LATTICES AND POLYHEDRA FROM GRAPHS

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Finally, I want to express my gratitude again to Stefan and also to Michael Joswig for reading this thesis. I hope you enjoy some of it, but I know this is quite some work and so I stumbled over something that Douglas Adams [1] said and which seemed adequate to me:

> "Would it save you a lot of time if I just gave up and went mad now?"

Thanks a lot,

Kolja

Contents

What is this thesis about?

The title "Lattices and Polyhedra from Graphs" of this thesis is general though describes quite well the aim of this thesis. Among the most important objects of this work are *distributive lattices* and *upper locally distributive lattices*. While distributive lattices certainly are one of the most studied lattice classes, also upper locally distributive lattices enjoy frequent reappearance in combinatorial order theory under many different names. Upper locally distributive lattices correspond to *antimatroids* and *abstract convex geometries* – objects of major importance in combinatorics.

Besides results of a purely lattice or order theoretic kind we present new characterizations of (upper locally) distributive lattices in terms of antichain-covers of posets, arc-colorings of digraphs, point sets in $\mathbb{N}^d$, vector addition languages, chip-firing games, and vertex and (integer) point sets of polyhedra. We exhibit links to a wide range of graph theoretical, combinatorial, and geometrical objects. With respect to the latter we study and characterize polyhedra which seen as subposets of the componentwise ordering of Euclidean space form (upper locally) distributive lattices.

Distributive lattice structures have been constructed on many sets of combinatorial objects, such as lozenge tilings, planar bipartite perfect matchings, planar orientations with prescribed outdegree, domino tilings, planar circular flow, orientations with prescribed number of backward arcs on cycles and several more. A common feature of all of them is that the Hasse diagram of the distributive lattice may be constructed applying *local transformations* to the objects. These local transformations lead to a *natural arc-coloring* of the diagram. For an example see the distributive lattice on the domino-tilings of a rectangular region on the side. The local transformation consists in flipping two tiles, which share a long side. In this work we present the first unifying generalization of all such instances of graph-related distributive lattices. We obtain a distributive lattice structure on the *tensions* of a digraph.
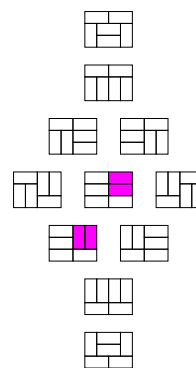
In order to provide a flavor of what we refer to as "unifying generalization", we show two consecutive steps of generalizing the domino tilings of a plane region, see the figure.

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The left-most part of the figure shows two domino tilings, which can be transformed into each other by a single flip of two neighboring tiles. In the middle of the figure we show how planar bipartite perfect matchings model domino tilings. The local transformation now corresponds to switching the matching on an alternating facial cycle. More generally, the right-most part of the figure shows how to interpret the preceeding objects as orientations of prescribed outdegree of a bipartite planar graph. Every yellow vertex has outdegree 1 and every blue vertex has outdegree $(\mathtt{deg} - 1)$. Reversing the orientation on directed facial cycles yields a distributive lattice structure on the set of orientations with these outdegree constraints.
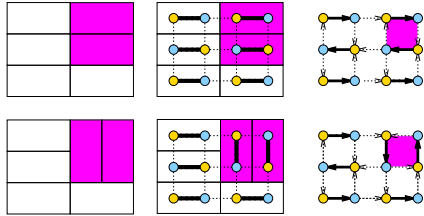
A particular interest of this work lies in embedding lattices into Euclidean space. The motivation is to combine geometrical and order-theoretical methods and perspectives. We investigate polyhedra, which seen as subposets of the componentwise ordering of Euclidean space form upper locally distributive or distributive lattices. In both cases we obtain full characterizations of these classes of polyhedra in terms of their description as intersection of bounded halfspaces.

In particular we obtain a polyhedral structure on known *discrete* distributive lattices on combinatorial objects as those mentioned above as integer points of *distributive* polytopes. A classical polytope which was defined in the spirit of combining discrete geometry and order-theory appears as a special case of our considerations, and thus might provide an idea of what kind of objects we will study: Given a poset $\mathcal{P}$, Stanley's *order polytope* $\mathsf{P}_{\mathcal{P}}$ may be defined as the convex hull of the characteristic vectors of the ideals of a poset $\mathcal{P}$.
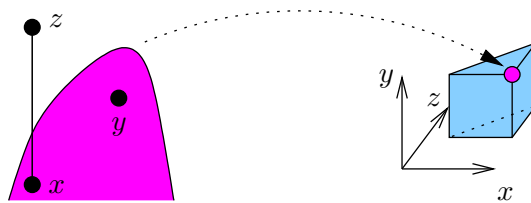
Figure 1: A poset $\mathcal{P}$ with an ideal on the left with its order polytope $\mathsf{P}_{\mathcal{P}}$ and the vertex the corresponding to the ideal, on the right.

Our characterization of *upper locally distributive* polyhedra opens connections to the theory of feasible polytopes of antimatroids. In the setting of *distributive* polyhedra we find graph objects that might be considered as the most general ones, which form a distributive lattice and carry a polyhedral structure. The connection to polytope theory links distributive lattices to generalized flows on digraphs. Thus, there is a link to important objects of com-

binatorial optimization. Moreover we exhibit new contributions to the theory of bicircular oriented matroids.

Large parts of the thesis are based on publications between 2008 and 2010 [40, 43, 41, 42, 54, 69]. In the following we give a rough overview over each single chapter. For more detailed introductions we refer to the first pages of the individual chapters.

Chapter 1: Lattices

The first chapter of the thesis is about lattices. It is based on papers [41, 42, 69] and includes joint work with Stefan Felsner. After giving a more detailed introduction into lattice theory and the chapter itself, we present some basic notation and vocabulary in Section 1.1.

The main result of Section 1.2 is a new representation result for general finite lattices. We provide a one-to-one correspondence between finite lattices and antichain-covered posets. As an application we strengthen a characterization of upper locally distributive lattices in terms of antichain-partitioned posets due to Nourine. The "smallest" special case of our theorem is the Fundamental Theorem of Finite Distributive Lattices alias Birkhoff's Theorem.

Section 1.3 proves three classes of combinatorial objects to be equivalent. We show that acyclic digraphs with a certain arc-coloring and unique source, cover-preserving join-sublattices of $\mathbb{N}^d$ and upper locally distributive lattices correspond to each other. The characterization turns out to be very useful in many applications where one actually wants to prove an (upper locally) distributive lattice structure on a given set of objects. Another application of this section is a generalization of Dilworth's Embedding Theorem for Distributive Lattices to upper locally distributive lattices.

In Section 1.4 we present the distributive lattice of Δ-tensions of a digraph. As mentioned before, many known distributive lattices coming from graphs are special cases of Δ-tensions. At the end of the section we show reductions to the most important special cases of Δ-tensions: Flow in planar graphs, prescribed outdegree orientations of planar graphs, and orientations with prescribed circular flow-difference of general graphs.

Section 1.5 is motivated by Björner and Lovász' chip-firing game on directed graphs. As an easy application of the results in the sections above, chip-firing games lead to upper locally distributive lattices. Moreover chip-firing games have a representation as vector addition languages. We capture the most important features of such languages to generalize the concept to *generalized chip-firing games*. In contrast to ordinary chip-firing games, the latter indeed are general enough to represent every upper locally distributive lattice. Moreover we show that every such lattice is representable as the intersection of finitely many chip-firing games.

We close the chapter with concluding remarks and open problems in Section 1.6.

Chapter 2: Polyhedra

This chapter is based on parts of [43, 69] and partial joint work with Stefan Felsner. After a brief introduction, make first observation about order-theoretic properties of convex subset of $\mathbb{R}^d$ with respect to the componentwise ordering of the space in Section 2.1. In particular we define upper locally distributive and distributive polyhedra.

As a basic ingredient Section 2.2 is devoted to affine Euclidean space satisfying poset properties. We characterize distributive affine space by a representation in terms of directed graphs. This is an important part of the characterizations of upper locally distributive and distributive polyhedra in the following sections.

In Section 2.3 we characterize upper locally distributive polyhedra via their description as nitersection of bounded halfspaces. We find relations of these polyhedra to *feasible polytopes* of antimatroids and draw connections to a membership problem discussed by Korte and Lovász. We show how to view every upper locally distributive polyhedron as the intersection of polyhedra associated to *chip-firing games*.

Section 2.4 develops the theory of distributive polyhedra. We obtain a characterization of their description as intersection of bounded halfspaces. We obtain that these polyhedra are dual to polyhedra of generalized digraph flows, i.e., flows on digraphs with lossy and gainy arcs. We establish a correspondence between distributive polyhedra and *generalized* tensions of digraphs yielding in a sense the most general distributive lattices arising from a graph, in Subsection 2.4.1. We show how to obtain the lattices from Section 1.4 as integer point lattices of special distributive polyhedra and prove that these polyhedra coincide with alcoved polytopes and polytropes, known in the literature before, in Subsection 2.4.2. The combinatorial model for general distributive polyhedra is closely related to oriented bicircular matroids. This will be made explicit in Subsection 2.4.3. As a special application we find the first distributive lattice on generalized flows of planar digraphs in Subsection 2.4.4.

Section 2.5 concludes with some open questions and further remarks.

Chapter 3: Cocircuit Graphs of Uniform Oriented Matroids

The last chapter is not strongly related to the rest of the thesis. It is based on the papers [54, 40] and is joint work with Stefan Felsner, Ricardo Gómez, Juan José Montellano-Ballesteros, and Ricardo Strausz. We present the first cubic-time algorithm which takes a graph as input and decides if the graph is the cocircuit graph of a uniform oriented matroid. In the affirmative case the algorithm returns the set of signed cocircuits of the oriented matroid. This improves an algorithm proposed by Babson, Finschi and Fukuda.

Moreover we strengthen a result of Montellano-Ballesteros and Strausz characterizing cocircuit graphs of uniform oriented matroids in terms of *crabbed* connectivity.

Chapter 1

Lattices

Lattices are posets with unique maximal lower bound and unique minimal upper bound for every pair of elements, see Definition 1.1.3. They are a classical research topic and frequently appear in many areas of mathematics, see [15]. Lattices are objects on the border line between order theory, combinatorics, and algebra. The latter is plausible for instance because lattices may be characterized as a ground set with two binary operations satisfying *commutative*, *associative* and *absorptive* laws. This interpretation of lattices plays an essential role in universal algebra.

For us the most important are relations between lattice theory and combinatorics, and there are many of them. A first reason for this is that every lattice can be represented as an inclusion-order on a set-system. Thus, many sets of combinatorial objects carry a specific lattice structure, e.g., *geometric lattices* correspond to simple matroids [13]; the divisors of a number [103] and the stable marriages of a bipartite graph [55] form *distributive lattices*; the inclusion-order on the normal subgroups of a group is a *modular lattice* [15]. In this chapter we will see many more examples of combinatorial objects forming a lattice. Some of these object classes will turn out general enough to actually represent the class of finite lattices.

Another natural link from lattices to combinatorics is viewing a poset as a *Hasse diagram*. One can study the particular properties of the Hasse diagrams of certain poset or lattice classes. A classical example would be *Birkhoff's criterion* to characterize upper semimodular lattices by their Hasse diagram [104].

The concept of upper locally distributive lattices (ULD) is central to this thesis. ULDs were first investigated by Dilworth [31] and many different lattice theoretical characterizations of ULDs are known. We stick to the original definition by Dilworth as *lattices with unique minimal meet-representations*, see Definition 1.1.6.

For a survey on the work on ULDs up to 1990 we refer to Monjardet [84].

