On the Duality of Semiantichains and Unichain Coverings

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Abstract

We study a min-max relation conjectured by Saks and West: For any two posets P and Q the size of a maximum semiantichain and the size of a minimum unichain covering in the product $P \times Q$ are equal. For positive we state conditions on P and Q that imply the min-max relation. Based on these conditions we identify some new families of posets where the conjecture holds and get easy proofs for several instances where the conjecture had been verified before. However, we also have examples showing that in general the min-max relation is false, i.e., we disprove the Saks-West conjecture.

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1 Introduction

This paper is about min-max relations with respect to chains and antichains in posets. In a poset, *chains* and *antichains* are sets of pairwise comparable and pairwise incomparable elements, respectively. By the *height* $h(P)$ and the *width* $w(P)$ of poset P we mean the size of a largest chain and a largest antichain in P, respectively. The product $P \times Q$ of two posets P and Q is an order defined on the product of their underlying sets by $(u, x) \leqslant_{P \times Q} (v, y)$ if and only if $u \leqslant_P v$ and $x \leqslant_Q y$.

Dilworth [2] proved that any poset P can be covered with a collection of $w(P)$ chains. Greene and Kleitman [5] generalized Dilworth's Theorem. A *k-antichain* in P is a subset of P which may be decomposed into k antichains. We denote the size of a maximal k-antichain of P by $d_k(P)$ or

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simply d_k if the poset is unambiguous from the context. The theorem of Greene and Kleitman says that for every k there is a chain-partition $\mathcal{C}$ of P such that $d_k(P) = \sum_{C \in \mathcal{C}} \min(k, |C|)$. In [12] Saks proves the theorem of Greene and Kleitman by showing the following equivalent statement:

Theorem 1.1. *In a product $C \times Q$ where C is a chain, the size of a maximum antichain A equals the size of a minimum chain covering with chains of the form $\{c\} \times C'$ and $C \times \{q\}$. In particular this number is $d_{|C|}(Q)$.*

The *Saks-West Conjecture* is about a generalization of Theorem 1.1. A chain in a product $P \times Q$ is a *unichain* if it is of the form $\{p\} \times C'$ or $C \times \{q\}$. A *semiantichain* is a set $S \subseteq P \times Q$ such that no two distinct elements of S are contained in an unichain. With this notation we are ready to state the Saks-West conjecture:

Conjecture 1. *In every product $P \times Q$ of two posets the size of a largest semiantichain equals the size of a smallest unichain covering.*

The conjecture had already been around for a while when it appeared in print [22]. Theorem 1.1 deals with the special case of conjecture where one of P and Q is a chain. Several partial results mostly regarding special classes of posets that satisfy the conjecture have been verified.

- Tovey and West [13] relate the problem to dual pairs of integer programs of packing and covering type and explain the interpretation as independence number and clique covering in the product of perfect graphs. Furthermore, they take first steps towards the investigation of posets with special chain- and antichain-decomposability properties. Using these they verify the conjecture for products of posets admitting a symmetric chain decomposition or a skew chain partition. This extends investigations of the largest Whitney numbers, i.e., width, of such products in [8] and [21].

- Tovey and West [14] deepen the study of the conjecture as a dual integrality statement in linear programs using a network flow approach.

- West [20] constructs special unichain coverings for posets with the nested saturation property. These are used to prove the conjecture for products $P_m \times P_m$ where P_k is a member from a special family of polyunsaturated posets introduced in [19].

- According to the abstract of [23], Wu provides another sufficient condition for posets to satisfy the conjecture.

- Liu and West [11] verify three special cases of the Saks-West Conjecture:
  - both posets have width at most 2.
  - both posets have height at most 2.
  - P is a *weak order* (a.k.a. ranking) and Q is a poset of height at most 2 whose comparability graph has no cycles.

