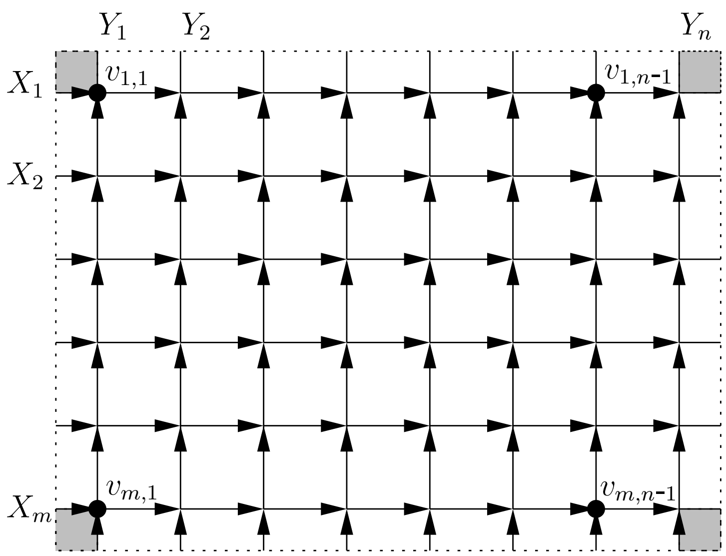

Figure 3.6: Reference orientation D of the torus square grid

Label the vertices of $T_{m,n}$ with the set $[m] \times [n]$. Start with $v_{1,1}$ in the upper left and continue labeling in matrix fashion. The letters $(X_i)_{1\leq i\leq m}$ and $(Y_j)_{1\leq j\leq n}$ stand for the horizontal and vertical **straight** cycles $(v_{i,1}, \ldots, v_{i,n})$ and $(v_{m,j}, \ldots, v_{1,j})$, respectively. They are considered as positively directed in D.

The shaded region in the corners of the torus drawing will be taken as the forbidden face with boundary $\mathbf{X}$. As a generating set of directed cycles for the α-reorientations of D, we claim $B := \mathcal{F}\backslash\{\mathbf{X}\} \cup \{X_1, Y_1\}$, where the orientation of the facial cycles in $\mathcal{F}$ again is obtained from running through their arcs in counter clockwise direction.

The first homology group of the torus is isomorphic to $\mathbb{Z}^2$ and $\{X_1, Y_1\}$ is a generating set of cycles, i.e. $\{[X_1], [Y_1]\}$=$\{e_1, e_2\}$, which minimally spans $\mathbb{Z}^2$. Thus B is a minimal spanning set of the cycle space of D, so B satisfies the necessary conditions given by Proposition 3.3.4 to generate all the Eulerian orientations of $T_{m,n}$. As $\{e_1, e_2\}$ is an independent set over $\mathbb{Z}$, the rows of B are integrally independent and we already have that $D_{\mathrm{ff}}(B)$ is acyclic.

Call a Eulerian E of any reorientation of D **straight** if $E = \bigcup_{i\in I} X_i \cup \bigcup_{j\in J} Y_j$ for some $I \subseteq [m]$ and $J \subseteq [n]$, i.e. E is a union of straight cycles. Therefore a straight Eulerian can also be written by the tuple of index sets (I, J), corresponding to the straight cycles involved. Denote by $\mathcal{S}$ the set of straight Eulerian arc sets of D. In order to explore the whole set of Eulerian arc sets of D, we will first analyse the structure of $\mathcal{S}$ as a subposet of $P_{\mathrm{ff}}(B)$.

Proposition 3.7.1. *Let $E, E' \in \mathcal{S}$ with index sets (I, J) and (I', J'). In the inherited order from $\mathrm{ff(B)}$ one has $E \leq_{\mathrm{ff}} E'$ if and only if (at least) one of the following three cases holds:*

1. *$I \neq [m]$ and $J \neq [n]$ and $I' \neq \emptyset \neq J'$ and there are injective maps*

$$\phi_I : I\backslash I' \hookrightarrow I'\backslash I$$
and
$$\phi_J : J\backslash J' \hookrightarrow J'\backslash J$$

   such that $i \leq \phi_I(i)$ and $j \leq \phi_J(j)$

2. *$I = I' = \emptyset$ or $I = I' = [m]$ and $J'\backslash J \subseteq \{1\}$*

3. *$J = J' = \emptyset$ or $J = J' = [n]$ and $I'\backslash I \subseteq \{1\}$*

Proof. We start with case $1.$:
"$\Leftarrow$":
The idea here is, that $I\backslash I'$ is the set of straight cycles that points to the left and has to be reversed and $I'\backslash I$ is the set of straight cycles that points to the right and has to be reversed. Under some conditions, taking one element of each of these sets forms a flippable Eulerian. We give an algorithm how these flips can be organized such that, with the help of X_1 and Y_1, the desired set can be flipped. The main idea can be read off Figure 3.6.

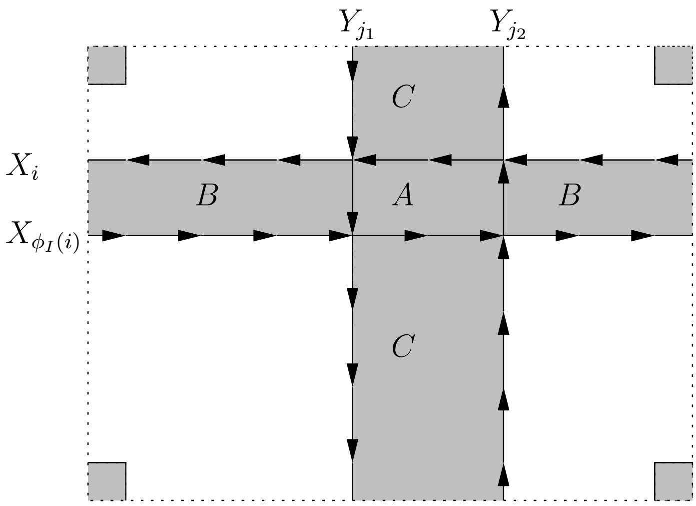

Figure 3.7: (A, B), (A, C), and (A, B, C, A) can be flipped

For every $i \in I\backslash I'$ we have that $X_i \cup X_{\phi_I(i)}$ is a 0-Eulerian, because it is the sum of the obvious 0-Eulerians A and B. The ccw orientation of A and B also induces that it is positively directed. As one of Y_{j_1} and Y_{j_2} points down and the other points up and by $i \leq \phi_I(i)$, flipping first A and then B, the orientation on $X_i \cup X_{\phi_I(i)}$ is reversed.
If $\phi_J(j_1) = j_2$ and some X_{i_1} points left and another X_{i_2} points right, then the analogue can be stated about $Y_{j_1} \cup Y_{\phi_J(j_2)}$.
If - as in the picture - both situations come together, then also $X_i \cup X_{\phi_I(i)} \cup Y_j \cup Y_{\phi_J(j)}$ is a positively directed 0-Eulerian and can be flipped via the sequence (A, B, C, A).
This is already the essence of the proof. It leads to an algorithm, that controls the consecutive application of such flips together with the flips of the special straight cycles X_1 and Y_1.
Now we present the algorithm. It takes $I, I', \phi_I, J, J', \phi_J$ satisfying the conditions given in 1. as input and constructs a pure flip sequence s with base point D^E, such that $(D^E)^s = D^{E'}$.
First we describe formally how to deal with the standard situation as depicted in Figure 3.7. For a subset K of I or J and $k \in K$ define Z_k to be X_k or Y_k respectively. In the following pseudo code fragments the statement s+=X stands for flipping X and adding it to s. The flips in FLIP are either of the type (A, B) or (A, C) exemplified by Figure 3.7 or they flip Z_1.

As depicted in Figure 3.7, the flips performed in FLIP are only possible if the set of straight cycles in $\{I, J\}\backslash\{K\}$ does not entirely point to one direction. In the MAIN part of the algorithm we will care about this property.
In the first **for**-scope FLIP takes $K\backslash K'$ and $\phi(K\backslash K')$ and reverses the orientation on these sets, by using the situation of Figure 3.7 in every step. In the second **for**-scope FLIP takes

Input : Row or column index sets K' and K that conform to condition 1. of the theorem. The set K' corresponds to a straight directed 0-Eulerian $\widetilde{E}'$ in a reorientation $D^{\widetilde{E}}$, such that $D(\widetilde{E})$ contains no directed pseudocut. The index set K' corresponds to $\widetilde{E}$
Output: The flip sequence s will be extended such that $D^s = (D^{\widetilde{E}})^{\widetilde{E}'}$

for $k \in K \backslash K'$ **do**
  s+=$(Z_k \cup Z_{\phi_K(k)})$;
for $k \in K' \backslash K \backslash \phi_K(K \backslash K') \backslash \{1\}$ **do**
  if Z_1 *positively directed* **then**
    s+=Z_1;
  s+=$(Z_1 \cup Z_k)$;
if $1 \in K' \backslash K$ **then**
  s+=Z_1;

Algorithm 1: FLIP

the remaining elements of K' except Z_1 and by orienting Z_1 appropriately it produces again the situation of Figure 3.7, such that $(Z_1 \cup Z_k)$ can be flipped. The last **if**-statement flips Z_1 again if necessary. This can always be done, because Z_1 is in B.
Thus after FLIP the orientation on $K \backslash K'$ and $K' \backslash K$ has been reversed.
In order to produce the situation that is expected by FLIP we need to care about the orientation of the whole set of straight cycles, which will be gained in MAIN.