ULDs have appeared under several different names, e.g., locally distributive lattices (Dilworth [33]), meet-distributive lattices (Jamison [59, 60], Edelman [34], Björner and Ziegler [22]), locally free lattices (Nakamura [86]). Following Avann [5], Monjardet [84], Stern [104] and others, we stick to the name ULD. The reason for the frequent reappearance of the concept is that there are many instances of ULDs, i.e., sets of combinatorial objects that can be *naturally* ordered to form an ULD, e.g.,

- Subtrees of a tree (Boulaye [24])
- Convex subsets of a poset (Birkhoff and Bennett [16])
- Convex subgraphs of an acyclic digraph (Pfaltz [91])
- Transitively oriented subgraphs of a transitively oriented digraph (Björner [17])
- ⋆ Convex sets of an abstract convex geometry (Edelman [34])
- ⋆ Pruning processes (Ardila and Maneva [4])
- Reachable configurations of a chip-firing game (Magnien, Phan, and Vuillon [79])
- ⋆ Learning spaces (Eppstein [36])
- ⋆ Feasible sets of an antimatroid (Korte [70])
- ⋆ Feasible multi-sets of an antimatroid with repetition (Björner and Ziegler [22])
- ⋆ Supports of a locally free, permutable, left-hereditary languages (Björner [21])

For sets in the list colored by magenta the reverse inclusion order yields a ULD. Those sets that are colored blue form ULDs under inclusion-order. The subtrees of a tree, the convex subsets of a poset, the convex subgraphs of an acyclic digraph, and the transitively oriented subgraphs of a transitively oriented digraph may all be modelled as the convex sets of an abstract convex geometry or equivalently as pruning processes. Indeed these last two classes of objects are universal for the class of ULDs. Therefore we labeled them with a star. The most important of these first examples is given by *convex geometries*, a combinatorial abstraction of convex sets in geometry.

A class which will come up later in this thesis is given by the chip-firing game. It is a classical discrete dynamical model, used in physics, economics and computer science. Learning spaces, feasible (multi-)sets of an antimatroid, and supports of a locally free, permutable, left-hereditary languages are universal for the class of ULDs. Therefore they are labelled with a star.

The most prominent among the blue entries of the list are *antimatroids* – a special case of *greedoids*. Antimatroids are set-systems such that the system of complements is an abstract convex geometry. Antimatroids and greedoids have many applications and connections in mathematics, see [72]. Glasserman and Yao [51] use antimatroids to model the ordering of events in discrete event simulation systems. They are also used to model progress towards a goal in artificial intelligence planning problems. In mathematical psychology, antimatroids have been used to describe feasible states of knowledge of a human learner.

A very important subclass of the class of ULDs is given by distributive lattices. Because of their nice structural properties and many applications distributive lattices count among the most important lattice classes. The following list gives some examples of objects carrying a natural distributive lattice structure.

- domino and lozenge tilings of a plane region (Rémila [97] and others based on Thurston [105])
- planar spanning trees (Gilmer and Litherland [48])
- planar bipartite perfect matchings (Lam and Zhang [73])
- planar bipartite d-factors (Felsner [39], Propp [92])
- Schnyder woods of a planar triangulation (Brehm [25])
- Eulerian orientations of a planar graph (Felsner [39])
- α-orientations of a planar graph (Felsner [39], Ossona de Mendez [88])
- k-fractional orientations with prescribed outdegree of a planar graph (Bernardi and Fusy [11])
- Schnyder decompositions of a plane d-angulations of girth d (Bernardi and Fusy [12])
- circular integer flows of a planar graph (Khuller, Naor and Klein [66])
- higher dimensional rhombic tilings (Linde, Moore, and Nordahl [77])
- c-orientations of a graph (Propp [92])

Generally, having a lattice structure on a set of objects may help in understanding the set or as Peter Panter puts it: "Ordnung muss sein!" [90]. A *distributive* lattice structure is particularly good:

An important technique for *random sampling* is *coupling from the past* (Propp, Wilson [94]). This way of analysing a Markov chain may be applied to distributive lattices (Propp [93]). *Enumerating* the elements of a distributive lattice, i.e., outputting all the elements while using little memory, can be done more efficiently on distributive lattices than on other underlying structures (Habib, Medina, Nourine, and Steiner [57]). The useful *FKG-inequality* of Fortuin, Kasteleyn, and Ginibre [46] and Ahlswede and Daykin's Four Functions Theorem [2], as well as their recently proved q-analogues due to Björner [18] and Christofides [26], respectively, are applicable only to distributive lattices.

In many of our results, the lattice structure is derived from a set of *local transformations*. As an example recall the distributive lattice on domino-tilings described in the introduction, where local transformations were given by flips of neighboring tiles. We obtain a correspondence of cover-relations in the lattice and applications of a transformation to one of our combinatorial objects. As a direct consequence the set carrying the lattice structure is *connected* with respect to these local transformations. In some cases modeling the cover-relations in the combinatorial object yields upper bounds on height and diameter of the lattice, e.g., the height of the lattice of c-orientations is quadratic in the size of the graph (Propp [92]). Since

every finite lattice has a unique minimal element we can conclude that our set of combinatorial objects has a unique element, where no more "downwards" transformation can be applied. When dealing with several sets each of them carrying an individual lattice structure, this *unique representant* can be used for *bijective counting* of the sets. One example for this is the bijective counting of tree-rooted maps and shuffles of parenthesis systems by Bernardi [9].

The present chapter is structured as follows:

Section 1.1 introduces basic notions and definitions needed throughout the whole thesis.

In Section 1.2 we present a vast generalization of Birkhoff's Theorem also known as *The Fundamental Theorem of Finite Distributive Lattices* to the class of all finite lattices. We establish a correspondence between finite lattices and special antichain-covered posets. This will in particular yield a characterization of upper locally distributive lattices in terms of antichain-partitioned posets, which strengthens a result of Nourine [87]. One application of this result will appear in Section 1.5 in connection with chip-firing games.

Section 1.3 provides a characterization of upper locally distributive lattices in terms of arc-colored acyclic digraphs. Our characterization of ULDs originates from a characterization in [67] of matrices whose *flip-flop* posets generate distributive lattices. It turned out that this tool yields handy proofs for the distributive lattice structure on several objects from graphs. In the applications the arc-colors correspond to the local transformations on combinatorial objects in a natural way. Moreover, we prove that cover-preserving join-sublattices of the componentwise ordering on $\mathbb{N}^d$ correspond to upper locally distributive lattices. This is a generalization of Dilworth's Embedding Theorem for distributive lattices [32]. Section 1.3 is a continuation of the first part of [42].

Section 1.4 – based on the second part of [42] – introduces a distributive lattices structure on the tensions of a directed graph. Tensions are classical objects in algebraic graph theory as they are dual to digraph flows. We provide a bijection to vertex-potentials, also known as height functions. Tensions are a unifying generalization of all the combinatorial sets of objects mentioned in the above list of distributive lattices. At the end of the section we show reductions to the most important special cases of Δ-tensions: Flow in planar graphs, prescribed outdegree orientations of planar graphs, and orientations with prescribed circular flow-difference of general graphs.

Section 1.5 deals with ways of representing ULDs in a more geometrical setting. Starting from the chip-firing game of Björner and Lovász we consider a generalization to vector-addition languages that still admit algebraic structures as sandpile group or sandpile monoid. We characterize the set of vector-addition languages which yield upper locally distributive lattice and call them *generalized chip-firing games*. We show that every upper locally distributive lattice can be represented by a generalized chip-firing games. Indeed, we can prove

that every upper locally distributive lattice is the *intersection* of finitely many ordinary chip-firing games. Parts of this chapter are based on the first part of [69].

1.1   Preliminaries for Posets and Lattices

The following is a brief, a self-contained introduction, restricted to the information needed to span the context of this thesis. We will mainly focus on the theory of finite lattices and finite posets. For further standard terminology we refer to Davey and Priestley [30].

A *poset* is a pair $\mathcal{P} = (E, \leq)$ of a ground set E and a binary relation $\leq$ on E satisfying for all $x, y, z \in E$ poset

1. $x \leq x$ (*reflexivity*)   reflexivity
2. $x \leq y$ and $y \leq x$ imply $x = y$ (*antisymmetry*)   antisymmetry
3. $x \leq y$ and $y \leq z$ imply $x \leq z$ (*transitivity*)   transitivity

The fundamental abuse of notation which we will repeatedly commit is the lack of distinction between E and $\mathcal{P}$, i.e., we will write $x \in \mathcal{P}$ instead of $x \in E$, $S \subseteq \mathcal{P}$ instead of $S \subseteq E$, etc. This does not mean that the ground set is not of importance. For instance an important class of posets are *inclusion-orders*. This means that E is a set of subsets of some   inclusion-order
set (also referrred to as *set-system*). For $X, Y \in E$ we define $X \leq Y$ if and only if $X \subseteq Y$.   set-system
We denote an inclusion order as $(E, \subseteq)$.

For a poset $\mathcal{P} = (E, \leq)$ its *dual poset* $\mathcal{P}^* = (E, \leq^*)$ is defined as $x \leq^* y :\Longleftrightarrow y \leq x$.   dual poset
Instead of $y \leq x$ we sometimes also write $x \geq y$. The dual poset of the inclusion order
$(E, \subseteq)$ is denoted by $(E, \supseteq)$.

If for $x, y \in \mathcal{P}$ we have $x \geq y$ or $x \leq y$, then we say that x and y are *comparable*.   comparability
Otherwise we say that x and y are *incomparable* denoted by $x \parallel y$. If $x \leq y$ and $x \neq y$, then   incomparability
we say that x is strictly smaller than y, denoted by $x < y$. We generally use $\ngeq, \nleq, \nparallel, \nless$ and
so on, for negating relations.

A set $I \subseteq \mathcal{P}$ is called an *ideal* if $x \leq y \in I$ implies $x \in I$. We collect the ideals of $\mathcal{P}$ ideal
in $\mathcal{I}(\mathcal{P})$. Given $S \subseteq \mathcal{P}$ we denote by $\downarrow S$ the ideal $\{x \in \mathcal{P} \mid \exists y \in S : x \leq y\}$. Dual to
an ideal we call $F \subseteq \mathcal{P}$ a *filter* if $x \geq y \in F$ implies $x \in F$. The set of filters of $\mathcal{P}$ is   filter
denoted by $\mathcal{F}(\mathcal{P})$. For $S \subseteq \mathcal{P}$ we denote by $\uparrow S$ the filter $\{x \in \mathcal{P} \mid \exists y \in S : x \geq y\}$. We
call $C \subseteq \mathcal{P}$ a *chain* if all elements of C are mutually comparable. A set $A \subseteq \mathcal{P}$ is called   chain
an *antichain* if all its elements are mutually incomparable. The set of *minimal elements* of   antichain
a subset $S \subseteq \mathcal{P}$ is denoted by $\mathrm{Min}(S) := \{x \in S \mid y \in S \Longrightarrow y \nless x\}$. Analogously, we   minimal element
define a *maximal element* of S and collect them in $\mathrm{Max}(S)$. For a finite poset $\mathcal{P}$ the *height*   maximal element
of $x \in \mathcal{P}$ is the cardinality of a longest chain C in $\mathcal{P}$ with $\mathrm{Max}(C) = \{x\}$.   height

For an element $x \in \mathcal{P}$ we will often use the expression x *is maximal with some property*. This means that $x \in \text{Max}(S)$, where $S \subseteq \mathcal{P}$ is the set of elements with that property. A first example of this is the following: We write $x \prec y$ if x is maximal with the property $x < y$. We then say that y *covers* x or that y is a *cover* of x or that x is a *cocover* of y. The

cover relation

directed graph $D_{\mathcal{P}} = (E, A)$ with $(x, y) \in A :\Longleftrightarrow x \prec y$ is called the *Hasse diagram* of $\mathcal{P}$.