This paper is organized as follows. In the last subsection of this introduction we connect semi-antichains and unichains to independent sets and clique covers in products of comparability graphs. This is used to reprove that products of posets of height 2 satisfy the conjecture. In Section 2 we study d- and c-decomposable posets. These are used to state conditions on P and Q that make $P \times Q$ satisfy the conjecture. The main result is Theorem 2.4, it allows us to reproduce known results as well as contribute new classes satisfying the conjecture. In Theorem 2.6 we show that

a new class of *rectangular* posets has the property that whenever P is rectangular, the conjecture holds for $P \times Q$ with arbitrary Q. For negative, in Section 3 we provide a counterexample to the Saks-West Conjecture. In particular we can produce an arbitrary large gap between the size of a largest semiantichain and the size of a smallest unichain covering. In Section 4 we comment on some natural dual versions of the Saks-West Conjecture raised by Trotter and West [15] and conclude with open problems.

1.1 Products of comparability graphs

For a graph G, let $\alpha(G)$ and $\theta(G)$ denote the size of the largest independent set in G and the minimum size of clique covering of G, respectively. The comparability graph of a poset $(P, \leq)$ is denoted G_P. The semiantichain conjecture has a nice reformulation in terms of products of two comparability graphs, where the product $G \square G'$ of two graphs has an edge $(v, v') \sim (u, u')$ iff $v = u$ and $v' \sim u'$ or $v \sim u$ and $v' = u'$. Note that in general $G_{P \times Q} \neq G_P \square G_Q$ indeed if $u <_P v$ and $x <_Q y$, then $(u, x) <_{P \times Q} (v, y)$ by transitivity, but in $G_P \square G_Q$ there is no edge between (u, x) and (v, y). In fact unichains in $P \times Q$ and cliques in $G_P \square G_Q$ are in bijection. Hence, the following holds:

- $\alpha(G_P \square G_Q)$ equals the size of the largest semiantichain in $P \times Q$.

- $\theta(G_P \square G_Q)$ equals the size of the minimum unichain decomposition of $P \times Q$.

Thus we can reformulate Conjecture 1 as:

Conjecture 2. *For any two posets P and Q it holds $\alpha(G_P \square G_Q) = \theta(G_P \square G_Q)$.*

Using this version of the conjecture we now show:

Proposition 1.2. *For posets P and Q of height at most 2 the conjecture is true.*

Proof. The comparability graph of a poset of height 2 is bipartite. Next we observe:

- If G and G' are bipartite graphs, then $G \square G'$ is again bipartite.

Bipartite graphs are perfect, hence in particular $\alpha(G) = \theta(G)$ for every bipartite graph. $\square$

Remark: The identity $\alpha(G) = \theta(G)$ used in the proof can also be obtained directly from Dilworth's theorem, we only have to observe that if G is bipartite, then $G = G_P$ for some poset P and that $\alpha(G) = w(P)$ while $\theta(G)$ equals the minimum size of a chain decomposition of P.

2 Constructions

In this section we obtain positive results for posets admitting special chain and antichain partitions. Dual to the concept of k-antichain we call a subset of P a k-chain if it is the union of k disjoint chains. Similarly to $d_k(P)$ we denote the size of a maximal k-chain of P by $c_k(P)$ or simply c_k. The main tool for our proof is Theorem 2.1, which has been obtained by Greene [4]. The theorem is a common generalization of the Theorem of Greene-Kleitman and its dual which has also be obtained by Greene [4]. Both theorems have been generalized in several directions and have been reproved using different methods. Surveys have been given by Greene and Kleitman [6] and West [18]. A more recent survey on a generalization to directed graphs is [1].