The procedure MAIN controls that, if any set K of either vertical other horizontal straight cycles is handed to FLIP, then the set of ”orthogonal” straight cycles does not point completely into the same direction. Therefore MAIN must distinguish some cases and decide whether I is flipped before J or viceversa. This can be done, because we have that I' and J' are non-empty and I and J are not the entire set.
Together with the correctness of FLIP one has that MAIN reverses the orientation on $I \backslash I', I' \backslash I, J \backslash J', J' \backslash J$, by the flip sequence s. So after applying the s to D^E we have obtained $D^{E'}$. This is the definition of $E \leq_{\text{ff}} E'$.

Now we show the ”$\Rightarrow$”-direction for the case where I, I' are neither both empty nor both full and J, J' are neither both empty nor both full.
So let I, I', J, J' be like that and $E \leq_{\text{ff}} E'$. It must be shown that this implies $I \neq [m]$ and $J \neq [n]$ and $I' \neq \emptyset \neq J'$, to see that we are in case 1. of the theorems statement.
Then we have to prove the existence of two injections $\phi_I : I \backslash I' \hookrightarrow I' \backslash I$ and $\phi_J : J \backslash J' \hookrightarrow J' \backslash J$ such that $i \leq \phi_I(i)$ and $j \leq \phi_J(j)$.
By definition $E \leq_{\text{ff}} E'$ means that there is a straight positively directed Eulerian $\widetilde{E}$ in D^E, such that $(D^E)^{\widetilde{E}} = D^{E'}$. As E and E' also $\widetilde{E}$ induces two index sets $\widetilde{I}$ and $\widetilde{J}$. We have $I \Delta \widetilde{I} = I'$. Define $\widetilde{I}^- := I \cap \widetilde{I}$ - the straight cycles of $\widetilde{E}$ that are pointing down in

Input : Sets and functions $I, I', \phi_I, J, J', \phi_J$ satisfying condition $1.$ of the theorem.
Output: A flip sequence s that leads from D^E to $D^{E'}$

```
s:=∅;
if I' = [m] and J' = [n] then
    if J = ∅ then
        s+=Y_1;
    FLIP(I\{φ_I^{-1}(m)}); FLIP(J\{φ_J^{-1}(n)});
    s+=(X_1 ∪ X_m ∪ Y_1 ∪ Y_n);
else
    if I' = [m] then
        if I = ∅ then
            s+=X_1;
        FLIP(J); FLIP(I);
    else
        if I = ∅ then
            s+=X_1;
        FLIP(J); FLIP(I);
```

Algorithm 2: MAIN

D^E and must be reversed. And call the rest - those that are pointing up in D^E and must be reversed: $\widetilde{I}^+ := I' \cap \widetilde{I}$.
Suppose that $|\widetilde{I}^-| > |\widetilde{I}^+|$. This means that some of the straight cycles indexed by $|\widetilde{I}^-|$ cannot be flipped as part of a straight 0-Eulerian together with some cycles indexed by elements of $|\widetilde{I}^+|$. So the remaining cycles can only be reversed as a part of a flip if some new wrongly oriented straight cycle is produced. To repair this a flop must be performed, which contradicts $E \leq_{\text{ff}} E'$.
Therefore we have $|\widetilde{I}^-| \leq |\widetilde{I}^+|$. This implies $|I| \leq |I'|$. Analogously we get $|J| \leq |J'|$. Together we are in case $1.$, i.e. $I \neq [m]$ and $J \neq [n]$ and $I' \neq \emptyset \neq J'$.
Supposing now that the injections as in $1.$ do not exist, means that $X_{\widetilde{I}^-}$ is not entirely contained in any positively directed straight Eulerian of $X_{\widetilde{I}}$. Thus any try to flip all the straight cycles in $\widetilde{I}^-$ would reorient some straight cycles, that were not desired. These would have to be flopped back later on, to obtain the orientation we are aiming for. So we cannot come from D^E to $D^{E'}$ by a sequence of flips.

Now we show the equivalence of $E \leq_{\text{ff}} E'$ and $J' \backslash J \subseteq \{1\}$ in case $2.$. It is easily seen that, because of the orientation on the $X_{[m]}$, every positively directed 0-Eulerian $\widetilde{E}$ in D^E has a directed pseudocut in $D^E(\widetilde{E})$, thus by Theorem 3.6.7 cannot be flip flopped by means of facial cycles. So the only Eulerians that can be flipped are Y_1 and X_1 if

$I' = \emptyset$. But flipping the latter, would have to be undone by some flop: either by flopping X_1 or by flipping some 0-Eulerian of the form $X_1 \cup X_i$, which then leaves X_i negatively directed. This would destroy directedness of the sequence we are aiming for, so $E \leq_{\text{ff}} E'$ if and only if $J' \backslash J \subseteq \{1\}$.

The equivalence in case 3. works completely analogously. $\square$

In order to understand the poset of straight Eulerians $(\mathcal{S}, \leq_{\text{ff}})$ we define an order $P_k := (\{0,1\}^k, \prec)$ such that for different $(0,1)$-vectors $x \prec y :\Leftrightarrow$ one of the following holds

1. $< x, \mathbf{1} >=< y, \mathbf{1} >$ and $x = (y(1), \ldots, y(i), 0, 1, y(i+3), \ldots, y(k)) \Leftrightarrow: x \prec_1 y$, or
2. $< x, \mathbf{1} >=< y, \mathbf{1} > -1$ and $x = (0, y(2), \ldots, y(k)) \Leftrightarrow: x \prec_2 y$

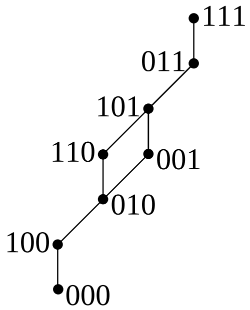

Figure 3.8: The poset P_3.

The poset P_k can be constructed from $\mathbf{0}$ on by two operations. In case 1. the vector y is obtained from x by switching a 10 to a 01. In case 2. the first entry of x has been changed from 0 to 1 in order to obtain y.

Identifying the sets I, I' and J, J' from Proposition 3.7.1 with the corresponding incidence vectors $x, x' \in \{0,1\}^m$ and $y, y' \in \{0,1\}^n$ respectively one gets:

$$(\mathcal{S}, \prec_{\text{ff}}) = (P_m \times P_n) \backslash \{(x,y) \prec_0 (x',y') \mid \{x,x'\} \in \{\{\mathbf{0}\},\{\mathbf{1}\}\} \textit{ or } \{y,y'\} \in \{\{\mathbf{0}\},\{\mathbf{1}\}\}\},$$

where it is understood that only the relations, not the elements are removed from the product.

To complete the picture of $P_{\text{ff}}(\mathcal{F} \backslash \{\mathbf{X}\} \cup \{X_1, Y_1\})$ we will see how the rest of the Eulerians of D fit into the pattern given by the straight Eulerians.