Hasse diagram

Because of antisymmetry of a poset a Hasse diagram has no directed cycles, i.e., is *acyclic*.

acyclic

Conversely, every acyclic digraph $D = (V, A)$ yields a poset $\mathcal{P}_D$ on V as its *transitive hull*,

transitive hull

i.e., $v \leq w$ if there is a directed (v, w)-path in D. If D is the Hasse diagram of $\mathcal{P}_D$, then we

transitively
reduced

call D *transitively reduced*.

Let $\mathcal{P} = (E, \leq)$, $\mathcal{Q} = (E', \leq')$ be two posets. A mapping φ from E to E' is said to be

order-preserving

- an *order-preserving* map if $x \leq y \Longrightarrow \varphi(x) \leq' \varphi(y)$ for all $x, y \in E$,

order-embedding

- an *order-embedding* if $x \leq y \Longleftrightarrow \varphi(x) \leq' \varphi(y)$ for all $x, y \in E$,

order-ismorphism

- an *order-ismorphism* if it is bijective and an embedding.

subposet

We say that $\mathcal{P}$ is a *subposet* of $\mathcal{Q}$ if and only if $E \subseteq E'$ and $x \leq y \Longleftrightarrow x \leq' y$ for all $x, y \in E$, i.e., the identity map of $\mathcal{P}$ is an order-embedding into $\mathcal{Q}$. In this case we call $\mathcal{P}$

induced subposet

the subposet of $\mathcal{Q}$ *induced* by E.

join

A minimal $z \in \mathcal{P}$ with $z \geq x, y$ is called a *join* of x, y. Dually, a maximal element $z \in \mathcal{P}$

meet

with $z \leq x, y$ is called a *meet* of x, y. If $|\text{Max}(\mathcal{P})| = 1$, then this means that $\mathcal{P}$ has a unique maximal element. We denote it by $\mathbf{1}_{\mathcal{P}}$. Dually, if $\mathcal{P}$ has a unique minimum, then we denote it by $\mathbf{0}_{\mathcal{P}}$. The existence of joins and meets and unique maxima and minima is closely related in finite posets.

Observation 1.1.1. Since if there were several maxima one could just take their join or meet, respectively, we have: A finite poset $\mathcal{P}$ has a join for every pair of elements if and only if $\mathcal{P}$ has a $\mathbf{1}_{\mathcal{P}}$. Dually, a finite $\mathcal{P}$ has a meet for every pair of elements if and only if $\mathcal{P}$ has a unique minimum $\mathbf{0}_{\mathcal{P}}$.

If in a poset $\mathcal{L}$ we have that every pair of elements has a *unique* join, then we call $\mathcal{L}$ a

semilattice

join-semilattice. The dual of a join-semilattice is called *meet-semilattice*. As an example it is easy to verify:

union-closed

Observation 1.1.2. An inclusion-order $(E, \subseteq)$ on a *union-closed* set-system E, i.e., if $X, Y \in E$, then also $X \cup Y \in E$, is a join-semilattice. The join of two sets is given by their

intersection-
closed

union. If E is *intersection-closed*, then $(E, \subseteq)$ is a meet-semilattice with the meet being set-intersection. Given a poset $\mathcal{P}$, the set-system $\mathcal{I}(\mathcal{P})$ is union-closed *and* intersection-closed, i.e., $(\mathcal{I}(\mathcal{P}), \subseteq)$ is a join- *and* a meet-semilattice.

The class of posets which are join- and meet-semilattices at the same time is of central importance for this entire thesis:

lattice

Definition 1.1.3. A poset $\mathcal{L}$ is called a *lattice* if every pair of elements of $\mathcal{L}$ has a unique join and a unique meet.

Often, we will denote lattices and semilattices by $\mathcal{L}$ and other posets by $\mathcal{P}$. If $x, y \leq z, w$, then if there is a unique join of x, y, then $x \vee y \leq z, w$. This yields the easy

Observation 1.1.4. A finite join-semilattice $\mathcal{L}$ which has meets for all pairs of elements has unique meets, i.e., $\mathcal{L}$ is a lattice. Dually a meet semilattice $\mathcal{L}$ with joins for all pairs of elements is a lattice.

In a lattice $\mathcal{L}$ we denote the join and meet of elements $x, y \in \mathcal{L}$ by $x \vee_{\mathcal{L}} y$ and $x \wedge_{\mathcal{L}} y$, respectively. If it is clear which lattice we are talking about, then we will usually drop the subindex $_{\mathcal{L}}$. Seen as binary operations join and meet in a lattice form *idempotent commutative semigroups*, i.e., for all $x, y, z \in \mathcal{L}$

semigroup

- $x \vee x = x$ (*idempotent*)   idempotent
- $x \vee y = y \vee x$ (*commutative*)   commutative
- $x \vee (y \vee z) = (x \vee y) \vee z$ (*associative*)   associative

and analogously for $\wedge$. In particular, they are associative binary operations. Thus an expression like $x_1 \vee \ldots \vee x_k$ makes sense and we will denote it as $\bigvee \{x_i \mid i \in [k]\}$. (Here and everywhere $[k]$ stands for $\{1, \ldots, k\}$.) The analogous abbreviation $\bigwedge S$ will be used for the meet of a set $S \subseteq \mathcal{L}$.

Let $\mathcal{L} = (E, \leq)$ and $\mathcal{L}' = (E', \leq')$ be lattices. We say that $\mathcal{L}$ is a *sublattice* of $\mathcal{L}'$ if $\mathcal{L}$ is a   sublattice subposet of $\mathcal{L}'$, and we have $x \vee_{\mathcal{L}} y = x \vee_{\mathcal{L}'} y$ and $x \wedge_{\mathcal{L}} y = x \wedge_{\mathcal{L}'} y$ for all $x, y \in E$.
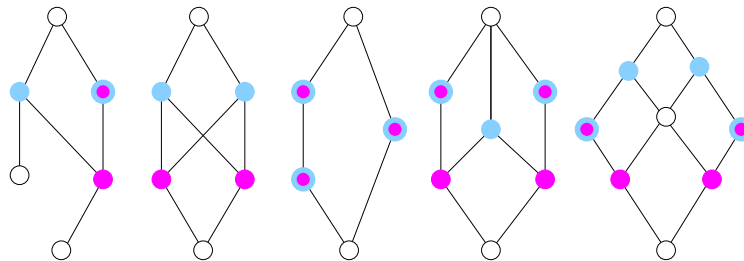
Figure 1.1: From left to right: join-semilattice; poset with unique minimum and maximum; lattice; upper locally distributive lattice; distributive lattice. Join-irreducible elements are colored magenta, meet-irreducibles are colored light blue. We will be consistent with this "color-code" through the entire thesis.

An element $j \in \mathcal{L}$ is called *join-irreducible* if it cannot be expressed as the join of a set   join-irreducible of elements not containing j. In the Hasse diagram join-irreducibles are those elements with exactly one incoming arc, i.e., a join-irreducible j has a unique cocover in $\mathcal{L}$. It is denoted

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by j^-. We write $\mathcal{J}(\mathcal{L})$ for the subposet of $\mathcal{L}$ induced by its join-irreducibles. Dually, one defines the poset of *meet-irreducibles* denoted by $\mathcal{M}(\mathcal{L})$. The unique cover of a meet-irreducible m in $\mathcal{L}$ is denoted by m^+. In a finite meet-semilattice (join-semilattice) $\mathcal{L}$ we set $\bigwedge \emptyset := \mathbf{1}_\mathcal{L}$ ($\bigvee \emptyset := \mathbf{0}_\mathcal{L}$). Hence maximum and minimum are not meet-irreducible and not join-irreducible, respectively. An important fact is that also every other element of a finite lattice may be expressed as a join of join-irreducibles By idempotence this is clear if it is a join-irreducible itself, otherwise it is a join of (not necessarily join-irreducible) elements below it. The conclusion follows by induction on the height. Dually, every element of a lattice is a meet of meet-irreducibles. More formally:

<div style="text-align: right">meet-irreducible</div>

Observation 1.1.5. In a finite join-semilattice every element $\ell \in \mathcal{L}$ is the join of join-irreducibles below it, i.e, $\ell = \bigvee(\downarrow\ell \cap \mathcal{J}(\mathcal{L}))$. Dually, in a finite meet-semilattice $\mathcal{L}$ we have $\ell = \bigwedge(\uparrow\ell \cap \mathcal{M}(\mathcal{L}))$ for all $\ell \in \mathcal{L}$.

The posets $\mathcal{J}(\mathcal{L})$ and $\mathcal{M}(\mathcal{L})$ are sufficient to encode a lattice. We will show one way to do this (Theorem 1.2.3), which specializes in a nice way to (upper locally) distributive lattices. The latter form indeed the lattice class being most vital to this thesis. It was first defined by Dilworth [31].

upper locally
distributive lattice
(ULD)

Definition 1.1.6. A finite lattice $\mathcal{L}$ is called *upper locally distributive (ULD)* if for every $\ell \in \mathcal{L}$ there is a unique inclusion-minimal set $M_\ell \subseteq \mathcal{M}(\mathcal{L})$ such that $\ell = \bigwedge M_\ell$.

lower locally
distributive lattice
(LLD)

The dual of a ULD is called *lower locally distributive (LLD)*. A special and important subclass of upper *and* lower locally distributive lattices are *distributive lattices*. They are of strong interest to this work. The following is their classical:

distributive lattice

Definition 1.1.7. A lattice $\mathcal{L}$ is called *distributive* if $k \vee (\ell \wedge m) = (k \vee \ell) \wedge (k \vee m)$ for all $k, \ell, m \in \mathcal{L}$.

It is one folklore lemma of distributive lattices that the definition could be equivalently stated using $k \wedge (\ell \vee m) = (k \wedge \ell) \vee (k \wedge m)$.

There are plenty of different characterizations and representations of distributive lattices. Most famously Birkhoff's Theorem [14] states a bijection between distributive lattices and posets, which furthermore yields a representation as union- and intersection-closed set-systems. This will be a corollary of the next section, stated as Theorem 1.2.1.

Another characterization states that a lattice is distributive if and only if it is upper and lower locally distributive. This was already shown by Dilworth in the first paper about ULDs [31]. We will obtain that characterization as a corollary of Section 1.3, stated as Theorem 1.3.22.

1.2   Generalizing Birkhoff's Theorem

In this section we will show a correspondence between finite lattices and finite posets covered by antichains. A very special case of this is Birkhoff's Theorem [14] also known as *The Fundamental Theorem of Finite Distributive Lattices*:

Theorem 1.2.1. *A finite lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L} \cong (\mathcal{I}(\mathcal{P}), \subseteq)$ for a finite poset $\mathcal{P}$. Moreover, $\mathcal{P} \cong \mathcal{J}(\mathcal{L})$.*

Note that Theorem 1.2.1 yields $\mathcal{L} \cong (\mathcal{I}(\mathcal{J}(\mathcal{L})), \subseteq)$ and $\mathcal{P} \cong \mathcal{J}(\mathcal{I}(\mathcal{P}), \subseteq)$ for every finite distributive lattice $\mathcal{L}$ and every finite poset $\mathcal{P}$. This is, mapping a finite poset $\mathcal{P}$ to the finite distributive lattice $(\mathcal{I}(\mathcal{P}), \subseteq)$ induces a one-to-one correspondence between isomorphism-classes of finite posets and isomorphism-classes of finite distributive lattices.