Theorem 2.1. *For any poset P there exists a partition $\lambda^P = \{\lambda_1^P \geq \ldots \geq \lambda_w^P\}$ of $|P|$ such that $c_k(P) = \lambda_1^P + \ldots + \lambda_k^P$ and $d_k(P) = \mu_1^P + \ldots + \mu_k^P$ for each k, where μ^P denotes the conjugate to λ^P.*

Following Viennot [16] we call the Ferrers diagram of λ^P the *Greene diagram* of P, it is denoted $G(P)$. A poset P is *d-decomposable* if it has an antichain partition $A_1, A_2, \ldots, A_h$ with $|\bigcup_{i=1}^{k} A_i| = d_k$ for each k. This is, $|A_k| = \mu_k^P$ for all k. Dual to the notion of d-decomposability we call P *c-decomposable* if it has a chain partition $C_1, C_2, \ldots, C_w$ with $|\bigcup_{i=1}^{k} C_i| = c_k$, i.e., $|C_k| = \lambda_k^P$ for all k. Chain partitions with this property have been referred to as *completely saturated*, see [13, 10].
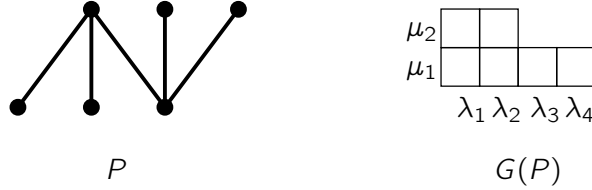

P $\qquad\qquad\qquad\qquad$ $G(P)$

Figure 1: A poset P with its Greene diagram $G(P)$. Note that P is *c*-decomposable but not *d*-decomposable.

For posets P and Q with families of disjoint antichains $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_\ell\}$, respectively, the set $A_1 \times B_1 \cup \ldots \cup A_{\min(k,\ell)} \times B_{\min(k,\ell)}$ is a semiantichain of $P \times Q$. A semiantichain that can be obtained this way is called *decomposable semiantichain*, see [13]. By our definitions we have the following:

Observation 2.2. *If P and Q are d-decomposable with height h_P and h_Q, then $P \times Q$ has a decomposable semiantichain of size*

$$\sum_{i=1}^{\min(h_P, h_Q)} \mu_i^P \mu_i^Q.$$

In order to construct unichain coverings for $P \times Q$ one can apply Theorem 1.1 repeatedly. The resulting coverings are called *quasi-decomposable* in [13]. More precisely:

Proposition 2.3. *In a product $P \times Q$ where $\mathcal{C}$ is a chain covering of P there is a unichain covering of size*

$$\sum_{C \in \mathcal{C}} d_{|C|}(Q).$$

Proof. Use Theorem 1.1 on every $C \times Q$ for $C \in \mathcal{C}$ to get a unichain covering of size $d_{|C|}(Q)$. The union of the resulting unichain coverings is a unichain covering of $P \times Q$. $\qquad\square$

The following theorem has already been noted implicitly by Tovey and West in [13].

Theorem 2.4. *If P is d-decomposable and c-decomposable and Q is d-decomposable, then the size of a maximum semiantichain and the size of a minimum unichain covering in the product $P \times Q$ are equal. The size of these is obtained by the two above constructions, i.e.,*

$$\sum_{i=1}^{\min(h_P, h_Q)} \mu_i^P \mu_i^Q = \sum_{j=1}^{w(P)} d_{\lambda_j^P}(Q).$$

Proof. Since P and Q are *d*-decomposable, there is a semiantichain of size $\sum_{i=1}^{\min(h_P, h_Q)} \mu_i^P \mu_i^Q$ by Observation 2.2. On the other hand if we take a chain covering $\mathcal{C}$ of P witnessing that P is *c*-decomposable we obtain a unichain covering of size $\sum_{j=1}^{w(P)} d_{\lambda_j^P}(Q)$ with Proposition 2.3. We have to prove that these values coincide. To this end consider the Greene diagrams $G(P)$ and $G(Q)$. Their