Proposition 3.7.2. *For every Eulerian E of D there are straight Eulerians E_1, E_2 such that $E_1 \leq E \leq E_2$. Moreover all the three E_1, E, E_2 are in the same homology class.*

Proof. The proof works by constructing a flip sequence from E to E_2 and a flop sequence from E to E_1. Both consist of reorientations of 0-Eulerians only.
E is a directed subgraph of D with vertex degrees 0, 2 or 4. In D^E the vertices with degree 2 in E have only two possibilities to look like with respect to E if E does not walk straight ahead. These vertices are called DL (down-left) and LD (left-down) vertices, respectively. They are depicted in Figure 3.9

Figure 3.9: DL-vertex and LD-vertex

It is rather obvious that the number of DL-vertices and LD-vertices on a fixed X_i must be equal. Therefore also on all E these numbers, denoted from now on by p, coincide.
Our proof will be an induction on p:
If $p = 0$ our Eulerian E is straight, so $E_1 = E = E_2$ proves the induction basis.
If $p > 0$, we construct $E' <_{\text{ff}} E$ with $p' < p$. Given a DL-vertex $v_{i,j}$ take the next LD-vertex to the right on X_i, say $v_{i,j'}$, and the next in downwards direction on Y_j, say $v_{i',j}$.
Locally two different situations can occur, depicted in Figure 3.10

In the picture, we draw only those lines that are at our concern. The dashed lines are those arcs, that are still oriented as in D. The other arcs are arcs in E (left side) or in E' (right side) respectively. It is possible that there are more vertices and arcs inside the shaded region. The important fact is, that there are no DL- and LD-vertices on the lines between $v_{i,j}$ and $v_{i,j'}$ and between $v_{i,j}$ and $v_{i',j}$, respectively. Hence, no arcs of E lie on these lines.
Case (1) reflects the situation where the heavily drawn directed path passing through $v_{i,j'}$, has a vertex $v_{.,j}$ before it possibly intersects, with the corresponding path, passing through $v_{i',j}$. So case (1) includes a similar picture, that is reflected on a diagonal axis.
Case (2) shows what we do, if both paths intersect before they cross the horizontal respectively vertical line induced by $v_{i,j}$.
In both cases, the bounding cycle of the shaded region, say F, is oriented in clockwise direction. If F is flopped the resulting Eulerian E' is smaller than E in the flip flop poset, and moreover it is in the same homology class, because F is a 0-Eulerian. Because the number of DL-vertices has been reduced by at least one by this flop one can apply the induction hypothesis and thus gets some straight Eulerian $E'' \leq E' \leq E$, which also lies in the same homology class with E.

The only problem that remains is, whether F can be flopped by our means.
We do not know, if the shaded region really is the interior of the directed cycle which we

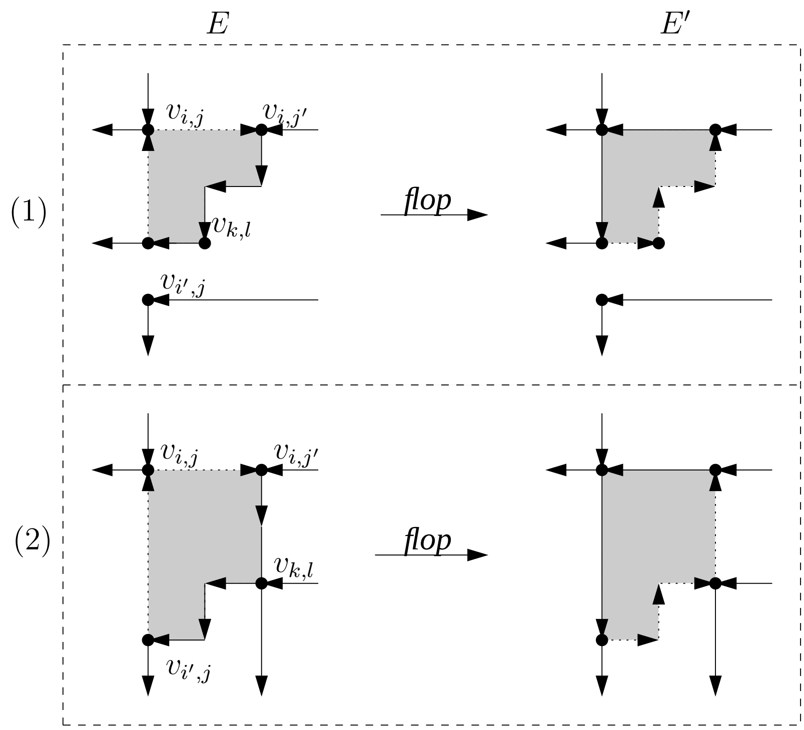

Figure 3.10: In both cases the flop of the shaded region reduces p

want to flop, or if it contains the forbidden facial cycle $\mathbf{X}$.
We will show that in every Eulerian with $p > 0$ there is a DL-vertex such that the induced negatively directed 0-Eulerian F has the shaded region as its interior. As this region is homeomorphic to a disk and by strong connectivity of the whole graph, F can be flopped, by flops of the facial cycles in its interior.

So take $v_{i,j}$ to be a DL-vertex of E which minimizes $min(i,j)$ among all of E's DL-vertices. The vertex $v_{i,j}$ induces a cycle F as in Figure 3.10. If the region that is induced by F contains $\mathbf{X}$, we cannot be in case (1) of Figure 3.10. Otherwise F has a "lower right" corner called $v_{k,l} \neq v_{i,j}$ in the figure, which is itself a DL-vertex. So as F's region contains $\mathbf{X}$ the vertex $v_{k,l}$ must lie between $v_{0,0}$ and $v_{i,j}$, i.e. $0 \leq k < i$ and $0 \leq l < j$, which contradicts the minimality of the choice of $v_{i,j}$.
So we concentrate on the case that our $min(i_1, j_1)$-minimizing v_{i_1,j_1} is of the second type of DL vertices (case (2) of Figure 3.10) and the region induced by the corresponding F_1 contains $\mathbf{X}$. This means we have a vertex v_{k_1,l_1} of degree four, which by supposing that $\mathbf{X}$ is contained in the shaded region, lies between $v_{0,0}$ and v_{i_1,j_1}, i.e. $0 \leq k_1 < i_1$ and $0 \leq l_1 < j_1$.
But as v_{k_1,l_1} has degree four there must be another DL-vertex $v_{k_1,j'}$ from v_{k_1,l_1} to the right, and one v_{i',l_1} downwards. Choose one of both, such that we stay with the minimum of k_1 and ℓ_1.

Call this DL-vertex v_{i_2,j_2}. It induces again a cycle F_2. Again, if F_2 induces a region that contains $\mathbf{X}$ again it cannot be of type (1). If F_2 is of type (2) we obtain a new degree four vertex v_{k_2,l_2}, with $0 \le k_2 < i_2$ and $0 \le l_2 < j_2$.
So if we could go on like this forever, we could construct a strictly decreasing sequence of is and js, such that $0 \le min(i_{r+1}, j_{r+1}) < min(i_r, j_r)$, for every r. So because of finiteness of the graph, this iteration must stop, i.e. lead to some DL vertex that induces a region, which does not contain the forbidden face.
Now we have constructed the desired $E' \le_{\text{ff}} E$, which we can apply the induction hypothesis to and obtain the straight $E_1 \le E$. The construction of E_2 works analogously by switching the roles of DL-vertices and LD-vertices. $\square$

Proposition 3.7 and Proposition 3.7.1 together give that $P_{\text{ff}}(\mathcal{F}\backslash\{\mathbf{X}\}\cup\{X_1, Y_1\})$ consist of distributive lattices on the homology classes of $\mathcal{E}(D, S_1)$, which are related by flips of X_1 and Y_1.

Chapter 4

Everybody is a Flip Flop

The question of how to get a distributive lattice on the α-orientations of cographic oriented matroids has already been solved in [12]. Moreover we have proved this result in Corollary 3.6.8. In Chapter 2 we have proved that every loop-free digraph is the flip flop graph of a sign matrix. In the present chapter we will prove, that every distributive lattice is a flip flop poset on the α-orientations of a cographic oriented matroid. Analogously to the questions raised after Theorem 2.1.3 in Chapter 2, we then analyze the structure of those cographic matroids, that generate the same distributive lattice. As justified by Section 3.5 we will treat α-orientations of totally cyclic cographic oriented matroids as c-reorientations of acyclic digraphs.

4.1 Every Distributive Lattice is the Flip Flop Poset of a Digraph

We will now describe a method that constructs out of a given distributive lattice L a set of digraphs $[D]$, that realize L as the flip flop poset on their c-reorientations.

So let L be a distributive lattice. Denote by $J(L)$ the sub poset of its join-irreducible elements. View $J(L)$ as a directed graph D' on the elements of $J(L)$, where

$$(u, v) \in A(D') :\Leftrightarrow u \prec_{J(L)} v.$$

Add a vertex $\top$ to D' and introduce arcs from the sources of D' (minima of $J(L)$) to $\top$. Call this new graph D. Denote by $[D]$ the set of digraphs that can be obtained from D by adding transitive arcs. Ordering the elements of $[D]$ by arc set inclusion, $[D]$ forms a boolean lattice. The minimal element of $[D]$ is D.

The construction is exemplified by Figure 4.1.