As an application of the main theorem of this section we will reprove Birkhoff's Theorem at the end of the section. However, the main motivation that led to this chapter is a result of Nourine establishing a partial generalization of Birkhoff's Theorem to ULDs and antichain-partitioned posets [87]. For the statement of Nourine's result we need one further definition. Let S be a subset of a poset $\mathcal{P}$ and $\mathcal{A}_{\mathcal{Q}} = \{A_y \mid y \in \mathcal{Q}\} \subseteq 2^{\mathcal{P}}$ a set of antichains, i.e., the antichains in $\mathcal{A}_{\mathcal{Q}}$ are indexed by the set $\mathcal{Q}$. We define the *fingerprint* of S in $\mathcal{A}_{\mathcal{Q}}$ as $fing_{\mathcal{A}_{\mathcal{Q}}}(S) := \{y \in \mathcal{Q} \mid S \cap A_y \neq \emptyset\}$. So given a poset $\mathcal{P}$ and a set of antichains indexed by a set $\mathcal{Q}$ the fingerprint takes subsets of $\mathcal{P}$ to subsets of $\mathcal{Q}$. Given a set $\mathcal{S}$ of subsets of $\mathcal{P}$ we write $fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{S})$ for $\{fing_{\mathcal{A}_{\mathcal{Q}}}(S) \mid S \in \mathcal{S}\}$.
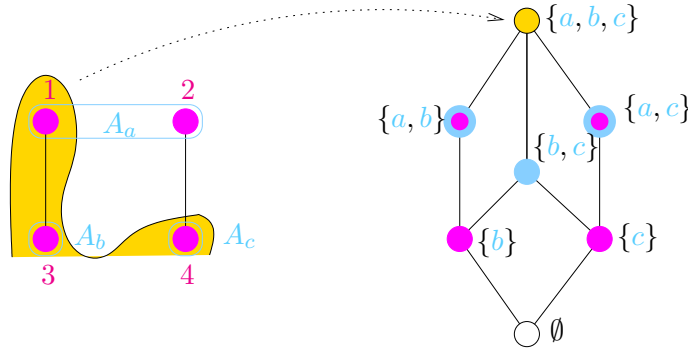
Figure 1.2: On the left: a poset $\mathcal{P}$ with ground set $\{1, 2, 3, 4\}$ and an antichain-partition $\mathcal{A}_{\mathcal{Q}} = \{A_a, A_b, A_c\}$, i.e., the index-set $\mathcal{Q}$ equals $\{a, b, c\}$. On the right: the corresponding ULD as inclusion order on the fingerprints of the ideals of $\mathcal{P}$. The golden ideal has fingerprint $\{a, b, c\}$.

Nourine's Theorem then reads:

Theorem 1.2.2. *A finite lattice $\mathcal{L}$ is a ULD if and only if $\mathcal{L} \cong (fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ for some poset $\mathcal{P}$ with antichain-partition $\mathcal{A}_{\mathcal{Q}}$.*

Nourine's Theorem is important for the study of ULDs. We combine Nourine's Theorem with join-sublattice embeddings of ULDs into the dominance order on $\mathbb{N}^d$ to obtain a generalization of Dilworth's Embedding Theorem for Finite Distributive Lattices to ULDs, see Theorem 1.3.18. At another point we will use Nourine's Theorem to prove that every ULD may be represented as *generalized chip-firing game*, see Theorem 1.5.10. But compare Birkhoff's Theorem with Nourine's Theorem: Birkhoff's Theorem yields that up to isomorphism there is a unique poset representing a given distributive lattice. Nourine's Theorem does not accomplish the analogue, i.e., there may exist "fairly different" antichain-partitoned posets all representing the same ULD. As an application of this section's results we will obtain a strengthening of Nourine's Theorem fully generalizing Birkhoff's Theorem: On the one hand we find a notion of isomorphisms of antichain-partitioned posets. On the other hand we find the class of *reduced* antichain-partitoned posets $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ such that mapping $(\mathcal{P}, \mathcal{A}_\mathcal{Q}) \mapsto (fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ induces a bijection between isomorphism-classes of reduced antichain-partitioned posets and isomorphism-classes of ULDs, see Theorem 1.2.24.

What we develop in the present section is actually much more general. We obtain a way of representing *every* finite lattice as inclusion-order on the fingerprints of the ideals of an antichain-covered poset. Moreover, we find the class of *good* antichain-covered posets, such that every finite lattice is represented by a member of this class which is unique up to *isomorphism*. Analogously to the case of Birkhoff's Theorem we obtain a one-to-one correspondence between isomorphism-classes of finite good antichain-covered posets and isomorphism-classes of finite lattices. This is the main result of the present section. We will define good antichain-covered posets and their isomorphisms later on in this section, see Definition 1.2.14 and Definition 1.2.18, respectively. Nevertheless, in order to give a more precise idea of the main result of this section we state it already:

Theorem 1.2.3. *A finite poset $\mathcal{L}$ is a lattice if and only if $\mathcal{L} \cong (fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ for a good antichain-covered poset $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$. Moreover, $(\mathcal{P}, \mathcal{A}_\mathcal{Q}) \cong (\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$.*

Note that in comparison to the case of ULDs we have to use antichain-covers instead of antichain-partitions. We hope that this result leads to generalizations of our results obtained with the help of Nourine's Theorem to more general lattice classes. Theorem 1.2.3 is similar to the finite case of *the basic theorem on concept lattices* [108] and to a theorem of Markowsky [80]. Nevertheless, the representation for lattices as described in Theorem 1.2.3 is essentially new.

We will now begin with the proof of Theorem 1.2.3, it will get quite technical. A pair $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ of a finite poset $\mathcal{P}$ and a set $\mathcal{A}_\mathcal{Q}$ of antichains of $\mathcal{P}$ is called an
antichain-covered
poset (ACP) *antichain-covered poset (ACP)* if for every $x \in \mathcal{P}$ there is at least one $y \in \mathcal{Q}$ such that $x \in A_y$, i.e., $\mathcal{A}_\mathcal{Q}$ is a cover of $\mathcal{P}$. First we show that the inclusion-order on the fingerprints of the ideals of an antichain-covered posets indeed is a lattice. This can be understood as the first part of Theorem 1.2.3. We start with an easy

Observation 1.2.4. Let $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ be an ACP and $S, S' \subseteq \mathcal{P}$. We have

$$fing_{\mathcal{A}_\mathcal{Q}}(S) \cup fing_{\mathcal{A}_\mathcal{Q}}(S') = fing_{\mathcal{A}_\mathcal{Q}}(S \cup S').$$

Now we can show:

Proposition 1.2.5. *Let $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ be an ACP. The inclusion-order $(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ is a lattice. More precisely, we show:*

- *the set-system $fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P}))$ is union-closed,*
- *for every $fing_{\mathcal{A}_\mathcal{Q}}(I)$ there is a unique inclusion-maximal ideal $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \in \mathcal{I}(\mathcal{P})$ such that $fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) = fing_{\mathcal{A}_\mathcal{Q}}(I)$, we call $fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}})$ distinguished ideal.*   distinguished ideal
- *We have $(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq) \cong (\{\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \mid I \in \mathcal{I}(\mathcal{P})\}, \subseteq)$,*
- *the set-system $\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_\mathcal{Q}} := \{\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \mid I \in \mathcal{I}(\mathcal{P})\}$ is intersection-closed.*

Proof. By Observation 1.2.4 for $I, I' \in \mathcal{I}(\mathcal{P})$ we have that $fing_{\mathcal{A}_\mathcal{Q}}(I) \cup fing_{\mathcal{A}_\mathcal{Q}}(I')$ equals $fing_{\mathcal{A}_\mathcal{Q}}(I \cup I')$. Since $I \cup I'$ is again an ideal of $\mathcal{P}$ the set-system $fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P}))$ is union-closed. Thus, by Observation 1.1.2 the inclusion-order $(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ is a join-semilattice and the join of two sets is their union.

By Observation 1.2.4, the union of a pair of ideals with the same fingerprint has again the same fingerprint. Thus, by Observation 1.1.1 the inclusion-order on these ideals has a unique maximum. Hence for every $fing_{\mathcal{A}_\mathcal{Q}}(I) \in fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P}))$ there is a unique inclusion-maximal $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \in \mathcal{I}(\mathcal{P})$ with $fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) = fing_{\mathcal{A}_\mathcal{Q}}(I)$, by Observation 1.1.1. This yields that the fingerprint is an inclusion-preserving bijection to the distinguished ideals, i.e., $(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq) \cong (\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_\mathcal{Q}}, \subseteq)$.

Now we show that $(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ is a meet-semilattice, by showing that $\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_\mathcal{Q}}$ is intersection-closed. Let $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \neq \lceil I' \rceil_{\mathcal{A}_\mathcal{Q}}$ be distinguished ideals. Since $\mathcal{I}(\mathcal{P})$ is intersection-closed the intersection of $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ and $\lceil I' \rceil_{\mathcal{A}_\mathcal{Q}}$ is an ideal again. Suppose it is not distinguished, i.e., there is an ideal $I'' \in \mathcal{I}(\mathcal{P})$ with $I'' \supsetneq \lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \cap \lceil I' \rceil_{\mathcal{A}_\mathcal{Q}}$ and $fing_{\mathcal{A}_\mathcal{Q}}(I'') = fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \cap \lceil I' \rceil_{\mathcal{A}_\mathcal{Q}})$. Then with Observation 1.2.4 we have

$$
\begin{aligned}
& fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) \\
=\ & fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \cup (\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \cap \lceil I' \rceil_{\mathcal{A}_\mathcal{Q}})) \\
=\ & fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) \cup fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \cap \lceil I' \rceil_{\mathcal{A}_\mathcal{Q}}) \\
=\ & fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) \cup fing_{\mathcal{A}_\mathcal{Q}}(I'') \\
=\ & fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \cup I'').
\end{aligned}
$$

Similarly we obtain $fing_{\mathcal{A}_\mathcal{Q}}(\lceil I' \rceil_{\mathcal{A}_\mathcal{Q}}) = fing_{\mathcal{A}_\mathcal{Q}}(\lceil I' \rceil_{\mathcal{A}_\mathcal{Q}} \cup I'')$. Since I'' is not contained in both $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ and $\lceil I' \rceil_{\mathcal{A}_\mathcal{Q}}$, this contradicts the maximality of at least one of $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ and $\lceil I' \rceil_{\mathcal{A}_\mathcal{Q}}$. Thus, $\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_\mathcal{Q}}$ is intersection-closed. Hence, by Observation 1.1.2 the inclusion-order $(\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_\mathcal{Q}}, \subseteq)$ is a meet-semilattice.

Since $(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ is a join-semilattice and is isomorphic to the meet-semilattice $(\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_\mathcal{Q}}, \subseteq)$ it is a lattice. $\square$

Before we proceed with the next part of the proof of Theorem 1.2.3 we look at the two set-systems representing the same lattice in the above proposition. They have different features. Since the set of distinguished ideals $\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_{\mathcal{Q}}}$ of an ACP $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ is intersection-closed, the inclusion-order $(\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_{\mathcal{Q}}}, \subseteq)$ is a meet-sublattice of the distributive lattice $(\mathcal{I}(\mathcal{P}), \subseteq)$.

We will now show that on the other hand $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ may be seen as join-subslattice of a distributive lattice which is given by the inclusion-order of ideals of a poset on the index-set $\mathcal{Q}$. So given an ACP $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ define a poset on the index set $\mathcal{Q}$ by $y \leq y' :\Longleftrightarrow \uparrow A_y \supseteq \uparrow A_{y'}$. We call the poset $\mathcal{Q}$ the *index-poset of* $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$.

<div style="margin-left:-6em">index-poset</div>

Proposition 1.2.6. *We have* $fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})) \subseteq \mathcal{I}(\mathcal{Q})$. *This is, the lattice* $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ *is a join-sublattice of the distributive lattice* $(\mathcal{I}(\mathcal{Q}), \subseteq)$.

