

A graph-theoretical axiomatization of oriented matroids

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Abstract

We characterize which systems of sign vectors are the cocircuits of an oriented matroid in terms of the cocircuit graph.

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1 Introduction

The cocircuit graph is a natural combinatorial object associated with an oriented matroid. For example, in the case of spherical pseudoline-arrangements,

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i.e., rank 3 oriented matroids, the vertices of the cocircuit graph are the intersection points of pseudolines and two points share an edge if they are adjacent on a pseudoline. More generally, the Topological Representation Theorem of Folkman and Lawrence [10] says that every oriented matroid can be represented as an arrangement of pseudospheres. The cocircuit graph is the 1-skeleton of this arrangement. In particular, the cocircuit graph is invariant under reorientation of the oriented matroid. Cordovil, Fukuda and Guedes de Oliveira [6] show that there are non-isomorphic oriented matroids having isomorphic cocircuit graphs. On the other hand Babson, Finschi and Fukuda [1] show that *uniform* oriented matroids are determined up to isomorphism by their cocircuit graph. Moreover they provide a polynomial time recognition algorithm for cocircuit graphs of uniform oriented matroids. In [12], Montellano-Ballesteros and Strausz give a characterization of uniform oriented matroids in view of sign labeled cocircuit graphs. This characterization is strengthened by Felsner, Gómez, Knauer, Montellano-Ballesteros and Strausz [9] and used to improve the recognition algorithm of [1].

In this paper we present a generalization and strengthening of the characterization of sign labeled cocircuit graphs of uniform oriented matroids of [9] to general oriented matroids. After introducing the necessary preliminaries in the next section, we prove the main theorem in the last section.

2 Preliminaries

Here we will only introduce the terminology necessary for proving our result, for a more general introduction into oriented matroids and proofs of the basic claims in the present section, see [4]. A *signed set* X on a ground set E is an ordered pair $X = (X^+, X^-)$ of disjoint subsets of E . For $e \in E$ we write $X(e) = +$ and $X(e) = -$ if $e \in X^+$ and $e \in X^-$, respectively, and $X(e) = 0$, otherwise. The *support* \underline{X} of a signed set X is the set $X^+ \cup X^-$. The *zero-support* of X is $X^0 := E \setminus \underline{X}$. By $-X$ we denote the signed set (X^-, X^+) . Given signed sets X, Y their *separator* is defined as $S(X, Y) := (X^+ \cap Y^-) \cup (X^- \cap Y^+)$.

Definition 2.1 A pair $\mathcal{M} = (E, \mathcal{C}^*)$ is called an *oriented matroid* on the ground set E with *cocircuits* \mathcal{C}^* if \mathcal{C}^* is a system of signed sets on the ground set E , satisfying the following axioms:

- (C0) $\emptyset \notin \mathcal{C}^*$
- (C1) $\mathcal{C}^* = -\mathcal{C}^*$
- (C2) if $X, Y \in \mathcal{C}^*$ and $\underline{X} \subseteq \underline{Y}$ then $X = \pm Y$

(C3) for all $X, Y \in \mathcal{C}^*$ with $X \neq \pm Y$ and $e \in S(X, Y)$ exists $Z \in \mathcal{C}^*$ with $Z(e) = 0$, $Z^+ \subseteq X^+ \cup Y^+$ and $Z^- \subseteq X^- \cup Y^-$

Given signed sets X, Y their *composition* is the signed set $X \circ Y$ defined as $(X^+ \cup (Y^+ \setminus X^-), X^- \cup (Y^- \setminus X^+))$. If \mathcal{S} is a system of signed sets we denote by $\mathcal{L}(\mathcal{S}) := \{X_1 \circ \dots \circ X_k \mid X_1, \dots, X_k \in \mathcal{S}\}$ the set of all (finite) compositions of \mathcal{S} . The empty set is considered as the empty composition of signed sets, and so $\emptyset \in \mathcal{L}(\mathcal{S})$. One can endow $\mathcal{L}(\mathcal{S})$ with a partial order relation where $Y \leq X$ if and only if $S(X, Y) = \emptyset$ and $\underline{Y} \subseteq \underline{X}$. Adding a global maximum $\hat{1}$ this partial order becomes a lattice denoted by $\mathcal{F}_{\text{big}}(\mathcal{S}) := (\mathcal{L}(\mathcal{S}) \cup \{\hat{1}\}, \leq)$. The lattice property is easily seen: the empty set is the global minimum and the unique join of two signed sets X, Y is $\hat{1}$ if $S(X, Y) \neq \emptyset$ and $X \circ Y$ otherwise.