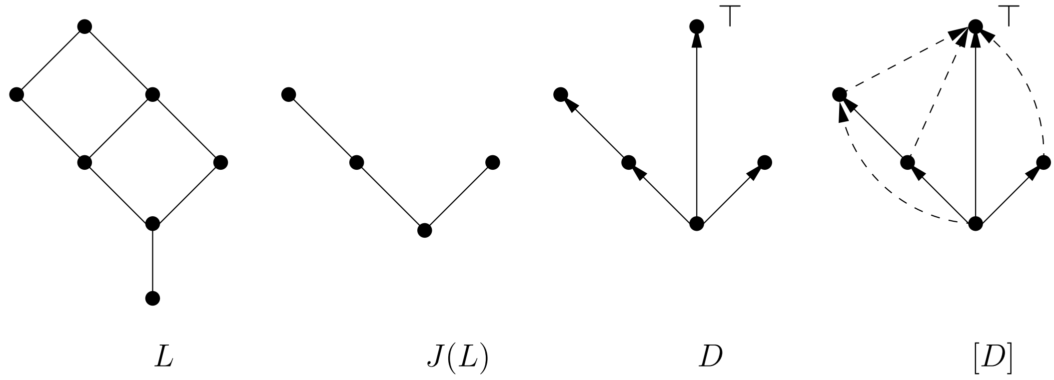

Figure 4.1: The construction of $[D]$ out of L. The dashed arcs in the drawing of $[D]$ stand for arcs that can be added to D.

The set of graphs $[D]$ induced by L has a nice property. Let $\widetilde{D} \in [D]$ and B the matrix, that is obtained from *Inc*$(\widetilde{D})$ by deleting the row that corresponds to the $\top$-vertex.

Theorem 4.1.1. *The flip flop poset $P_{\mathrm{ff}}(B)$ is isomorphic to the distributive lattice L.*

Proof. This result is a special case of the theory developed in Section 4.2. Anyways, here is a proof:
As shown by Corollary 2.2.6, $P_{\mathrm{ff}}(B)$ is a distributive lattice. As B has no negatively directed rows, B corresponds to the minimum of $P_{\mathrm{ff}}(B)$. So every element of $P_{\mathrm{ff}}(B)$ can be reached by a flip sequence based at B. By the fundamental theorem of distributive lattices, (see [5], pp171), it is enough to show that the distributive lattice $\mathcal{O}(J(L)) = L$ of ideals of $J(L)$ is isomorphic to $P_{\mathrm{ff}}(B)$.
First, the rows that come from the sources of $\widetilde{D}$ are positively directed in B. But as the sources are connected to the forbidden vertex $\top$, after flipping them once, they cannot have a positively directed vertex cut again.
Second, any row can be flipped only after the rows, that correspond to its predecessors in $\widetilde{D}$, have been flipped. So iteratively every row can be flipped at least as often as the sources.
This means that every row can be flipped at least once in a flip sequence. And it tells us that the set of vertices that have been flipped in any flip sequence s corresponds to an ideal of $J(L)$.
To see now, that we get any ideal of $J(L)$ this way, take an antichain A in $J(L)$. We try to flip all the vertices $\widetilde{A}$ of $\widetilde{D}$ that correspond to A and none of their successors. So take all the vertices that lie on directed paths from sources of $\widetilde{D}$ to elements of $\widetilde{A}$. This vertex set induces a directed cut in $\widetilde{D}$. Directed cuts correspond to c-reorientations of $\widetilde{D}$, which by Corollary 3.6.8 can be reversed by a flip sequence based at B. The rows in this sequence corresponds to the vertices in one of the sides of the cut - the interior of the cut. One of the sides contains $\top$, thus cannot be flipped. The other side of the

cut corresponds exactly to the vertices between the sources of $\widetilde{D}$ and $\widetilde{A}$. So $\widetilde{A}$ can be flipped and the vertices flipped in this flip sequence correspond to the ideal of A in $J(L)$. □

The construction for $[D]$ used by the theorem does not generally give the only graphs which flip flop generate L. We will see in the next section that the graphs in $[D]$ are vertex number maximal for doing the job.

4.2 Towards a Structure on Digraphs with the Same Flip Flop Poset

We try to analyze the entire set of digraphs that generate a given flip flop poset. Therefore, our aim is to look more precisely how a given digraph D produces a flip flop poset on its c-reorientations. We have seen in Chapter 2 that flip flop posets as the ones coming from the vertex cuts of a graph are integral, i.e. are naturally embedded into $\mathbb{Z}^{|V(D)|-1}$. Every orientations is mapped to an integral point, that counts for every vertex how many times it has been flipped. Heading towards a structure on the set of digraphs that generate a given distributive lattice, it is useful to observe how the digraph embeds the generated distributive lattice into some $\mathbb{Z}^{|V(D)|-1}$.

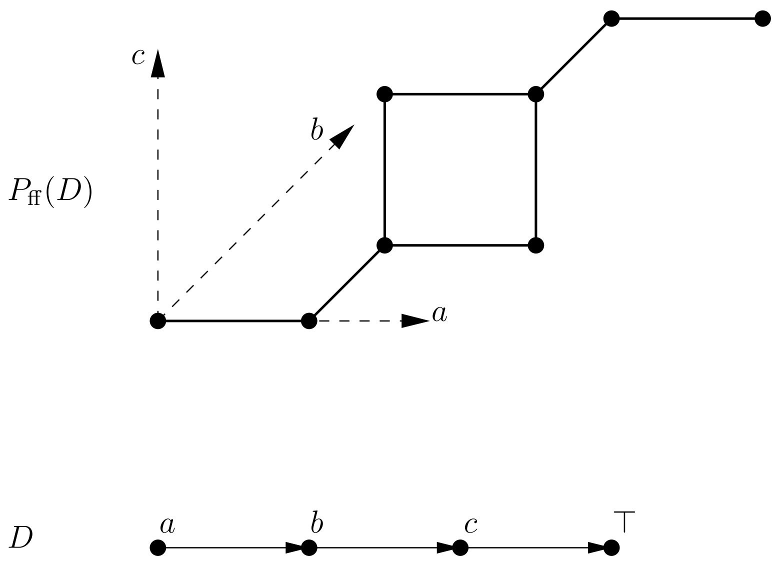

Figure 4.2: A digraph D with associated embedded flip flop poset.

First observe that we can restrict us to acyclic digraphs with unique sink at the forbidden vertex. Let D be an acyclic directed graph, with a forbidden vertex $\top$. All the

c-reorientations of D are elements of D's flip flop poset. Starting the flip flopping at any of these orientations generates the same distributive lattice. The difference between the different embeddings is, that the respective starting orientation is mapped to the $\mathbf{0} \in \mathbb{Z}^m$. So the embeddings coming from any $D' \in$ *reor*$_c(D)$ are all translations of another.

Therefore let the orientation of D be the minimum orientation of the associated flip flop poset. This means that D has exactly one sink, which sits at the forbidden vertex $\top$. From now on we will consider D always like that and denote by $P_{\text{ff}}(D)$ the distributive lattice induced by flip flops of vertex cuts different from the one of $\top$. This way we get an embedding of $P_{\text{ff}}(D)$ whose minimal element is $\mathbf{0} \in \mathbb{Z}^{|V(D)|-1}$.

To understand now the embeddings of flip flop posets we must characterize when a vertex can be flipped the kth time in terms of the other vertex flips. By doing so, the following lemmas will lead to a description of the embedding.

For every vertex $v \in V(D)\backslash\{\top\}$ there is a directed $(v, \top)$-path in D. Denote by $\downarrow v$ the set of vertices w such that there is a directed (w, v)-path and by $\uparrow v$ the the vertices such that a directed (v, w)-path exists.

Lemma 4.2.1. *To flip $v \in V(D)\backslash\{\top\}$ exactly k times every vertex in $\downarrow v$ has to be flipped exactly k times before.*

Proof. Let $w \in \downarrow v$. We proceed by induction on $dist(w, v)$, the length of the shortest directed (w, v)-path in D.
If $dist(w, v) = 1$, we have $a = (w, v) \in A(D)$. Thus, each time v wants to be flipped, w had to be flipped before, because otherwise a would point into v and the vertex cut of v could not be positively directed. On the other hand w cannot have been flipped more than once, without flipping v in between.
If $dist(w, v) > 1$ choose w' as a vertex that is the last vertex on a shortest (w, v)-path before arriving at v. To flip v's vertex cut k times w' must be flipped k times, and as $dist(w, w') < dist(w, v)$, by induction hypothesis, w hast to be flipped exactly as often as w'. □

Lemma 4.2.2. *To flip $v \in V(D)\backslash\{\top\}$ exactly k times every vertex in $w \in \uparrow v$ has to be flipped exactly $k - dist(v, w)$ times before.*

Proof. Let $w \in \uparrow v$. Again, we proceed by induction on $dist(v, w)$.
If $dist(v, w) = 1$, we have $a = (v, w) \in A(D)$. Thus, each time v wants to be flipped, w had to be flipped once after the last time v was flipped, because otherwise a would still point into v and the vertex cut of v could not be positively directed. Obviously w cannot have been flipped more than once, as v would have to be flipped in between to make this possible. Thus before the kth flip of v can be performed, the vertex cut of w has been flipped exactly $k - 1$ times.