Proof. Let $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ be a distinguished ideal of $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ and $fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}) \subseteq \mathcal{Q}$ its fingerprint. We show that $fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}})$ is an ideal of the index-poset $\mathcal{Q}$. So let $y \in fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}})$, i.e., there is $x \in \lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} \cap A_y$. Now take a $y' \leq y$. Hence by definition $\uparrow A_y \subseteq \uparrow A_{y'}$, i.e., there is an $x' \in A_{y'}$ with $x' \leq x$. Since $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ is an ideal and $x \in \lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ also $x' \in \lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$. Thus, $A_{y'} \cap \lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} \neq \emptyset$ and $y' \in fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}})$. We have shown that $fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}})$ is an ideal of $\mathcal{Q}$, and thus $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ may be seen as a subposet of the inclusion-order on $\mathcal{I}(\mathcal{Q})$. Since by Proposition 1.2.5 the set of fingerprints is union-closed $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ and by Observation 1.1.2 the union is the join of both set-systems, $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ is a join-sublattice of $(\mathcal{I}(\mathcal{Q}), \subseteq)$. $\square$

Any $fing_{\mathcal{A}_{\mathcal{Q}}}(I)$ may clearly be represented as the union of fingerprints $\bigcup_{x \in I} fing_{\mathcal{A}_{\mathcal{Q}}}(\downarrow x)$. (Define the union over an empty index-set as empty.) The following is an analogue statement for distinguished ideals which will be useful at several points in this section. We denote by $\overline{S}$ the complement $E \backslash S$ of a subset $S \subseteq E$.

Lemma 1.2.7. *Let* $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ *be an ACP. An ideal* $I \in \mathcal{I}(\mathcal{P})$ *is distinguished if and only if* $I = \bigcap_{y \in F} \overline{\uparrow A_y}$ *for a filter* F *of* $\mathcal{Q}$. *We set* $\bigcap_{y \in \emptyset} \overline{\uparrow A_y} := \mathcal{P}$.

Proof. "$\Longleftarrow$": Clearly $\mathcal{P}$ is a distinguished ideal, so assume $F \neq \emptyset$. Observe that $\overline{\uparrow A_y}$ is a distinguished ideal: All elements that might be added to $\overline{\uparrow A_y}$ while maintaining an ideal increase the fingerprint by at least y. By Proposition 1.2.5 their intersection is distinguished, too.

"$\Longrightarrow$": If $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} = \mathcal{P}$, i.e., $\overline{\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}} = \emptyset$ we are done by taking $F = \emptyset$. Otherwise, since adding any element to a distinguished ideal $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ increases its fingerprint we have $\overline{\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}} = \bigcup_{y \notin fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}})} \uparrow A_y$. By Proposition 1.2.6 the set $fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}) \in \mathcal{I}(\mathcal{Q})$ and consequently for the index-set of the union on the right-hand side we have $\overline{fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}})} \in \mathcal{F}(\mathcal{Q})$. Applying the complement on both sides of the equation we obtain the result. $\square$

The index-poset $\mathcal{Q}$ will be of importance through the rest of this chapter. Indeed it is in a certain duality to $\mathcal{P}$, which will be explained in more detail in the last subsection of this section.

We will now return to the proof of Theorem 1.2.3. We have shown that given an ACP $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ the inclusion-order $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ is a finite lattice. Next we show that for every finite lattice $\mathcal{L}$ there is a $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ such that $\mathcal{L} \cong (fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$. We start with some basic lemmas.

Lemma 1.2.8. *Let* ℓ, ℓ' *be elements of a finite lattice* $\mathcal{L}$. *We have* $\downarrow\!\ell \cap \mathcal{J}(\mathcal{L}) \subseteq \downarrow\!\ell' \cap \mathcal{J}(\mathcal{L}) \Longleftrightarrow \ell \leq \ell'$ *and dually* $\uparrow\!\ell \cap \mathcal{M}(\mathcal{L}) \subseteq \uparrow\!\ell' \cap \mathcal{M}(\mathcal{L}) \Longleftrightarrow \ell \geq \ell'$.

Proof. Because of duality we only prove the first part of the statement. For "$\Longleftarrow$" note that $\ell \leq \ell'$ implies $\downarrow\!\ell \subseteq \downarrow\!\ell'$ and thus $\downarrow\!\ell \cap \mathcal{J}(\mathcal{L}) \subseteq \downarrow\!\ell' \cap \mathcal{J}(\mathcal{L})$.

For "$\Longrightarrow$" let $\downarrow\!\ell \cap \mathcal{J}(\mathcal{L}) \subseteq \downarrow\!\ell' \cap \mathcal{J}(\mathcal{L})$. This implies $\bigvee(\downarrow\!\ell \cap \mathcal{J}(\mathcal{L})) \leq \bigvee(\downarrow\!\ell' \cap \mathcal{J}(\mathcal{L}))$. By Observation 1.1.5 the sides of that inequality equal ℓ and ℓ', respectively, i.e., $\ell \leq \ell'$. $\qquad\square$

Lemma 1.2.9. *Let* ℓ, ℓ' *be elements of a lattice* $\mathcal{L}$. *We have that the set of minima* $\mathrm{Min}(\downarrow\!\ell\backslash\downarrow\!\ell')$ *is a subset of* $\mathcal{J}(\mathcal{L})$ *and dually* $\mathrm{Max}(\uparrow\!\ell\backslash\uparrow\!\ell') \subseteq \mathcal{M}(\mathcal{L})$.

Proof. Because of duality we only prove the first part of the statement. Let $\ell, \ell' \in \mathcal{L}$. We can assume $\downarrow\!\ell\backslash\downarrow\!\ell'\backslash\mathcal{J}(\mathcal{L}) \neq \emptyset$ since otherwise the statement is trivially true. So take an $\ell'' \in \downarrow\!\ell\backslash\downarrow\!\ell'$ which is not join-irreducible. By Observation 1.1.5 ℓ'' may be represented as a join of join-irreducibles below ℓ'', i.e, $\ell'' = j_1 \vee \ldots \vee j_k$ and $j_i < \ell''$ for all $i \in [k]$. Since $\ell'' \leq \ell$ all the j_i are in $\downarrow\!\ell$. Observe that $\downarrow\!\ell'$ is closed under taking joins. Hence at least one j_i is in $\downarrow\!\ell\backslash\downarrow\!\ell'$. If $\ell'' \in \mathrm{Min}(\downarrow\!\ell\backslash\downarrow\!\ell')$ there cannot be such a $j_i < \ell''$, i.e., ℓ'' itself must be a join-irreducible. $\qquad\square$

We will now define an ACP representing $\mathcal{L}$ as the inclusion-order on the fingerprints of its ideals. For every $m \in \mathcal{M}(\mathcal{L})$ set $A_m := \{j \in \mathcal{J}(\mathcal{L}) \mid m \in \uparrow\!j^-\backslash\uparrow\!j\}$ and let $\mathcal{A}_{\mathcal{M}(\mathcal{L})} := \{A_m \mid m \in \mathcal{M}(\mathcal{L})\}$. For an example of this construction consider Figure 1.3.
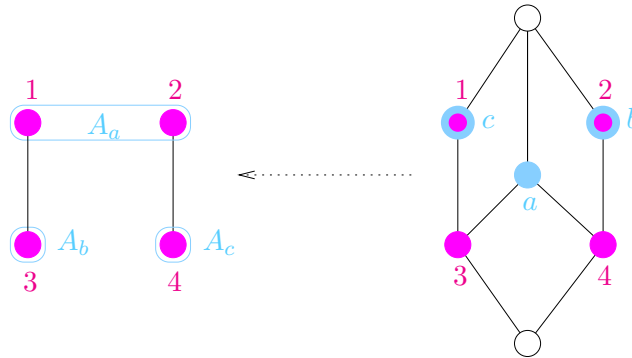

Figure 1.3: On the right: the ULD $\mathcal{L}$ from Figure 1.2. Meet-irreducibles are light blue and join-irreducibles are magenta. On the left: the ACP $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$

Remark 1.2.10. Note that since $\mathcal{J}(\mathcal{L})$ and $\mathcal{M}(\mathcal{L})$ are subposets of $\mathcal{L}$ when applying $\uparrow$, $\downarrow$ or the complement to subsets or elements of $\mathcal{J}(\mathcal{L})$ and $\mathcal{M}(\mathcal{L})$ it is not *ad hoc* clear which ground set we are considering. We do not want to define new notation and new subindeces. Generally we will view $\uparrow S$, $\downarrow S$, and $\overline{S}$ as subsets of $\mathcal{L}$ even if S is a subset of $\mathcal{J}(\mathcal{L})$ or $\mathcal{M}(\mathcal{L})$. In order to avoid confusion, we point out the only two exceptions to this rule:

- If $\uparrow S$, $\downarrow S$, or $\overline{S}$ appears as the argument of the fingerprint, e.g., $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j)$, then we consider it as a subset of $\mathcal{J}(\mathcal{L})$.
- If $S \subseteq \mathcal{A}_{\mathcal{M}(\mathcal{L})}$, then $\uparrow S$, $\downarrow S$, or $\overline{S}$ and their compositions are considered as subset of $\mathcal{J}(\mathcal{L})$, e.g., $\overline{\uparrow A_m}$ for $A_m \in \mathcal{A}_{\mathcal{M}(\mathcal{L})}$.
- If $S = fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(I)$, then $\uparrow S$, $\downarrow S$, or $\overline{S}$ and their compositions are considered as subset of $\mathcal{M}(\mathcal{L})$.

Before proving that $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ is an ACP we need another lemma.

Lemma 1.2.11. *Let $j \in \mathcal{J}(\mathcal{L})$ and $m \in \mathcal{M}(\mathcal{L})$. We have $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j) = \overline{\uparrow j} \cap \mathcal{M}(\mathcal{L})$ and $\uparrow A_m = \overline{\downarrow m} \cap \mathcal{J}(\mathcal{L})$.*

Proof. First we show $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j) \subseteq \overline{\uparrow j} \cap \mathcal{M}(\mathcal{L})$. If $m \in fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j)$, then by definition there is a $j' \leq j$ with $m \in \uparrow j'^{-} \setminus \uparrow j'$. In particular $j' \not\leq m$ and thus by transitivity $j \not\leq m$. This is, $m \in \overline{\uparrow j} \cap \mathcal{M}(\mathcal{L})$.

To show $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j) \supseteq \overline{\uparrow j} \cap \mathcal{M}(\mathcal{L})$ let $m \in \overline{\uparrow j} \cap \mathcal{M}(\mathcal{L})$. This particularly means $\downarrow j \setminus \downarrow m \neq \emptyset$ and by Lemma 1.2.9 the set $\mathrm{Min}(\downarrow j \setminus \downarrow m) \cap \mathcal{J}(\mathcal{L})$ is non-empty. So take an element $j' \in \mathrm{Min}(\downarrow j \setminus \downarrow m) \cap \mathcal{J}(\mathcal{L})$. It satisfies $m \in \uparrow j'^{-} \setminus \uparrow j'$ and $j' \leq j$. By definition this means, $m \in fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j)$.

The proof of the second statement is very similar: First we show $\uparrow A_m \subseteq \overline{\downarrow m} \cap \mathcal{J}(\mathcal{L})$. Let $j \in \uparrow A_m$. Hence there is a $j' \leq j$ with $j' \in A_m$, i.e., $m \in \uparrow j'^{-} \setminus \uparrow j'$. In particular $j' \not\leq m$ hence $j \not\leq m$. Thus, $j \in \overline{\downarrow m} \cap \mathcal{J}(\mathcal{L})$.

To show $\uparrow A_m \supseteq \overline{\downarrow m} \cap \mathcal{J}(\mathcal{L})$ let $j \in \overline{\downarrow m} \cap \mathcal{J}(\mathcal{L})$. In particular $\downarrow j \setminus \downarrow m \neq \emptyset$ and by Lemma 1.2.9 the set $\mathrm{Min}(\downarrow j \setminus \downarrow m) \cap \mathcal{J}(\mathcal{L})$ is non-empty. Any element j' in $\mathrm{Min}(\downarrow j \setminus \downarrow m) \cap \mathcal{J}(\mathcal{L})$ satisfies $m \in \uparrow j'^{-} \setminus \uparrow j'$ and $j' \leq j$. This is, $j \in \uparrow A_m$. $\qquad\square$

Proposition 1.2.12. *Let $\mathcal{L}$ be a finite lattice. The pair $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ is an ACP with index-poset $\mathcal{M}(\mathcal{L})$.*

Proof. To see that $\mathcal{A}_{\mathcal{M}(\mathcal{L})}$ consists of antichains take join-irreducibles $j' < j \in A_m$. We have $j' \leq j^{-}$ so $j' \leq m$. Thus, $m \notin \uparrow j'^{-} \setminus \uparrow j'$ which means $j' \notin A_m$.