There are two important undirected graphs associated to $\mathcal{F}_{\text{big}}(\mathcal{S})$ – one on its atoms and one on its coatoms. So the first is a graph $G(\mathcal{S})$ with vertex set \mathcal{S} such that two signed sets $X, Y \in \mathcal{S}$ are connected by an edge if and only if there is $Z \in \mathcal{L}(\mathcal{S}) \cup \{\hat{1}\}$ such that X, Y are the only elements of \mathcal{S} with $X, Y \leq Z$.

The second graph is called the *tope graph* $G(\mathcal{T})$. It is defined on the set \mathcal{T} of coatoms of $\mathcal{F}_{\text{big}}(\mathcal{S})$. The elements of \mathcal{T} are called *topes*. Topes $S, T \in \mathcal{T}$ are contained in an edge of $G(\mathcal{T})$ if and only if there is $Z \in \mathcal{L}(\mathcal{S})$ such that S, T are the only elements of \mathcal{T} with $X, Y \geq Z$.

Let G be any graph on a system \mathcal{R} of signed sets with ground set E . For $X_1, \dots, X_k \in \mathcal{R}$ we denote by $[X_1, \dots, X_k]$ the subgraph of G induced by $\{Z \in \mathcal{R} \mid Z(e) \in \{0, X_1(e), \dots, X_k(e)\} \text{ for all } e \in E\}$. We call $[X_1, \dots, X_k]$ the *crabbed hull* of X_1, \dots, X_k . An (X, Y) -path in G is called *crabbed* if it is contained in $[X, Y]$.

If \mathcal{S} is the system of cocircuits \mathcal{C}^* of an oriented matroid \mathcal{M} , then the elements of $\mathcal{L}(\mathcal{C}^*)$ are called the *covectors* of \mathcal{M} . Moreover, $\mathcal{F}_{\text{big}}(\mathcal{C}^*)$ is a *graded* lattice with rank function r , see [4]. In this case $\mathcal{F}_{\text{big}}(\mathcal{C}^*)$ is called *the big face lattice* of \mathcal{M} . The *rank* $\text{rk}(\mathcal{M})$ of \mathcal{M} is defined as $r(\hat{1}) - 1$, i.e., one less than the rank of $\mathcal{F}_{\text{big}}(\mathcal{C}^*)$. Moreover, $G(\mathcal{C}^*)$ is called the *cocircuit graph* of \mathcal{M} .

One important oriented matroid operation is the *contraction*, again proofs for its properties can be found in [4]. For a subset $A \subseteq E$, the contraction of A yields an oriented matroid \mathcal{M}/A on the ground set $E \setminus A$ with $\mathcal{C}^*/A := \{X \setminus A \mid X \in \mathcal{C}^* \text{ and } A \subseteq X^0\}$. The set $\mathcal{L}(\mathcal{C}^*/A)$ is $\{X \setminus A \mid X \in \mathcal{L}(\mathcal{C}^*) \text{ and } A \subseteq X^0\}$. Furthermore, for $U \in \mathcal{L}(\mathcal{C}^*)$ we have $\text{rk}(\mathcal{M}/U^0) = r(U)$, where $r(U)$ is the rank of U in $\mathcal{F}_{\text{big}}(\mathcal{C}^*)$.

3 Result

In order to prove Theorem 3.3 we need two lemmas. The first one is about tope graphs of oriented matroids. Tope graphs of oriented matroids are a special class of *partial cubes* [11]. We will make use of a particular consequence of this, which in our terminology reads like:

Lemma 3.1 ([5]) *Let \mathcal{M} be an oriented matroid with topes \mathcal{T} . For all $U, V \in \mathcal{T}$ there is a crabbed (U, V) -path in $G(\mathcal{T})$.*

The second lemma establishes a connection between tope graph and co-circuit graph. As an application of a theorem of Barnette [2], Cordovil and Fukuda prove:

Lemma 3.2 ([6]) *Let \mathcal{M} be an oriented matroid of rank r and $U \in \mathcal{T}$ a tope of \mathcal{M} . The graph $G(U)$ induced by $\{X \in \mathcal{C}^* \mid X \circ U = U\}$ in $G(\mathcal{C}^*)$ is $(r - 1)$ -connected.*

Together this enables us to prove a graph-theoretical axiomatization of oriented matroids:

Theorem 3.3 *Let \mathcal{S} be a set of sign vectors satisfying (C0)–(C2) then the following are equivalent*

- (i) \mathcal{S} is the system \mathcal{C}^* of cocircuits of an oriented matroid \mathcal{M} ,
- (ii) for any $X_1, \dots, X_k \in \mathcal{S}$ the crabbed hull $[X_1, \dots, X_k]$ is an induced subgraph of $G(\mathcal{S})$ of connectivity $h(X_1 \circ \dots \circ X_k) - 1$, where h is the height-function of $\mathcal{F}_{\text{big}}(\mathcal{S})$,
- (iii) for all $X, Y \in \mathcal{S}$ with $X \neq \pm Y$ there is a crabbed (X, Y) -path in $G(\mathcal{S})$.