If $dist(v, w) > 1$ choose w' as a vertex that is the first vertex on a shortest (v, w)-path after leaving v. To flip v's vertex cut k times w' must be flipped $k - 1$ times, and as $dist(w', w) < dist(v, w)$, by induction hypothesis, w hast to be flipped exactly $k - 1 - dist(w', w) = k - dist(v, w)$ times. □

Denote by $N^-(v)$ the set of vertices that point to v and by $N^+(v)$ the set of vertices v is pointing to. During the whole chapter these terms refer to the starting orientation of D. Moreover $N(v) := N^+(v) \cup N^-(v)$ is called the set of **neighbors** of v.

Lemma 4.2.3. *If we can flip every $w \in N^-(v)$ at least k times and every $u \in N^+(v)$ at least $k - 1$ times, then we can flip v at least k times.*

Proof. For every $u \in N^+(v)$ and $w \in N^-(v)$ denote by u^{k-1} and w^k some orientations where u has been flipped $k - 1$ times or w has been flipped k times, respectively.
Take the orientation of D that is the join of all these orientations with respect to the integral embedding. By Lemma 4.2.2 and Lemma 4.2.1, in order to generate this orientations v has been flipped at least $k - 1$ times. Suppose that v has been flipped exactly $k - 1$ times. By Lemma 4.2.2 and Lemma 4.2.1 the entire $N^+(v)$ has been flipped exactly $k - 1$ times and $N^-(v)$ has been flipped exactly k times. Moreover the vertex cuts of the elements of $N^+(v)$ have been flipped once since the last flip of v. The same holds for the vertex cuts of the elements of $N^-(v)$. But this means that v's vertex cut is positively directed again and can be flipped the kth time. □

The three above Lemmas give rise to a new definition. Introduce for every arc (v, w) in D the oppositely directed arc (w, v). Let the original arcs of D have length 1 and the new auxiliary arcs length 0. Denote by $\pi(v)$ the distance from v to $\top$ in this new graph. We call the function $\pi : V \to \mathbb{Z}_{\geq 0}$ **the potential function of** D.

Lemma 4.2.4. *For every $v \in V$ the value $\pi(v)$ gives the maximal number of v occurring in a flip sequence.*

Proof. First we show that v can be flipped at most $\pi(v)$ times.
Assume that v can be flipped $k > \pi(v)$ times. By the definition of π, there is a shortest path of auxiliary and original arcs with length $\pi(v)$ from v to $\top$. Applying Lemma 4.2.2 and Lemma 4.2.1 along this path one gets that $\top$ has to be flipped, which is impossible.

Now we must prove that v indeed can be flipped $\pi(v)$ times. We proceed by induction on $\pi(v)$.
If $\pi(v) = 0$ we have $v = \top$ and nothing to show.
If $\pi(v) > 0$, first assume to have a v such that $N^+(v)$ consists only of vertices with potential $\pi(v) - 1$. After flipping all the vertices in $N^+(v)$ exactly $\pi(v) - 1$ times, by Lemma 4.2.1 all $\downarrow v \cup \{v\}$ has been flipped at least $\pi(v) - 1$ times.

First observe that for every element $w \in (\downarrow v \cup \{v\})$ we have $\pi(w) \geq \pi(v)$. So every element in $N^+(w)$ can be flipped at least $\pi(v) - 1$ times.
Therefore, if any element $w \in \downarrow v \cup \{v\}$ has been flipped more than $\pi(v) - 1$ we can flip v at least $\pi(v)$ times by applying Lemma 4.2.3 along a directed (w, v)-path.
So suppose all $\downarrow v \cup \{v\}$ has been flipped exactly $\pi(v) - 1$ times. The vertices in $\downarrow v \cup \{v\}$ with longest paths to v (the sources of D in $\downarrow v \cup \{v\}$) have empty N^-. So with the above observation they can be flipped again by Lemma 4.2.3.
This way one can reverse the orientation of v the $\pi(v)$th time.
If not every vertex in $N^+(v)$ has potential $\pi(v) - 1$, we can move towards $\top$ without changing the potential until we arrive at a vertex v' with this property. By Lemma 4.2.1 we obtain, that v can be flipped $\pi(v)$ times. $\square$

Theorem 4.2.5. *For D as above the embedded distributive lattice $P_{\text{ff}}(D)$ is isomorphic to the dominance order on the integral point set given by*

$$\{0 \leq z \leq \pi \mid (v, w) \in A(D) \Rightarrow 0 \leq z(v) - z(w) \leq 1\} \subseteq \mathbb{Z}^{V(D)\setminus\top}.$$

Proof. The isomorphism works by identification of the orientations of D with the vectors $0 \leq z \leq \pi$ that count for every vertex how many times it has been flipped.
Injectivity is obvious, and as $P_{\text{ff}}(D)$ is integral, we have an order-embedding into $\mathbb{Z}^{V(D)\setminus\top}$.
By Lemma 4.2.4, Lemma 4.2.2, and Lemma 4.2.1 we have that $P_{\text{ff}}(D)$ is indeed embedded into $\{0 \leq z \leq \pi \mid (v, w) \in A(D) \Rightarrow 0 \leq z(v) - z(w) \leq 1\} \subseteq \mathbb{Z}^{V(D)\setminus\top}$.
To prove surjectivity let $z \in \{0 \leq z \leq \pi \mid (v, w) \in A(D) \Rightarrow 0 \leq z(v) - z(w) \leq 1\}$. For every vertex $v \neq \top$ by Lemma 4.2.4 and distributivity there is a minimal orientation x_v with $x_v(v) = z(v)$ and $x_v \leq z$. We can take the join $\bigvee_{v \in V(D)\setminus\{\top\}} x_v$ of all these orientations which is still smaller or equal than z, thus by the choice of the orientations x_v it is the same as z. We have obtained surjectivity. $\square$

Now that we have a way to write down the embedded distributive lattice coming from a digraph, we will try to investigate the set of graphs that have the same distributive flip flop lattice, by comparing the embeddings they lead to. Some lead to the same embedding, some do not.

For a distributive lattice L there is a correspondence of the embeddings of L into some $\mathbb{Z}^m$ and the set of chain partitions of $J(L)$. Given a chain partitioned poset $(P, \{C_i\}_{i \in [m]})$ the corresponding distributive lattice of ideals $\mathcal{O}(P)$ will be embedded into $\mathbb{Z}^m$ the following way. Map every ideal I of P to the vector $z_I \in \mathbb{Z}^m$, where $z_I(i) := |I \cap C_i|$. The inverse consists of putting a join irreducible $x \succ y$ in L into the chain C_i if $x - y = e_i$.

In the following we will give a characterization of those chain partitions that correspond to the embedded flip flops acyclic digraphs.

$(P, \{C_a, C_b, C_c\})$

D

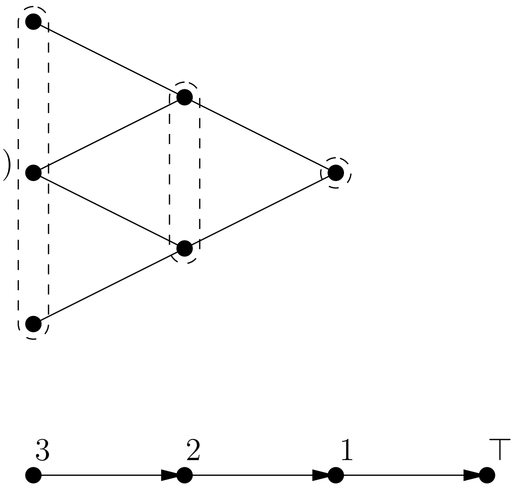

Figure 4.3: On the one hand a good plissée partitioned poset $(P, \{C_a, C_b, C_c\})$, that comes from the embedded distributive lattice of Figure 4.2, with its projection D. On the other hand an acyclic graph D with is potential poset carrying the canonical chain partiton.

For any poset P, we call a chain partition $\{C_i\}_{1 \leq i \leq k}$ **plissée** for every $i, j \in [k]$ we have that $C_i \cup C_j$ contains a cover relation implies $C_i \cup C_j$ is a alternating chain between C_i and C_j. In order to approach the properties of chain partitioned posets coming from embedded flip flop posets, the idea of this definition is to reflect, that adjacent vertices of a digraph can only be flipped in an alternating fashion.

Having P together with a plissée partition $\{C_i\}_{1 \leq i \leq k}$, we define **the projection of** $(P, \{C_i\}_{1 \leq i \leq k})$ as the directed graph $\Delta_{(P, \{C_i\}_{1 \leq i \leq k})} = (V, A)$, where

$$V := \{Min(C_i) \mid 1 \leq i \leq k\} \cup \{\top\}$$

and
$(v, w) \in A :\Leftrightarrow$ either $v <_P w$ and $C_v \cup C_w$ alternates between both chains or $|C_v| = 1$ and $w = \top$.