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merge $G(P, Q)$ (see Figure 2) is the set of unit-boxes at coordinates (i, j, k) with $j \leq w(P) = \mu_1^P$, $i \leq \min(\lambda_j^P, h_Q)$, and $k \leq \mu_i^Q$. Counting the boxes in $G(P, Q)$ by i-slices we obtain the left hand side of the formula. A given j-slice contains $\mu_1^Q + \ldots + \mu_{\min(\lambda_j^P, h_Q)}^Q = d_{\lambda_j^P}(Q)$ boxes. Thus counting the boxes in $G(P, Q)$ by j-slices yields the right hand side of our formula. This concludes the proof. $\square$

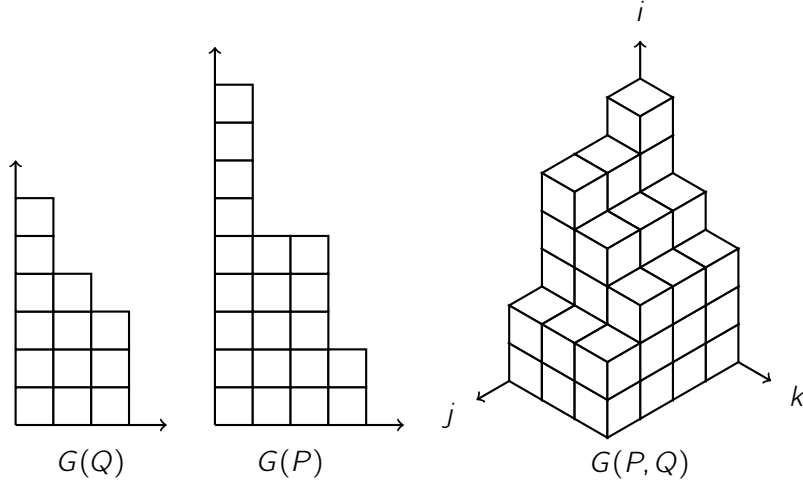$$G(Q) \qquad G(P) \qquad G(P, Q)$$

Figure 2: Merge of two Greene diagrams.

Theorem 2.4 includes some interesting cases for the min-max relation that have been known but also adds a few new cases. These instances follow from proofs that certain classes of posets are d-decomposable, respectively c-decomposable.

A graded poset P whose ranks yield an antichain partition witnessing that P is d-decomposable is called *strongly Sperner*, see [9]. For emphasis we repeat

- Strongly Sperner posets are d-decomposable.

For a chain C in P denote by $r(C)$ the set of ranks used by C. A chain-partition $\mathcal{C}$ of P is called *nested* if for each $C, C' \in \mathcal{C}$ we have $r(C) \subseteq r(C')$ if $|C| \leq |C'|$. The most examples of nested chain partitions are symmetric chain partitions. In [9] Griggs observes that nested chain-partitions are completely saturated and that posets admitting a nested chain partition are strongly Sperner. Hence we have the following

- Posets that have a nested chain partition are d-decomposable and c-decomposable.

The fact that products of posets with nested chain partitions satisfy the Saks-West Conjecture was shown by West [20]. A special class of strongly Sperner posets are *LYM posets*. A conjecture of Griggs [7] that remains open [3, 17] and seems interesting in our context is that LYM posets are c-decomposable.

- Orders of width at most 3 are d-decomposable.

Proof. Since P has width at most 3 we have $\mu_i \in \{1, 2, 3\}$ for all $i \leq h_P$. Let a, b, c be the numbers of 3s, 2s, and 1s in $\mu_1, \ldots, \mu_k$, respectively. We will find an antichain partition of P such that a antichains will be of size 3, b antichains will be of size 2 and c antichains will have size 1. Let $A \subseteq P$ be a maximum $(a + b)$-antichain. From Greene's Theorem (Thm. 2.1) we know that $|A| = 3a + 2b$. Since $|P - A| = c$ we can partition this set into c antichains of size 1. Now consider a partition $\bigcup_{i=1}^{h_A} B_i$ of A such that B_i is the set of minimal points in $B_i \cup \ldots \cup B_{h_A}$. Since $|B_i| \leqslant 3$ we may