Proof. (i) \implies (ii): Let $U := X_1 \circ \dots \circ X_k$ be a covector of rank $\text{rk}' := h(X_1 \circ \dots \circ X_k)$ and X, Y cocircuits in $[X_1, \dots, X_k]$. Contract U^0 obtaining $\mathcal{M}' := \mathcal{M}/U^0$ of rank rk' , and topes \mathcal{T}' . Since $U^0 \subseteq X^0, Y^0, X_i^0$ for $i = 1, \dots, k$ the contraction does not affect the set of cocircuits we are considering other than changing the ground set. We denote them with respect to the smaller ground set by X', Y', X'_i for $i = 1, \dots, k$. In particular, for the crabbed hull we are considering we have $[X_1, \dots, X_k] \cong [X'_1, \dots, X'_k]$. Now U' is a tope of \mathcal{M}' and so are $V' := X' \circ U'$ and $W' := Y' \circ U'$. By Lemma 3.1 there is a crabbed (V', W') -path $P = (V' = T_1, \dots, T_k = W')$ in $G(\mathcal{T}')$. The graphs $G(T_i)$ are all contained in $[X'_1, \dots, X'_k]$ and $(r - 1)$ -connected by Lemma 3.2. Consecutive T_i and T_{i+1} differ only with respect to the sign of a single element – say e . Thus, the intersection of $G(T_i)$ and $G(T_{i+1})$ is the graph of a tope $T_{i,i+1}$ of the rank $\text{rk}' - 1$ oriented matroid \mathcal{M}'/e . By Lemma 3.2 the graph

$G(T_{i,i+1})$ is $(\text{rk}' - 1)$ -connected. Hence, in particular $G(T_i)$ and $G(T_{i+1})$ share at least $\text{rk}' - 1$ vertices. Together with Menger's theorem (see e.g. [8]) this yields that the graph $G(T_1) \cup \dots \cup G(T_k)$ is $(\text{rk}' - 1)$ -connected. In particular there are $\text{rk}' - 1$ internally disjoint paths connecting X' and Y' in $[X'_1, \dots, X'_k]$ and thus the analogue holds for X and Y in $[X_1, \dots, X_k]$. Hence $[X_1, \dots, X_k]$ is $(h(X_1 \circ \dots \circ X_k) - 1)$ -connected.

(ii) \implies (iii): If $X \neq \pm Y$ then $(h(X \circ Y) - 1) > 0$. Hence $[X, Y]$ is connected and there is a crabbed (X, Y) -path in $G(\mathcal{S})$.

(iii) \implies (i): We have to show that (C3) holds for \mathcal{S} . Let $X, Y \in \mathcal{S}$ with $X \neq \pm Y$ and $e \in S(X, Y)$. Let P be a crabbed (X, Y) -path. Since adjacent cocircuits have empty separator, there must be $Z \in P$ with $Z(e) = 0$. Since P is crabbed Z also satisfies $Z^+ \subseteq X^+ \cup Y^+$ and $Z^- \subseteq X^- \cup Y^-$. \square

It shall be mentioned that the “(i) \implies (ii)”-part of the proof is only a slight generalization of a result in [6]. But there the characterizing quality of (ii) was not noted. Furthermore we remark that the connectivity in (ii) is best-possible, since in uniform oriented matroids X_i has exactly $h(X_1 \circ \dots \circ X_k) - 1$ neighbors in $[X_1, \dots, X_k]$.

Even if the cocircuit graph does not uniquely determine the oriented matroid, Theorem 3.3 might lead to an efficient recognition algorithm for cocircuit graphs of general oriented matroids, as its uniform specialization did in [9].

Apart from contributing yet another axiomatization of oriented matroids, a big goal would be to characterize cocircuit graphs in purely graph-theoretic terms, i.e., excluding any information about signed sets. We see our result as a step into that direction.

A somewhat dual question arises, when considering tope-graphs of oriented matroids, which in contrast to cocircuit graphs do determine the oriented matroid up to isomorphism, as shown by Björner, Edelman, and Ziegler [3]. Is there a theorem analogous to Theorem 3.3 for signed tope-graphs? It is open to characterize tope-graphs of oriented matroids in purely graph-theoretical terms.

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