A plissée partition $(P, \{C_i\})$ is called **good** if the potential function π of the $\Delta_{(P, \{C_i\}_{1 \leq i \leq k})}$ coincides with the values $|C_v|$ that are naturally assigned to the vertices of the projection of $(P, \{C_i\})$.

Given an acyclic digraph D with unique sink $\top$ and potential function π, we define its **potential poset**, as the set $\Pi_D := \{v_i \mid 1 \leq i \leq \pi(v), v \in V(D)\}$ together with the order relation transitively induced by

$$v_i \leq w_j :\Leftarrow i \leq j \textit{ and } ((v, w) \in A(D) \textit{ or } v = w).$$

The potential poset carries a canonical chain partition

$$\{C_v \mid v \in V\} := \{\{v_i \mid 1 \leq i \leq \pi(v)\} \mid v \in V(D)\}.$$

We want to understand the maps between the classes of acyclic directed graphs with unique sink, good plissée partitioned posets, and embedded flip flop posets. They are depicted in Figure 4.4.

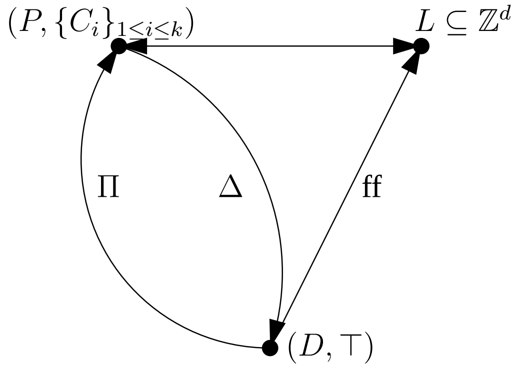

Figure 4.4: The classes of objects involved and the maps between them.

Theorem 4.2.6. *Let D be acyclic with unique sink $\top$. Let $(J(P_{\mathrm{ff}}(D)), \{C_i\}_{i\in[m]})$ be the chain partitioned poset coming from the embedded flip flop poset $P_{\mathrm{ff}}(D)$. Then $(J(P_{\mathrm{ff}}(D)), \{C_i\}_{i\in[m]})$ is isomorphic to$(\Pi(D), \{C_v \mid v \in V\})$ the potential poset of D with its canonical chain partition.*

Proof. We establish a correspondence between the elements of $(J(P_{\mathrm{ff}}(D)), \{C_i\}_{i\in[m]})$ and $(\Pi(D), \{C_v \mid v \in V\})$. It must preserve the respective chain partitions. The idea is to map the join-irreducible orientation v^k of $P_{\mathrm{ff}}(D)$ to the element v_k of $\Pi(D)$. Here v^k stands for the orientation, where v has been flipped exactly k times and is the only vertex with negatively directed vertex cut.

So first observe that in any orientation, where v has just been flipped exactly k times ($1 \leq k \leq \pi(v)$), one can flop other vertex cuts until v is the only floppable. This way one obtains a join-irreducible orientation v^k for every $k \in [\pi(v)]$.
Suppose there were two incomparable join-irreducible orientations v^k and $(v^k)'$ where v has been flipped exactly k times. Take their meet $v^k \wedge (v^k)'$, which corresponds to the meet of the dominance order in $\mathbb{Z}^{|V(D)|-1}$. In $v^k \wedge (v^k)'$ the vertex cut of v has been flipped k times, as well. So the last flip on flip sequences from $v^k \wedge (v^k)'$ to v^k and $(v^k)'$ cannot be the flip of v's vertex cut, so v^k and $(v^k)'$ are not join-irreducible.
We have already obtained an ismorphism between the elements of $(J(P_{\mathrm{ff}}(D)), \{C_i\}_{i\in[m]})$ and $(\Pi(D), \{C_v \mid v \in V\})$, that preserves the chain partitions.

It remains to check if it is an order isomorphism, i.e. $v_k \prec w_l \Leftrightarrow v^k \prec w^l$.
For both directions we clearly have $v \neq w$.

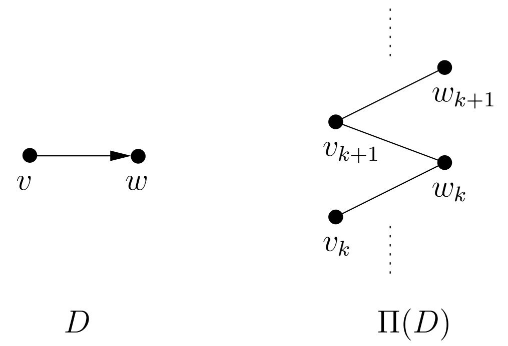

Figure 4.5: Candidates for covering relations in $\Pi(D)$

Moreover by looking at the construction of $\Pi(D)$ (Figure 4.5), one observes that

$$(((v,w) \in A(D) \textit{ and } l = k) \textit{ or } ((w,v) \in A(D) \textit{ and } l = k+1)) \Rightarrow v_k < w_l.$$

and

$$(((v,w) \in A(D) \textit{ and } l = k) \textit{ or } ((w,v) \in A(D) \textit{ and } l = k+1)) \Leftarrow v_k \prec w_l.$$

So we have $v_k \prec w_l$ if and only if there is no longer (v_k, w_l)-path in $\Pi(D)$ consisting of this kind of relations.

We start with "$\Rightarrow$":
Let $v_k \prec w_l$ in $\Pi(D)$. Together with Lemma 4.2.1 and Lemma 4.2.2 having

$$((v,w) \in A(D) \textit{ and } l = k) \textit{ or } ((w,v) \in A(D) \textit{ and } l = k+1))$$

clearly implies $v^k < w^l$. In order to show $v^k \prec w^l$ we observe the following:
If we have $v^k \prec u^m$ in $(J(P_{\mathrm{ff}}(D)), \{C_i\}_{i\in[m]})$ then a pure flip sequence s leading from v^k to u^m must flip a vertex cut x which is incident to the vertex cut of v, in order to destroy the negativity of v's vertex cut. Take x to be the first such vertex in s and look at the orientation just after this flip. Now flop negatively directed vertex cuts different from x, until x is the only negatively directed, i.e. we stand at x^i. The arcs of v, which are not incident to x have not been reoriented during this process, because they point into v. So flopping x gives an orientation, where v is negatively directed and has been flipped exactly k times. As argumented above, this orientation is bigger than v^k.
Therefore one obtains a join-irreducible orientation x^i with $v^k < x^i < u^m$. So $v^k \prec u^m$ implies that u must be incident to v and with Lemma 4.2.1 and Lemma 4.2.2 we get

$$((v,u) \in A(D) \textit{ and } m = k) \textit{ or } ((u,v) \in A(D) \textit{ and } m = k+1)).$$

So suppose there were $v^k \prec u_1^{m_1} \prec \ldots \prec u_n^{m_n} \prec w^l$. Then one has a (v_k, w_l)-path in $\Pi(D)$ consisting of the kind of relations that was forbidden.

For "$\Leftarrow$ it is enough to show $v^k \prec w^l \Rightarrow v_k < w_l$. But as just observed $v^k \prec w^l$ implies that w must be incident to v and

$$((v, w) \in A(D) \textit{ and } l = k) \textit{ or } ((w, v) \in A(D) \textit{ and } l = k + 1)).$$

This implies $v_k < w_l$.

Theorem 4.2.7. *The canonical chain partition of the potential poset, is a good plissée partition.*

Proof. Let D be a graph with unique sink $\top$. It is easy to see that $(\Pi(D), \{C_v \mid v \in V(D)\})$ is a plissée partitioned poset. Let $D' = \Delta_{(\Pi(D),\{C_v|v\in V(D)\})}$ be the projection of the potential poset of D. Obviously $V(D') = V(D)$ and $A(D') \supseteq A(D)$.
To see that the plissée partition is good, we must show that the potential function π' of D' equals the potential function π of D. As $A(D') \supseteq A(D)$ we have $\pi' \leq \pi$.
Suppose there is an arc $a = (u, v)$ in D' but not in D, which comes from an alternating $(C_u \cup C_v)$-chain in $\Pi(D)$ and lets π' be smaller than π. There are two possibilities how this can happen.
On the one hand the new arc can lower $\pi(u)$, i.e. $\pi(v) + 1 < \pi(u)$. So by reasons of cardinality $C_u \cup C_v$ cannot have been an alternating chain.
On the other hand introducing a could lower $\pi(v)$, i.e. $\pi(u) < \pi(v)$. But this again contradicts the fact that $C_u \cup C_v$ is an alternating chain with minmal element in C_u. $\square$

Theorem 4.2.8. *A poset P with a chain partition $\{C_i\}_{1\leq i\leq k}$ is the chain partitioned poset of join-irreducibles of an embedded distributive lattice $P_{\text{ff}}(D)$ if and only if $\{C_i\}_{1\leq i\leq k}$ is a good plissée partition.*

Proof. We begin with " $\Rightarrow$ ":
Let P be the the chain partitioned poset of join-irreducibles of an embedded distributive lattice $P_{\text{ff}}(D)$. By Theorem 4.2.6 it is the potential poset with canonical chain partition coming from D. By Theorem 4.2.7 this is a good plissée partition.
" $\Leftarrow$ ":
Let P be a poset with a chain partition $\{C_i\}_{1\leq i\leq k}$ that is a good plissée partition. It is the potential poset of its own projection, which by Theorem 4.2.6 comes from the corresponding embedded flip flop poset. $\square$

The plissée partition of $J(L)$ we have used to prove Theorem 4.1.1, is the only one that generally exists for every poset. It consists of singletons only.