consider a', b', c' as the numbers of 3s, 2s, and 1s in all $|B_i|$ (for $i = 1, \ldots, h_A$). With these numbers we have $|A| = 3a' + 2b' + c' = 3a + 2b$ and $h_A = a' + b' + c' = a + b$. Note that $a' \leqslant a$, otherwise we would have an $(a+1)$-antichain of size $3(a+1) > d^A_{a+1}$. Since $|A| - 2h_A = a = a' - c'$ we obtain $c' = 0$, $a' = a$ and $b' = b$. This completes the proof. $\qquad\square$

- Series-parallel orders are d-decomposable and c-decomposable.

- Weak orders are d-decomposable and c-decomposable.

Since weak orders are a subclass of series-parallel orders the second item follows from the first which is implied by the following lemma.

Lemma 2.5. *If P and P' are d-decomposable (resp. c-decomposable), then the same holds for their series composition $P * Q$ and their parallel composition $P + Q$.*

Proof. For d-decomposability let $\mathcal{A} = \{A_1, \ldots, A_{h(P)}\}$ and $\mathcal{A}' = \{A'_1, \ldots, A'_{h(P')}\}$ be witnesses for d-decomposability of P and P', respectively. Ordering the antichains of $\mathcal{A} \cup \mathcal{A}'$ by size yields a witness for d-decomposability of $P * P'$. For the parallel composition $P + P'$ note that $(A, A') \longleftrightarrow A \cup A'$ is a bijection between pairs of antichains with $A \subseteq P$ and $A' \subseteq P'$ and antichains in $P + P'$. Therefore, $d_k(P + P') = d_k(P) + d_k(P')$ and the antichain partition $\{A_1 \cup A'_1, \ldots, A_h \cup A'_h\}$ of $P + P'$ proves d-decomposability (we let $h = \max(h(P), h(P'))$ and use empty antichains for indices exceeding the height).

For c-decomposability the same proof applies but with roles changed between $P + P'$ and $P * P'$, i.e., ordering the chains of $\mathcal{C} \cup \mathcal{C}'$ by size yields a witness for c-decomposability of $P + P'$ while c-decomposability of $P * P'$ is witnessed by the chain partition $\{C_1 \cup C'_1, \ldots, C_w \cup C'_w\}$. $\qquad\square$

With the next result we provide a rather general extension of Theorem 1.1, i.e., we exhibit a class of posets such that every product with one of the factors from the class satisfies the conjecture.

A poset P is *rectangular* if P contains a poset L consisting of the disjoint union of w chains of length h and P is contained in a weak order U of height h with levels of size w. Here containment is meant as an inclusion among binary relations. Note that since they may contain maximal chains of size $< h$ rectangular posets need not be graded. Still there is natural concept of rank and of a nested chain-decomposition, these can be used to show that rectangular posets are c-and d-decomposable. Even more can be said:

Theorem 2.6. *In a product $P \times Q$ where P is rectangular of width w and height h the size of a largest semiantichain equals the size of a smallest unichain covering. Moreover, this number is $w d_h(Q)$.*

Proof. P contains a poset L consisting of the disjoint union of w chains of length h and P is contained in a weak order U of height h with levels of size w. Using Proposition 2.3 we obtain a unichain covering of $L \times Q$ of size $\sum_{i=1}^{w} d_h(Q) = w d_h(Q)$. This unichain covering is also a unichain covering of $P \times Q$ of the required size. On the other hand in $U \times Q$ we can find a decomposable semiantichain as a product of the ranks of U with the antichain decomposition $B_1, \ldots, B_h$ of a maximal h-antichain in Q. The size of this semiantichain is then $\sum_{i=1}^{h} w|B_i| = w d_h(Q)$ in $U \times Q$. The semiantichain of $U \times Q$ is also a semiantichain of $P \times Q$. This concludes the proof. $\qquad\square$

3 A bad example

To simplify the analysis of the counterexample we use the following property of weak orders which may be of independent interest.