In order to prove that $\mathcal{A}_{\mathcal{M}(\mathcal{L})}$ is a set suppose it is not. This is, there are two antichains $A_m = A_{m'}$. This implies $\uparrow A_m = \uparrow A_{m'}$ which by Lemma 1.2.11 implies $\overline{\downarrow m} \cap \mathcal{J}(\mathcal{L}) = \overline{\downarrow m'} \cap \mathcal{J}(\mathcal{L})$. This is equivalent to $\downarrow m \cap \mathcal{J}(\mathcal{L}) = \downarrow m' \cap \mathcal{J}(\mathcal{L})$. Thus, $\bigvee(\downarrow m \cap \mathcal{J}(\mathcal{L})) = \bigvee(\downarrow m' \cap \mathcal{J}(\mathcal{L}))$, where by Observation 1.1.5 both sides equal m and m', respectively, i.e., we have $m = m'$.

In order to show that $\mathcal{A}_{\mathcal{M}(\mathcal{L})}$ covers $\mathcal{J}(\mathcal{L})$ let $j \in \mathcal{J}(\mathcal{L})$. The second part of Lemma 1.2.9 yields that the non-empty set $\mathrm{Max}(\uparrow j^- \backslash \uparrow j)$ contains at least one meet-irreducible m. By definition $m \in \uparrow j^- \backslash \uparrow j$ is equivalent to $j \in A_m$.

The last thing to prove is that $\mathcal{M}(\mathcal{L})$ is isomorphic to the index-poset of $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$. But by Lemma 1.2.8 and the second part of Lemma 1.2.11 we have

$$m \leq m' \Longleftrightarrow \downarrow m \cap \mathcal{J}(\mathcal{L}) \subseteq \downarrow m' \cap \mathcal{J}(\mathcal{L}) \Longleftrightarrow \uparrow A_m \supseteq \uparrow A_{m'}.$$

$\square$

We are ready to prove the next part of Theorem 1.2.3, i.e., that for every finite lattice there is an ACP representing it:

Proposition 1.2.13. *Let $\mathcal{L}$ be a finite lattice. We have $\mathcal{L} \cong (fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\mathcal{I}(\mathcal{J}(\mathcal{L}))), \subseteq)$.*

Proof. As a candidate for an order-isomorphism from $(fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\mathcal{I}(\mathcal{J}(\mathcal{L}))), \subseteq)$ to $\mathcal{L}$ define $\varphi : fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(I) \mapsto \bigvee I$. To see that φ is well-defined we use Lemma 1.2.11 and calculate

$$
\begin{aligned}
& \bigvee I \\
= & \bigwedge(\uparrow(\bigvee I) \cap \mathcal{M}(\mathcal{L})) \\
= & \bigwedge(\bigcap_{j \in I} \uparrow j \cap \mathcal{M}(\mathcal{L})) \\
= & \bigwedge(\bigcap_{j \in I} \overline{fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j)}) \\
= & \bigwedge(\overline{\bigcup_{j \in I} fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j)}) \\
= & \bigwedge(\overline{fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(I)}).
\end{aligned}
$$

Hence, φ does not depend on the choice of I, but only on $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(I)$. Clearly, φ is order-preserving. As inverse mapping we claim $\varphi^{-1} : \ell \mapsto fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow \ell \cap \mathcal{J}(\mathcal{L}))$ for $\ell \in \mathcal{L}$. Also φ^{-1} is order-preserving by Lemma 1.2.8. Now $\varphi \circ \varphi^{-1}(\ell) = \bigvee\{j \in \mathcal{J}(\mathcal{L}) \mid j \leq \ell\} = \ell$. Thus it remains to show that $\varphi^{-1} \circ \varphi = id$.

We have to show $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow \bigvee I \cap \mathcal{J}(\mathcal{L})) = fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(I)$. Since $\downarrow \bigvee I \cap \mathcal{J}(\mathcal{L}) \supseteq I$ the direction "$\supseteq$" is clear. For "$\subseteq$" let $j \leq \bigvee I$ and $m \in fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j)$. By Lemma 1.2.11 we have $j \not\leq m$, so $\bigvee I \not\leq m$. Hence there must be some $j' \in I$ with $j' \not\leq m$. Thus by Lemma 1.2.11, $m \in fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j') \subseteq fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(I)$. $\square$

We have shown that every finite lattice can be represented by an ACP $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$. But there are many "fairly different looking" ACPs representing the same lattice. See for example Figure 1.4. We will now define *good* ACPs and prove that ACPs of the form $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ are good. Afterwards we will show that up to isomorphism $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ is the only good ACP representing $\mathcal{L}$.

Definition 1.2.14. We call an ACP $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ *good* if   good ACP

    1. $x \parallel x' \Longrightarrow fing_{\mathcal{A}_{\mathcal{Q}}}(\downarrow x) \parallel fing_{\mathcal{A}_{\mathcal{Q}}}(\downarrow x')$,
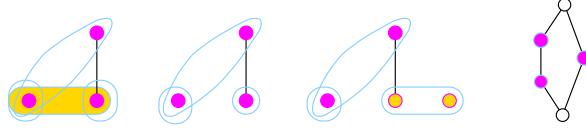
Figure 1.4: Three ACPs representing the lattice on the right. Only the ACP in the middle is good. The golden antichain in the left-most ACP violates part 2. of Definition 1.2.14. The golden elements of the right-most ACP violate part 1. of Definition 1.2.14.

2. $\forall y \in \mathcal{Q} \, \exists x \in A_y : fing_{\mathcal{A}_{\mathcal{Q}}}(\overline{\uparrow A_y} \cup \{x\}) = fing_{\mathcal{A}_{\mathcal{Q}}}(\overline{\uparrow A_y}) \cup \{y\}$.

Proposition 1.2.15. *Let $\mathcal{L}$ be a finite lattice. The ACP $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ is good.*

Proof. We start proving part 1. of Definition 1.2.14. Let $j, j' \in \mathcal{J}(\mathcal{L})$ and say $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j) \subseteq fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j')$. By Lemma 1.2.11 this is the same as $\uparrow j \cap \mathcal{M}(\mathcal{L}) \supseteq \uparrow j' \cap \mathcal{M}(\mathcal{L})$. Now Lemma 1.2.8 yields $j \leq j'$.

It remains part 2. of Definition 1.2.14. So we have to show that every $A_m \in \mathcal{A}_{\mathcal{M}(\mathcal{L})}$ contains a j such that $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\overline{\uparrow A_m} \cup \{j\}) = fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\overline{\uparrow A_m}) \cup \{m\}$. So let $A_m \in \mathcal{A}_{\mathcal{M}(\mathcal{L})}$. By Lemma 1.2.9 we can choose a join-irreducible $j \in \mathrm{Min}(\downarrow m^+ \backslash \downarrow m)$. In particular $m \geq j^-$ and $m \not\geq j$ and thus $j \in A_m$. We have $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\overline{\uparrow A_m} \cup \{j\}) \supseteq fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\overline{\uparrow A_m}) \cup \{m\}$. Let us see what happens if $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(j)$ contains more elements than m. Assume now that $j \in A_{m'}$.

If $m' \not> m$, then by definition some element $y \in A_{m'}$ is not contained in $\uparrow A_m$. Hence $y \in \overline{\uparrow A_m}$ and $m' \in fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\overline{\uparrow A_m})$.

If $m' > m$, then $m' \geq m^+$. Since $j \in \downarrow m^+ \backslash \downarrow m$ this implies $m' \geq j$ and in particular, $m' \notin \uparrow j^- \backslash \uparrow j$, i.e., $j \notin A_{m'}$ – a contradiction.

We have shown $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\overline{\uparrow A_m} \cup \{j\}) = fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\overline{\uparrow A_m}) \cup \{m\}$. $\square$

The last part of Theorem 1.2.3 that remains to be shown is, that every good ACP $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ with $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq) \cong \mathcal{L}$ is isomorphic to $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$, where ACP isomorphisms still has to be defined. First we need two more lemmas.

Lemma 1.2.16. *If $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ is good, then*

$$\mathcal{P} \cong (fing_{\mathcal{A}_{\mathcal{Q}}}(\{\downarrow x \mid x \in \mathcal{P}\}), \subseteq) = \mathcal{J}(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq).$$

Proof. It is a direct consequence of part 1. of Definition 1.2.14 that mapping $x \in \mathcal{P}$ to $fing_{\mathcal{A}_{\mathcal{Q}}}(\downarrow x) \in (fing_{\mathcal{A}_{\mathcal{Q}}}(\{\downarrow x \mid x \in \mathcal{P}\}), \subseteq)$ is an order-isomorphism.

For $(fing_{\mathcal{A}_{\mathcal{Q}}}(\{\downarrow x \mid x \in \mathcal{P}\}), \subseteq) = \mathcal{J}(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ we show equality of the ground sets. This is enough since the order is defined equivalently on both sides. First prove "$\subseteq$". Since $\mathcal{A}_{\mathcal{Q}}$ consists of antichains and is a cover of $\mathcal{P}$, adding $x' > x$ to $\downarrow x$ increases the

fingerprint. Since $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ is good, also adding $x' \parallel x$ to $\downarrow x$ increases the fingerprint. We have that $\downarrow x$ is a distinguished ideal.

Now suppose $fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x)$ is not join-irreducible, i.e., there are I and I' with $fing_{\mathcal{A}_\mathcal{Q}}(I), fing_{\mathcal{A}_\mathcal{Q}}(I') \neq fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x)$ but $fing_{\mathcal{A}_\mathcal{Q}}(I) \cup fing_{\mathcal{A}_\mathcal{Q}}(I') = fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x)$. Thus, $fing_{\mathcal{A}_\mathcal{Q}}(I \cup I') = fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x)$ by Observation 1.2.4. Since $\downarrow x$ is distinguished we have $I \cup I' \subseteq \downarrow x$. Since $\mathcal{A}_\mathcal{Q}$ consists of antichains at least one of I, I' contains x and hence equals $\downarrow x$ – a contradiction.

For "$\supseteq$" let $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ be a distinguished ideal such that $fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) \in \mathcal{J}(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$. Observe that $\emptyset \notin \mathcal{J}(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$, because $\emptyset = fing_{\mathcal{A}_\mathcal{Q}}(\emptyset)$ is the minimum of $(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ and thus is not join-irreducible by definition. Hence $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ is not empty. Suppose $\text{Max}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) = \{x_1, \ldots, x_k\}$ with $k > 1$, because otherwise $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} = \downarrow x_1$. Since $I = \downarrow x_1 \cup \ldots \cup \downarrow x_k$ we have $fing_{\mathcal{A}_\mathcal{Q}}(I) = fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x_1) \cup \ldots \cup fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x_k)$ by Observation 1.2.4. Using the first part of $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ being good we have that $fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x_1), \ldots, fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x_k)$ are mutually incomparable. Since $fing_{\mathcal{A}_\mathcal{Q}}(I) \supsetneq fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x_1), \ldots, fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x_k)$ we have $fing_{\mathcal{A}_\mathcal{Q}}(I) \neq fing_{\mathcal{A}_\mathcal{Q}}(\downarrow x_i)$ for all $i \in [k]$. Hence $fing_{\mathcal{A}_\mathcal{Q}}(I)$ is not join-irreducible – a contradiction. $\qquad\square$

Lemma 1.2.17. *If $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ is good then*

$$\mathcal{Q} \cong (fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow \mathcal{A}_\mathcal{Q}}), \subseteq) = \mathcal{M}(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq),$$

where $\overline{\uparrow \mathcal{A}_\mathcal{Q}} := \{\overline{\uparrow A_y} \mid A_y \in \mathcal{A}_\mathcal{Q}\}$.