Recall that our goal is to understand the set of digraphs that generate the same distributive lattice. We indeed are closer now to what we wanted, as we can say that the

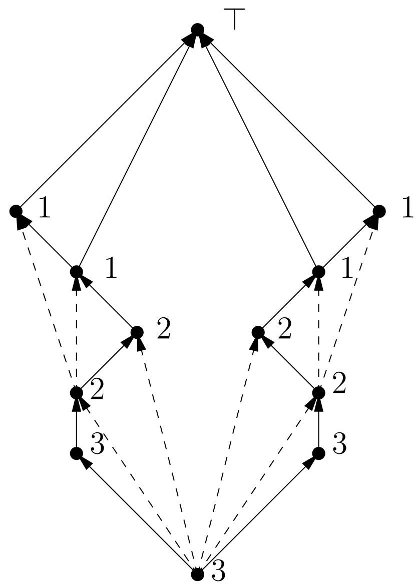

Figure 4.6: Thin projection of a plissée partitioned poset. The desired π-values are written next to the vertices. Arcs that can be added in order to achieve the right potential are drawn dashed. Only one outgoing dashed arc per vertex is needed to repair the potential function.

”essentially different” graphs, that generate our distributive lattice L correspond to the good plissée partitions of $J(L)$.

First we consider the set of ”essentially equal” graphs - those that have the same partitioned potential poset. As in Section 4.1, given a good plissée partitioned poset, we denote them as $[D] := \Pi^{-1}((J(L), \{C_i\}_{1\leq i\leq k}))$. Are they ordered as nicely with respect to arc set inclusion as in Section 4.1?

We have seen in the proof of Theorem 4.2.7 that the projection Δ of a plissée partitioned poset gives the arc maximal digraph among the essentially equal graphs in $[\Delta]$.

Now what can we say about an element $D \in [\Delta]$? Obviously neither the vertex set of two graphs that generate the same chain partitioned poset nor their potential functions can differ. We have seen in the proof of Theorem 4.2.7 that we can add an arc $a = (u, v)$ to such a graph D without leaving $[\Delta]$ if this does not change the potential poset $\Pi(D)$. This can be guaranteed if the corresponding alternating chain $C_u \cup C_v$ consists of transitive arcs only. So adding arcs to D without changing the corresponding chain partition can be done in any order. The graphs in $[\Delta]$ that can be obtained by arc adding from D form a Boolean lattice under arc set inclusion, with minimum D.

Analogously, deleting an arc $a = (u, v)$ is only allowed if the deletion of a does not change π. The problem is, that this cannot be assured by only requiring the corresponding chain $C_u \cup C_v$ to consist of transitive arcs only.

The additional condition for an arc to be allowed to be deleted destroys the Boolean that we had in Theorem 4.1.1. How do the minima of $[\Delta]$ look like? Still there is a fairly easy way to construct the minima of $[\Delta]$, and we know that from these on we have a Boolean lattice in upwards direction.

Given the chain partitioned potential poset $(\Pi(D), \{C_v \mid v \in V(D)\})$, we define the **thin projection** as $\Delta'((\Pi(D), \{C_v \mid v \in V(D)\}))$, which is the graph obtained from the projection, by deleting all the arcs $a = (u, v)$ that come from $C_u \cup C_v$-chains consisting of transitive arcs only. The potential function of the thin projection does not generally coincide with the potential function of the projection π. The arcs that can be added to repair the potential function without changing the potential poset are transitive arcs of the form $a = (u, v)$ with $\pi(u) = \pi(v) + 1$. So the minimal such arc sets take only vertices u which satisfy $\pi(\downarrow u) > \pi(u)$ and introduce some arc of the given form.

In general there is no unique inclusion minmum among these arc sets as exemplified by the thin projection in Figure 4.6.

Now we turn to ”essentially different” graphs. What can we say about them?

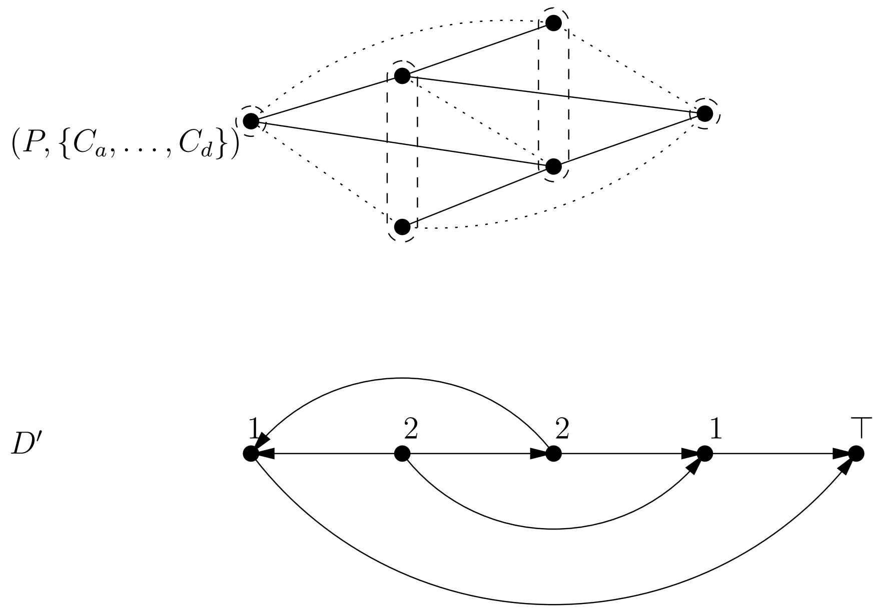

Figure 4.7: Another good plissée partition of the poset P from Figure 4.7 with its projection D'. Transitive arcs that appear in alternating chains are drawn dotted.

For instance the graphs $[D]$ constructed for Theorem 4.1.1 are the vertex number maximal graphs as every chain partition has less chains than the singleton partition. Some posets admit no other plissée partition than that. Take for instance height-two posets. So in this case, we know the unique $[D]$ that generates a given height-two poset.

Take on the other hand the poset in Figure 4.3. There it is drawn with a partition different from the singleton partition. Figure 4.7 shows another one.

Our general aim is to find some order structure carried by the different $[D]$ or equivalently among the set of plissée partitions of a given poset. This could be a way to obtain in some sense small representatives for a given distributive lattice.

A related teasing question is how to characterize classes of distributive lattices that arise from special classes of directed graphs.

One first example into that direction is, that not every distributive lattice comes from the c-reorientations of a planar graph, which equivalently means that not every distributive lattice comes from the α-orientations of a planar graph.

To see this, consider the height-three poset drawn in Figure 4.8. We call it $P_{3,3,3}$.

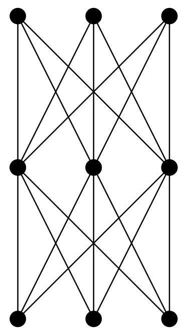

Figure 4.8: Every plissée partition of the poset $P_{3,3,3}$ comes from a graph that has $K_{3,3}$ as a underlying subgraph.

Every chain in a plissée partiton of $P_{3,3,3}$ can have at most 2 elements. One from the top level and one from the bottom level. The thin projection is in all the cases an orientation of a subgraph of $K_{3,3,3}$ that has $K_{3,3}$ as underlying subgraph. More precisely there are up to isomorphism 4 different plissée partitions given by their thin projections $D_0, \dots, D_3$. Counting with i the number of two-element chains in the partition $D_i \cong K_{3,3,3-i}$. So no D that has $P_{3,3,3}$ as its potential poset can be planar.

Conclusions

In this thesis we have developed some theory that particularly opened up a bouquet of questions. We summarize three of the most intriguing directions to further investigation.