Proposition 3.1. *If P is a weak order and Q is an arbitrary poset, then the maximal size of a semiantichain in $P \times Q$ can be expressed as $\sum_{i=1}^{k} \mu_i^P \cdot |B_i|$ where $B_1, B_2, \ldots, B_k$ is a family of disjoint antichains in Q.*

Proof. Let S be a semiantichain in $P \times Q$. For any $X \subseteq P$ denote by $S(X) := \{q \in Q \mid p \in X, (p, q) \in S\}$. Recall that for any $p \in P$ the set $S(\{p\})$ (or shortly $S(p)$) is an antichain in Q. Now take a level $A_i = \{p_1, \ldots, p_k\}$ of P and let B_i be a maximum antichain among $S(p_1), \ldots, S(p_k)$. Replacing $\{p_1\} \times S(p_1), \ldots, \{p_k\} \times S(p_k)$ in S by $A_i \times B_i$ we obtain S' with $|S'| \geq |S|$. Moreover, since P is a weak order the $S(A_i)$ are mutually disjoint. This remains true in S'. Thus S' is a semiantichain. Applying this operation level by level we construct a decomposable semiantichain of the desired size. $\square$
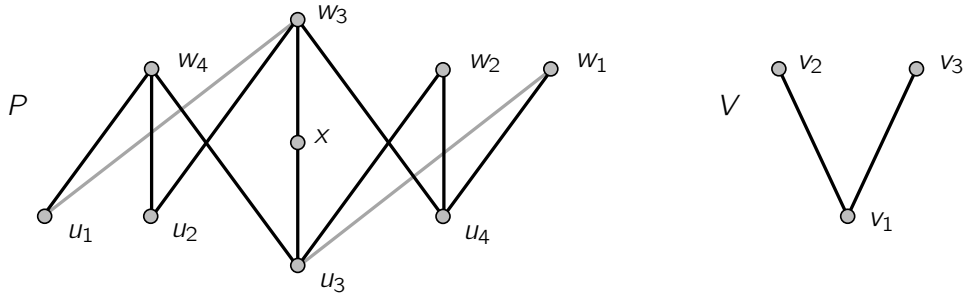

Figure 3: A pair (P, V) of posets disproving the conjecture. The comparabilities depicted in gray are optional. The argument does not depend on whether they belong to P or not.

Let P and V be the posets shown on in Figure 3. Since V is a weak-order we can use Proposition 3.1 to determine the size of a maximum semiantichain in $P \times V$. We just need to maximize formula $2a + b$ where a and b are the sizes of non-intersecting antichains in P. Starting with the observation that there is a unique antichain of size 5 in P it is easily seen that the optimal value is 12 and can be attained as $2 \cdot 5 + 2$ or as $2 \cdot 4 + 4$. We focus on the following maximum semiantichain:

$$S = \{u_1, u_2, u_3, u_4\} \times \{v_2, v_3\} \cup \{w_1, w_2, w_3, w_4\} \times \{v_1\}$$

If there is an optimal unichain covering of size 12 then every unichain has to contain one element an element of S. This implies that the points (x, v_i) have to be in three different unichains. These three unichains can cover all points (u_3, v_i) and (w_3, v_i) from S. To cover $\{u_1, u_2, w_4\} \times V$ we need at least 5 unichains. Another 5 unichains are needed for $\{w_1, w_2, u_4\} \times V$. Therefore, any cover of $P \times V$ consists of at least 13 unichains.