Proof. We start by showing $\mathcal{Q} \cong (fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow \mathcal{A}_\mathcal{Q}}), \subseteq)$. Let $y, y' \in \mathcal{Q}$. We have $y \leq y' :\Longleftrightarrow \uparrow A_y \supseteq \uparrow A_{y'} \Longleftrightarrow \overline{\uparrow A_y} \subseteq \overline{\uparrow A_{y'}}$. Ideals of the form $\overline{\uparrow A_z}$ are distinguished by Lemma 1.2.7. Thus, by Proposition 1.2.5 mapping $\overline{\uparrow A_z}$ to $fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow A_z})$ is an order-embedding since it is obviously surjective. We have proved $\mathcal{Q} \cong (fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow \mathcal{A}_\mathcal{Q}}), \subseteq)$.

For $(fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow \mathcal{A}_\mathcal{Q}}), \subseteq) = \mathcal{M}(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ we only need to show equality of the ground sets. Start by proving "$\subseteq$", i.e., let $fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow A_y}) \in fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow \mathcal{A}_\mathcal{Q}})$. Since $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ is good there is some $x \in A_y$ such that $fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow A_y} \cup \{x\}) = fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow A_y}) \cup \{y\}$. Hence $fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow A_y} \cup \{x\})$ is the unique cover of $fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow A_y})$ in $(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$, i.e., $fing_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow A_y})$ is a meet-irreducible.

For the "$\supseteq$"-direction let $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ be a distinguished ideal such that $fing_{\mathcal{A}_\mathcal{Q}}(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) \in \mathcal{M}(fing_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$. By Lemma 1.2.7 we may represent $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ as meet of distinguished ideals i.e., $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} = \bigcap_{y \in F} \overline{\uparrow A_y}$ for some set $F \subseteq \mathcal{Q}$. Since $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ is meet-irreducible it is not the maximum $\mathcal{P}$ of $(\lceil \mathcal{I}(P) \rceil_{\mathcal{A}_\mathcal{Q}}, \subseteq)$ and consequently $F \neq \emptyset$. Thus, because $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ is meet-irreducible we have $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} = \overline{\uparrow A_y}$ for some $y \in F$.

$\qquad\square$

Definition 1.2.18. Let $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ and $(\mathcal{P}', \mathcal{A}_{\mathcal{Q}'})$ be ACPs. A mapping $\varphi : \mathcal{P} \to \mathcal{P}'$ is called an *ACP-isomorphism* if φ is an order-isomorphism and $\varphi(A_y) \in \mathcal{A}_{\mathcal{Q}'} \Longleftrightarrow A_y \in \mathcal{A}_\mathcal{Q}$.

ACP-isomorphism

This is, φ preserves the order and the antichain-partititon. Note that φ also induces an order-isomorphism of the index-posets. After this fairly natural definition we first check that isomorphic lattices will be represented by isomorphic good ACPs. Otherwise the definition would not be so good.

Proposition 1.2.19. *Let $\mathcal{L}, \mathcal{L}'$ be isomorphic finite lattices. The ACPs $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ and $(\mathcal{J}(\mathcal{L}'), \mathcal{A}_{\mathcal{M}(\mathcal{L}')})$ are isomorphic.*

Proof. Let $\varphi : \mathcal{L} \to \mathcal{L}'$ be an order-isomorphism. We show that φ induces an ACP-isomorphism from $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ to $(\mathcal{J}(\mathcal{L}'), \mathcal{A}_{\mathcal{M}(\mathcal{L}')})$. Clearly, φ induces an order-isomorphism of $\mathcal{J}(\mathcal{L})$ and $\mathcal{J}(\mathcal{L}')$ and also of $\mathcal{M}(\mathcal{L})$ and $\mathcal{M}(\mathcal{L}')$. It remains to show that $A_m \in \mathcal{A}_{\mathcal{M}(\mathcal{L})} \iff \varphi(A_m) \in \mathcal{A}_{\mathcal{M}(\mathcal{L}')}$. But $A_m \in \mathcal{A}_{\mathcal{M}(\mathcal{L})}$ by definition is equivalent to $A_m = \{j \in \mathcal{J}(\mathcal{L}) \mid m \in {\uparrow}j^- \backslash {\uparrow}j\}$ for $m \in \mathcal{M}(\mathcal{L})$. We apply φ and since it is an order-isomorphism of join- and meet-irreducibles we obtain the equivalent statement $\varphi(A_m) = \{\varphi(j) \in \mathcal{J}(\mathcal{L}') \mid \varphi(m) \in {\uparrow}\varphi(j)^- \backslash {\uparrow}\varphi(j)\}$ for $\varphi(m) \in \mathcal{M}(\mathcal{L}')$. This is equivalent to $\varphi(A_m) = A_{\varphi(m)} \in \mathcal{A}_{\mathcal{M}(\mathcal{L}')}$. $\qquad\square$

We are ready to prove the last part of Theorem 1.2.3.

Proposition 1.2.20. *Let $\mathcal{L}$ be a finite lattice and $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ a good ACP such that $\mathcal{L} \cong (\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$. We have $(\mathcal{P}, \mathcal{A}_\mathcal{Q}) \cong (\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$.*

Proof. By Proposition 1.2.19 it is enough to show

$$(\mathcal{P}, \mathcal{A}_\mathcal{Q}) \cong (\mathcal{J}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq), \mathcal{A}_{\mathcal{M}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)}).$$

So let $\varphi : \mathcal{P} \to \mathcal{J}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ be the map defined as $\varphi(x) := \text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x)$. By Lemma 1.2.16 φ is an order-isomorphism so we take φ as our candidate for the ACP-isomorphism. We still need to show that φ induces an isomorphism of the antichain-partitions $\mathcal{A}_\mathcal{Q}$ and $\mathcal{A}_{\mathcal{M}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)}$.

Let $A_y \in \mathcal{A}_\mathcal{Q}$. We want to prove that $\varphi(A_y) \in \mathcal{A}_{\mathcal{M}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)}$. Recall that an element of $\mathcal{A}_{\mathcal{M}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)}$ looks like $A_m = \{j \in \mathcal{J}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq) \mid m \in {\uparrow}j^- \backslash {\uparrow}j\}$ for a meet-irreducible $m \in \mathcal{M}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$. Now since every join-irreducible is of the form $\text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x)$ the unique cocover may be written as $\text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x \backslash \{x\})$. By Lemma 1.2.17 meet-irreducibles correspond to elements of the form $\text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\downarrow}A_y})$. Hence every antichain $A_{\text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\downarrow}A_y})} \in \mathcal{A}_{\mathcal{M}(\text{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)}$ is of the form

$$\{\text{fing}_{\mathcal{A}_\mathcal{Q}}(x) \mid \text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x \backslash \{x\}) \subseteq \text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\downarrow}A_y}) \text{ and } \text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x) \not\subseteq \text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\downarrow}A_y})\}$$

We show that $\varphi(A_y) = \{\text{fing}_{\mathcal{A}_\mathcal{Q}}(x) \mid x \in A_y\}$ equals $A_{\text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\downarrow}A_y})}$. For "$\subseteq$" note that $x \in A_y \implies \text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x) \not\subseteq \text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\uparrow}A_y})$ and $\text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x \backslash \{x\}) \subseteq \text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\uparrow}A_y})$.

For "$\supseteq$" observe that $\text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x) \not\subseteq \text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\uparrow}A_y})$ implies $x \in {\uparrow}A_y$. On the other hand $\text{fing}_{\mathcal{A}_\mathcal{Q}}({\downarrow}x \backslash \{x\}) \subseteq \text{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{{\uparrow}A_y})$ means that all cocovers of x are in $\overline{{\uparrow}A_y}$. Hence $x \in A_y$

We have shown equality. Since by Lemma 1.2.17 all meet-irreducibles of $(\mathit{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ are of the form $\mathit{fing}_{\mathcal{A}_\mathcal{Q}}(\overline{\uparrow A_y})$, we may represent all elements of $\mathcal{A}_{\mathcal{M}(\mathit{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)}$ in the above fashion. This is, $\varphi(A_y) \in \mathcal{A}_{\mathcal{M}(\mathit{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)}$ if and only if $A_y \in \mathcal{A}_\mathcal{Q}$. $\qquad\square$

Let us plug everything together to resume how we proved Theorem 1.2.3. In Proposition 1.2.5 we have shown that a finite poset $\mathcal{L}$ is a lattice if $\mathcal{L} \cong (\mathit{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$ for some (good) antichain-covered poset $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$. On the other hand Propsition 1.2.13 shows that every finite $\mathcal{L}$ is isomorphic to $(\mathit{fing}_{A_{\mathcal{M}(\mathcal{L})}}(\mathcal{I}(\mathcal{J}(\mathcal{L}))), \subseteq)$, where $(\mathcal{J}(\mathcal{L}), A_{\mathcal{M}(\mathcal{L})})$ is a good ACP, by Propition 1.2.15. Finally, Proposition 1.2.20 shows that if $\mathcal{L} \cong (\mathit{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$, then $(\mathcal{P}, \mathcal{A}_\mathcal{Q}) \cong (\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$.

To obtain that the map, which takes ACPs to the inclusion-orders on the fingerprints of their ideals indeed induces a one-to-one correspondence between isomorphism-classes of good ACPs and isomorphism-classes of finite lattices, we have to prove one last thing. By Propsition 1.2.19 we know that isomorphic lattices cannot come from non-isomorphic good ACPs but we have to show that isomorphic good ACPs yield isomorphic lattices.

Proposition 1.2.21. *Let $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ and $(\mathcal{P}', \mathcal{A}_{\mathcal{Q}'})$ be ACPs. If $(\mathcal{P}, \mathcal{A}_\mathcal{Q}) \cong (\mathcal{P}', \mathcal{A}_{\mathcal{Q}'})$, then $(\mathit{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq) \cong (\mathit{fing}_{\mathcal{A}_{\mathcal{Q}'}}(\mathcal{I}(\mathcal{P}')), \subseteq)$.*

Proof. Let $\varphi : \mathcal{P} \to \mathcal{P}'$ an isomorphism of $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ and $(\mathcal{P}', \mathcal{A}_{\mathcal{Q}'})$. We will prove that $\phi : \lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \mapsto \varphi(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}})$ is an isomorphism of $(\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_\mathcal{Q}}, \subseteq)$ and $(\lceil \mathcal{I}(\mathcal{P}') \rceil_{\mathcal{A}_{\mathcal{Q}'}}, \subseteq)$. This is enough by Proposition 1.2.5.

The first and most difficult thing to prove here is, that $\varphi(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}})$ is indeed a distinguished ideal of $(\mathcal{P}', \mathcal{A}_{\mathcal{Q}'})$: We will use that φ induces an order-isomorphism φ' of $\mathcal{Q}$ and $\mathcal{Q}'$, by

$$y \leq y' :\Leftrightarrow \uparrow A_y \supseteq \uparrow A_{y'} \Leftrightarrow \varphi(\uparrow A_y) \supseteq \varphi(\uparrow A_{y'}) \Leftrightarrow: \uparrow A_{\varphi'(y)} \supseteq \uparrow A_{\varphi'(y')} \Leftrightarrow \varphi'(y) \leq \varphi'(y').$$

Moreover we apply Lemma 1.2.7 to obtain that an ideal is distinguished with respect to $\mathcal{A}_\mathcal{Q}$ if and only if $I = \bigcap_{y \in F} \overline{\uparrow A_y}$ for a filter F of $\mathcal{Q}$. We calculate:

$$
\begin{aligned}
& & \varphi(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) & = & \varphi(\textstyle\bigcap_{y \in F} \overline{\uparrow A_y}) \\
& = & \varphi(\mathcal{P} \backslash \textstyle\bigcup_{y \in F} \uparrow A_y) & = & \varphi(\mathcal{P}) \backslash \textstyle\bigcup_{y \in F} \varphi(\uparrow A_y) \\
& = & \mathcal{P}' \backslash \textstyle\bigcup_{\varphi'(y) \in \varphi'(F)} \uparrow A_{\varphi'(y)}
\end{aligned}
$$

Since φ is an isomorphism $\varphi(F)$ is a filter of $\mathcal{Q}'$, and we obtain a distinguished ideal by Lemma 1.2.7.