The first couple of questions are related to the universality of flip flop sequences:

In Chapter 4 we have shown that every distributive lattice comes from the flip flops of a digraph. As every digraph leads to an embedded distributive lattice, we have characterized the embeddings, that come from digraphs. Our aim is to find some order structure among the directed graphs, which generate the same flip flop poset. It would be very interesting to find a set of directed operations transforming one digraph to another, while leaving the generated distributive lattice invariant. A first step could be to analyze the subposet of the partition lattice induced by the good plissée partitions of a poset. In particular the minima of the resulting partial order are teasing to be understood. This would lead to small representatives for embedded flip flop posets of digraphs.

Moreover it would be interesting to characterize classes of distributive lattices, that are generated by certain classes of graphs. Outerplanar graphs, interval graphs and planar graphs are teasing candidates.

A question concerning a bigger class of flip flop posets is the following. As commented in Proposition 2.2.2, every integral flip flop poset is embedded in some $\mathbb{Z}^m$, such that its elements form the vertices of a polytope. Only very few subposets of $\mathbb{Z}^m$ have this special property. The characterization of embedded flip flop posets coming from graphs in Chapter 4 could maybe be extended in the sense that every (in the above sense) polytopally embedded poset is an integral flip flop poset.

A last question concerning universality is in terms of flip flop graphs. Theorem 2.1.3 proves that every connected loop-free digraph is the flip flop graph generated by some sign matrix. Moreover the arc set of every flip flop graph comes with a natural matching partition. The matching partition that arises from the construction in Theorem 2.1.3 is the trivial one, i.e. every matching consists of a single arc. Can the matching partitions of a digraph, that arise from flip flop sequences be characterized?

Another type of question arises, when attempting to find a small generating sign matrix for a fixed set of reorientations:

Particularly for α-reorientations of graphic oriented matroids, i.e. non-planar digraphs, this is an interesting question. Take such a digraph as embedded into an orientable surface. We want to extend the set of facial cycles by Eulerians that are not 0-homologous. A ”good” extension has to provide two properties:

By Proposition 3.3.4, we know that the extended set must span the digraphs cycle space. This condition is easy to handle by requiring the homology classes of the new Eulerians to span the first Homology group. But we have seen that this is not sufficient. Theorem 3.6.7 characterizes all those 0-Eulerians E which cannot be reversed by facial flip flops. The extension has to be able to repair all these. It is an open question, to identify or construct sets of cycles, which satisfy both these properties.

Suppose we could find ”good” generating systems for the α-reorientations of graphic oriented matroids. We have understood the α-reorientations of $\mathbf{R10}$ and cographic oriented matroids. Moreover we have seen that 1-sums of regular oriented matroids conserve the flip flop structure. The last step to establish a theory of α-reorientations of regular matroids, would be to investigate whether a ”good” flip flop structure on the α-reorientations of two regular oriented matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ leads to a structure on the i-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$, for $i \in \{2, 3\}$.

A last open problem diverges a little more from the subjects treated in the thesis:

We have seen that we cannot generalize the theory of α-orientations to the set of all oriented matroids. As in general oriented matroids there is no canonical choice for a representative of two cocircuits $X, -X$, we have no analogue to in- and outdegree. So we must broaden the set of orientations at our concern. Such a bigger set of graph orientations which is suitable to be generalized consists of those orientations that fix the absolute value of the difference of in- and outdegree. In oriented matroids this invariant coincides with a cocircuit parameter called *imbalance* or *log-discrepancy* in [10]. It is a teasing question whether - specializing back - these graph orientations carry an order structure similar to those on α-reorientations.

Zusammenfassung

Thema dieser Diplomarbeit sind partielle Ordnungen auf Orientierungen und Reorientierungen gerichteter Strukturen. Anlass dazu geben Arbeiten von Felsner [7] und Propp [12].

Felsner konstruiert einen distributiven Verband auf den Orientierungen eines planaren Graphen, die knotenweise denselben Ausgrad haben (α-Orientierungen). Die α-Orientierungen eines Graphen verallgemeinern f-Faktoren, spannende Bäume, eulersche Orientierungen und Schnyder-Wälder.

Propp gibt eine Methode zur Erzeugung eines distributiven Verbands auf den Orientierungen eines (nicht notwendigerweise planaren) Graphen an, deren Invariante die Anzahl der Vorwärtskanten in Kreisen ist (c-Orientierungen).

Die dieser Diplomarbeit zugrundeliegende Motivation besteht in der Frage, wie weit und mit welchen Einschränkungen man Felsners und Propps Ergebnisse auf orientierte Matroide übertragen kann. Es stellt sich heraus, dass ein Verallgemeinerung möglich ist, jedoch zu einer Theorie führt, die nicht mehr so schön ist wie in [7, 12]. Deshalb konzentrieren wir uns ab einem bestimmten Punkt auf spezielle Klassen orientierter Matroide.

Indem wir die Orientierungen eines ungerichteten Graphen mit den Reorientierungen eines gerichteten Graphen identifizieren, ermöglichen wir eine Zusammenführung und Verallgemeinerung der von Felsner und Propp betrachteten Strukturen. Außerdem übertragen wir die Invarianten der untersuchten Reorientierungsklassen in die Terminologie orientierter Matroide und zeigen, dass sie in diesem Sinne dual zueinander sind.

Desweiteren werden die Erzeugungsmethoden (Flip-Flop-Folgen) der distributiven Verbände in [7, 12] reformuliert, um zu zeigen, dass es sich im Wesentlichen um ein und dieselbe handelt. Diese kann nicht nur auf gerichtete Graphen sondern auf beliebige $(1, -1, 0)$-Matrizen angewandt werden. Da orientierte Matroide als $(1, -1, 0)$-Matrizen darstellbar sind, verspricht diese Theorie in unserem Sinne anwendbar zu sein. Flip-Flop-Folgen auf $(1, -1, 0)$-Matrizen generieren nicht mehr nur Hasse-Diagramme distributiver Verbände sondern fast beliebige gerichtete Graphen. Tatsáchlich läßt sich jeder zusammenhängende, schlingenfreie gerichtete Graph als Ergebnis einer solchen Konstruktion darstellen.

Diesem Kontrollverlust entgehend finden wir hinreichende Kriterien für die erzeugenden Matrizen, sodass der resultierende Graph das Hasse-Diagramm eines distributiven Verbandes ist. Zusätzlich ergibt sich eine natürliche Einbettung in das höherdimensionale ganzzahlige Gitter. Als Korollar erhalten wir die Distributivität der Verbände in [7] und [12] nebst deren Einbettung. Diesem Korollar liegt in beiden Fällen eine 2-Basis des Kreis- respektive Schnittraums zugrunde.

Im Folgenden vereinen wir die Erkenntnisse aus der Anlayse der Flip-Flop-Folgen mit geeigneten Verallgemeinung von α-Reorientierungen auf allgemeine orientierte Matroide. Wir zeigen die Existenz eines Flip-Flop-Erzeugendensystems für die α-Reorientierungen beliebiger orientierter Matroide. Im Weiteren stellen wir fest, dass eine strengere Analogie zum Graphenfall im Kontext von Kreisbasen nur bei regulären orientierten Matroiden möglich ist. Dies nehmen wir zum Anlass, unsere Untersuchungen auf eben jene Matroidklasse zu spezialisieren.

Aus Seymours Dekompositionstheorem für reguläre Matroide [13] ergibt sich die Analyse der drei orientierten Splitter: $\mathbf{R10}$, graphische und cographische orientierte Matroide.

- Der Matroid $\mathbf{R10}$ hat eine endliche Anzahl von Reorientierungen, die wir per Computer enumerieren.
- Aus der Theorie der α-Reorientierungen **graphischer orientierter Matroide** erhalten wir als Korollar zusammen mit der bereits bewiesenen Disrtibutivität die Hauptptheoreme aus [7] und [12]. Ein weiteres positives Ergebnis ist eine Halbordnung auf den eulerschen Orientierungen des quadratischen Torusgitters. Ein allgemeines Konstruktionsverfahren für schöne Verbände auf diesen α-Reorientierungen zu finden, bleibt ein offenes Problem.
- Die α-Reorientierungen **cographischer orientierter Matroide** entsprechen den in [12] untersuchten Orientierungen. Deshalb tragen sie die Struktur eines distributiven Verbandes.

Als letztes Ergebnis beweisen wir, dass jeder distributive Verband via Flip-Flop-Folgen aus den α-Reorientierungen eines cographischen orientierten Matroids entspringt. Schließlich beschreiben wir die Menge aller cographischer orientierter Matroide, die einen gegebenen distributiven Verband erzeugen und unternehmen erste Schritte hin zu einer Ordnungsstruktur auf dieser Menge.

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