The above construction can be modified to make the gap between a maximum semiantichain and a minimum unichain covering arbitrary large. To see that, just replace V by a height 2 weak order V' with k minima and $k + 1$ maxima. Now consider P' arising from P by blowing up the antichains $\{u_1, u_2\}$ and $\{w_1, w_2\}$ to antichains of size $k + 1$ (again edges between elements from the u-antichain and w_3 as well as edges between elements from the w-antichain and u_3 are optional). Along the lines of the above proof the following can be shown:

Remark 3.2. *The gap between the size of a maximum semiantichain and a minimum unichain covering in $P' \times V'$ is k.*

Recall that there is no gap if one factor of the product is rectangular (see Theorem 2.6). The factor V' is *almost* rectangular but the gap is large.

The class of two-dimensional posets was considered in [11] as the next candidate class for verifying the conjecture. However, in the above construction both factors are two-dimensional.

In a computer search we have identified the poset P shown in Figure 3 (respectively the four posets that can be obtained by choosing a selection of the gray edges) as the unique minimal examples for a P such that $P \times V$ is a counterexample to the conjecture. One can check that all posets with at most 5 elements are both d-decomposable and c-decomposable. It follows from Theorem 2.4 that their products have semiantichains and unichain decompositions of the same size. Also, clearly a posets of size at most 2 is either a chain or an antichain. Hence we obtain:

Corollary 3.3. *The four examples for P represented by the left part of Figure 3 together with V and the dual V^d are the only counterexample to the conjecture with at most 27 elements.*

In the conference version of the paper we have been using a poset P with 13 elements to show that the conjecture fails for $P \times V$.

4 Further comments

4.1 A question of Trotter and West

In [15] it is shown that the size of a minimum semichain covering equals the size of a largest uniantichain. They continue to define a *uniantichain* as an antichain in $P \times Q$ in which one of the coordinates is fixed, and a *semichain* is a set $T \subseteq P \times Q$ such that no two distinct elements of T are contained in an uniantichain. They state the open problem whether the size of a minimum uniantichain covering always equals the size of a largest semichain. Recall the version of the Saks-West conjecture stated as Conjecture 2. A uniantichain is a clique in $\overline{G_P} \square \overline{G_Q}$ where $\overline{G_s}$ denotes the complement of the comparability graph of poset S. The question about minimum uniantichain coverings and maximal semichains in products therefore translates to the question whether

$$\alpha(\overline{G_P} \square \overline{G_Q}) = \theta(\overline{G_P} \square \overline{G_Q})$$

holds for all posets P and Q. We have:

- This duality does not hold in general.

Proof. Let P^* and V^* be the two-dimensional conjugates of P and V from the previous section. The definition of conjugates implies $\overline{G_{P^*}} = \overline{G_P}$ and $\overline{G_{V^*}} = \overline{G_V}$, hence, $P^* \times V^*$ is an example of a product where the cover requires more uniantichains than the size of a maximum semichain. $\square$

Our positive results may also be "dualized" to provide conditions for equality and classes of posets where the size of a minimum uniantichain covering always equals the size of a largest semichain. We exemplify this by stating the dual of Theorem 2.4:

- If P is c-decomposable and d-decomposable and Q is c-decomposable, then the size of a maximum semichain and the size of a minimum uniantichain covering in the product $P \times Q$ are equal. The size of these is

$$\sum_{i=1}^{\min(w_P, w_Q)} \lambda_i^P \lambda_i^Q = \sum_{j=1}^{h(P)} c_{\mu_j^P}(Q).$$

4.2 Open problems

In the present paper the concept of d-decomposability is of some importance. This notion is also quite natural in the context of Greene-Kleitman Theory. We wonder if there is any "nice" characterization of d-decomposable posets. Let P be the six-element poset on $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ with $x_i \leqslant y_2$ and $x_2 \leqslant y_i$ for all $i = 1, 2, 3$. Is it true that any poset excluding P as an induced subposet is d-decomposable?

Another set of questions arises when considering complexity issues. How hard are the optimization problems of determining the size of a largest semiantichain or a smallest unichain covering for a given $P \times Q$? What is the complexity of deciding whether $P \times Q$ satisfies the Saks-West Conjecture? In particular, what can be said about the above questions in the case that Q is the poset V from Figure 3.

5 Acknowledgments

References

[1] I. Ben-Arroyo Hartman, *Berge's conjecture on directed path partitions—a survey*, Discrete Math., 306 (2006), pp. 2498–2514.

[2] R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. (2), 51 (1950), pp. 161–166.

[3] E. Escamilla, A. Nicolae, P. Salerno, S. Shahriari, and J. Tirrell, *On nested chain decompositions of normalized matching posets of rank 3*, Order, 28 (2011), pp. 357–373.

[4] C. Greene, *Some partitions associated with a partially ordered set*, J. Combinatorial Theory Ser. A, 20 (1976), pp. 69–79.

[5] C. Greene and D. J. Kleitman, *The structure of Sperner k-families*, J. Combinatorial Theory Ser. A, 20 (1976), pp. 41–68.

[6] C. Greene and D. J. Kleitman, *Proof techniques in the theory of finite sets*, in Studies in combinatorics, vol. 17 of MAA Stud. Math., Math. Assoc. America, Washington, D.C., 1978, pp. 22–79.

[7] J. R. Griggs, *Sufficient conditions for a symmetric chain order*, SIAM J. Appl. Math., 32 (1977), pp. 807–809.

[8] J. R. Griggs, *Symmetric Chain Orders, Sperner Theorems, and Loop Matchings*, PhD thesis, MIT, 1977.

[9] J. R. Griggs, *On chains and Sperner k-families in ranked posets*, J. Combin. Theory Ser. A, 28 (1980), pp. 156–168.

[10] J. R. Griggs, *Matchings, cutsets, and chain partitions in graded posets*, Discrete Math., 144 (1995), pp. 33–46.

[11] Q. Liu and D. B. West, *Duality for semiantichains and unichain coverings in products of special posets*, Order, 25 (2008), pp. 359–367.

[12] M. Saks, *A short proof of the existence of k-saturated partitions of partially ordered sets*, Adv. in Math., 33 (1979), pp. 207–211.

[13] C. A. Tovey and D. B. West, *Semiantichains and unichain coverings in direct products of partial orders*, SIAM J. Algebraic Discrete Methods, 2 (1981), pp. 295–305.

[14] C. A. Tovey and D. B. West, *Networks and chain coverings in partial orders and their products*, Order, 2 (1985), pp. 49–60.

[15] L. E. Trotter, Jr. and D. B. West, *Two easy duality theorems for product partial orders*, Discrete Appl. Math., 16 (1987), pp. 283–286.

[16] X. G. Viennot, *Chain and antichain families, grids and Young tableaux*, in Orders: description and roles (L'Arbresle, 1982), vol. 99 of North-Holland Math. Stud., North-Holland, Amsterdam, 1984, pp. 409–463.

[17] Y. Wang, *Nested chain partitions of LYM posets*, Discrete Appl. Math., 145 (2005), pp. 493–497.

[18] D. B. West, *Parameters of partial orders and graphs: packing, covering, and representation*, in Graphs and Order, I. Rival, ed., D. Reidel Dordrecht, 1985, pp. 267–350.

[19] D. B. West, *Poly-unsaturated posets: the Greene-Kleitman theorem is best possible*, Journal of Combinatorial Theory. Series A, 41 (1986), pp. 105–116.

[20] D. B. West, *Unichain coverings in partial orders with the nested saturation property*, Discrete Math., 63 (1987), pp. 297–303.

[21] D. B. West and D. J. Kleitman, *Skew chain orders and sets of rectangles*, Discrete Math., 27 (1979), pp. 99–102.

[22] D. B. West and M. Saks, *Research problem 10*, Discrete Math., 38 (1982), p. 126.

[23] C. Wu, *On relationships between semi-antichains and unichain coverings in discrete mathematics*, Chinese Quart. J. Math., 13 (1998), pp. 44–48.