The above chain of arguments could equivalently be applied to φ^{-1}. Thus ϕ is a bijection of $\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_\mathcal{Q}}$ and $\lceil \mathcal{I}(\mathcal{P}') \rceil_{\mathcal{A}_{\mathcal{Q}'}}$. That ϕ is an order-embedding follows from a straightforward equivalence transformation yielding $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \subseteq \lceil I \rceil'_{\mathcal{A}_\mathcal{Q}} \Leftrightarrow \varphi(\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}) \subseteq \varphi(\lceil I \rceil'_{\mathcal{A}_\mathcal{Q}})$. $\qquad\square$

We restate the result as a new theorem:

Theorem 1.2.22. *Isomorphism-classes of finite lattices and of good ACPs are in one-to-one-correspondence, induced by $\mathcal{L} \mapsto (\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ and its inverse $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}}) \mapsto (fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(P)), \subseteq)$.*

How Theorem 1.2.3 specializes back to Birkhoff's Theorem and a strengthening of Nourine's Theorem will be shown in the next subsection.

1.2.1 Applications

In this section we will refine Theorem 1.2.3 to specific classes of finite lattices. As promised in the beginning of the section we first prove a strengthening of Nourine's Theorem (Theorem 1.2.2). For that matter we call an antichain-partitioned poset $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ *reduced* if

reduced
antichain-partition

$$x \parallel x' \implies fing_{\mathcal{A}_{\mathcal{Q}}}(\downarrow x) \parallel fing_{\mathcal{A}_{\mathcal{Q}}}(\downarrow x').$$

We introduced this definition because it is more economical than *good*, but:

Observation 1.2.23. An antichain-*partitioned* poset is good if and only if it is reduced.

We can now prove:

Theorem 1.2.24. *A finite lattice $\mathcal{L}$ is a ULD if and only if $\mathcal{L} \cong (fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ for some poset $\mathcal{P}$ with reduced antichain-partition $\mathcal{A}_{\mathcal{Q}}$. Moreover, $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}}) \cong (\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$.*

Proof. By Observation 1.2.23 an antichain-partitioned poset is a good if and only if it is reduced. Thus by Theorem 1.2.3 it is enough to show that if $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ is antichain-partitioned, then $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ is a ULD and that if $\mathcal{L}$ is a ULD, then $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ is an antichain-partitioned poset.

So let $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ be an antichain-partitioned poset. By Proposition 1.2.5 $(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$ is isomorphic to the inclusion-order $(\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_{\mathcal{Q}}}, \subseteq)$, where the meet coincides with set-intersection. We prove that $(\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_{\mathcal{Q}}}, \subseteq)$ is a ULD.

Let $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} \in \lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_{\mathcal{Q}}}$. We claim that $M := \{\overline{\uparrow A_y} \mid y \in fing_{\mathcal{A}_{\mathcal{Q}}}(\mathrm{Min}(\overline{\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}}))\}$ is the $M_{\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}}$ of the definition of ULD (Definition 1.1.6), i.e., $M_{\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}}$ is the unique inclusion-minimal set of meet-irreducibles such that $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} = \bigcap M_{\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}}$. Recall that ideals of the form $\overline{\uparrow A_y}$ are distinguished ideals and correspond to $\mathcal{M}(fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$, by Lemma 1.2.17. Hence, $M \subseteq \mathcal{M}(\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_{\mathcal{Q}}}, \subseteq)$.

We start by showing that $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ is indeed the meet of M. First, we show that $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} \subseteq \bigcap M$. Let $\overline{\uparrow A_y} \in M$. If there were $x \in A_y \cap \lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ then either adding $A_y \cap \mathrm{Min}(\overline{\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}})$ to $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ does not increase the fingerprint of $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ which contradicts maximality of $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}$ or $A_y \cap \mathrm{Min}(\overline{\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}}})$ must intersect some other $A_{y'} \in \mathcal{A}_{\mathcal{Q}} \backslash \{A_y\}$ with $y' \in fing_{\mathcal{A}_{\mathcal{Q}}}(\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}})$ contradicting that $\mathcal{A}_{\mathcal{Q}}$ is a partition. Thus $A_y \cap \lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} = \emptyset$ which implies $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} \subseteq \overline{\uparrow A_y}$. We obtain $\lceil I \rceil_{\mathcal{A}_{\mathcal{Q}}} \subseteq \bigcap M$.

In order to to show $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \supseteq \bigcap M$, let $x \in \bigcap M$. This is equivalent to $x \notin \bigcup_{y \in fing_{\mathcal{A}_\mathcal{Q}}(\mathrm{Min}(\overline{\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}}))} \uparrow A_y$. In particular $x \notin \uparrow \mathrm{Min}(\overline{\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}})$ and consequently $x \in \lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$. We have proved $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} = \bigcap M$.

Now suppose M is not the unique inclusion-minimal with $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} = \bigcap M$, i.e., there is another set of meet-irreducibles M' whose intersection is $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$ and there is some meet-irreducible $\overline{\uparrow A_y} \in M \backslash M'$. By the choice of M at least one of the elements $x \in A_y$ has all its predecessors in $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$. Now if $\overline{\uparrow A_{y'}} \in M'$ then $x \in \overline{\uparrow A_{y'}}$, otherwise $x \in \uparrow A_{y'}$ but since $\mathcal{A}_\mathcal{Q}$ is a partition and $x \in A_y$ we have $x \notin A_{y'}$. Thus, $A_{y'} \cap \lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \neq \emptyset$. This implies $\lceil I \rceil_{\mathcal{A}_\mathcal{Q}} \nsubseteq \overline{\uparrow A_{y'}}$. Hence $x \in \overline{\uparrow A_{y'}}$ for all $\overline{\uparrow A_{y'}} \in M'$. Thus $x \in \bigcap M'$ – a contradiction because $x \notin \lceil I \rceil_{\mathcal{A}_\mathcal{Q}}$.

Let on the other hand $\mathcal{L}$ be a ULD and $j \in \mathcal{J}(\mathcal{L})$, i.e., for every $\ell \in \mathcal{L}$ there is a unique inclusion-minimal set $M_\ell \subseteq \mathcal{M}(\mathcal{L})$ such that $\bigwedge M_\ell = \ell$. Suppose that $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ is not an antichain-partitioned poset. We know that it is an ACP by Proposition 1.2.15, hence the problem must be that $\mathcal{A}_{\mathcal{M}(\mathcal{L})}$ is no partition. So for some $j \in \mathcal{J}(\mathcal{L})$ we have two meet-irreducibles $m, m' \in \Uparrow j^- \backslash \Uparrow j$. This implies $m \wedge j = m' \wedge j = j^-$. Thus, the set M_{j^-} must satisfy $M_{j^-} \subseteq (M_j \cup \{m\}) \cap (M_j \cup \{m'\})$ and $m, m' \notin M_j$. Thus $M_{j^-} \subseteq M_j$, thus $\bigwedge M_{j^-} \geq \bigwedge M_j$ which means $j^- \geq j$ – a contradiction. $\square$

By Theorem 1.2.22 there is indeed a one-to-one-correspondence between finite ULDs and posets with reduced antichain-partition. We will now reprove Birkhoff's Theorem [14]. It was stated as Theorem 1.2.1 in the beginning of the section. As Nourine's Theorem it is a refinement of Theorem 1.2.3. We restate for convenience:

Theorem 1.2.1. *A finite lattice $\mathcal{L}$ is distributive if and only if $\mathcal{L} \cong (\mathcal{I}(\mathcal{P}), \subseteq)$ for a poset $\mathcal{P}$. Moreover, $\mathcal{P} \cong \mathcal{J}(\mathcal{L})$.*

Proof. Let $A_\mathcal{Q}$ be the singleton-partition of $\mathcal{P}$, i.e., all antichains consist of a single element. Thus we may identify $\mathcal{Q}$ and $\mathcal{P}$. Hence we identify $fing_{\mathcal{A}_\mathcal{Q}}(I)$ with I and $(\mathcal{P}, A_\mathcal{Q})$ with $\mathcal{P}$. Note that the singleton-partition is good and we can apply Theorem 1.2.3. We show that $(\mathcal{I}(\mathcal{P}), \subseteq)$ is distributive and that if $\mathcal{L}$ is distributive, then $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ is a singleton-partitioned poset.

For $(\mathcal{I}(\mathcal{P}), \subseteq)$ we know by Observation 1.1.2 that meet and join of this lattice are intersection and union. It is straight forward to check that $\cap$ and $\cup$ satisfy the distributive laws.

On the other hand, let $\mathcal{L}$ be distributive. We know that $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$ is an ACP by Proposition 1.2.15, so if it is no singleton-partitioned poset there are $j, k \in \mathcal{J}(\mathcal{L})$ and a meet-irreducible $m \in (\Uparrow j^- \backslash \Uparrow j) \cap (\Uparrow k^- \backslash \Uparrow k)$, i.e., an antchain A_m of size at least 2. If $k > j$, then $k^- \geq j$ so $m \geq k^- \geq j$, i.e., m cannot lie in $(\Uparrow j^- \backslash \Uparrow j) \cap (\Uparrow k^- \backslash \Uparrow k)$. Thus j and k are incomparable. Hence $j \wedge k = j^- \wedge k^-$. Since $j^-, k^- \leq m$ by assumption we calculate $(j \wedge k) \vee m = (j^- \wedge k^-) \vee m = m$. On the other hand since $j, k \parallel m$ we have $(j \vee m), (k \vee m) \neq m$ and since m is meet-irreducible we have $m \neq (j \vee m) \wedge (k \vee m)$. This contradicts distributivity. $\square$

Question 1.2.25. It would be interesting to characterize other lattice classes in terms of their representation as antichain-covered posets. One reason for this interest is that these representations are more economical. One example where no such characterization is known, are upper semi-modular lattices. Another class of particular interest to us are lattices whose Hasse-diagram admits an arc-coloring, such that all maximal chains between two elements have the same multiset of colors. Moreover, the outgoing arcs of an element are colored mutually different. We call these properties together the *colored Jordan-Dedekind chain condition*. Both lattice classes are natural generalizations of ULDs.

Question 1.2.26. There is a generalization of Birkhoff's Theorem due to Dilworth [32], called *Dilworth's Embedding Theorem*. It establishes a correspondence between cover-preserving sublattice embeddings of finite distributive lattices into $\mathbb{N}^d$ and chain-partitions of posets. It would be interesting to generalize this result to more general lattice classes. One such generalization to cover-preserving join-sublattice embeddings of finite ULDs into $\mathbb{N}^d$ and chain-partitions of antichain-partitioned posets is obtained at the end of the next section, see Theorem 1.3.18.

1.2.2 Duality

Before we continue with a new type of results in the next section, in this subsection we will remark, that there is a "dual" way of characterizing finite lattices by antichain-covered posets. These results will not be used further on and we mention them just because, they somehow complete the picture presented so far in this section. We sketch this different way of representing finite lattices in the following. The basic idea is to switch the role of $\mathcal{P}$ and the index-poset $\mathcal{Q}$.

Given a good ACP $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ we define $A_x := \mathrm{Max}\{y \in \mathcal{Q} \mid x \in \uparrow A_y\}$ for every $x \in \mathcal{P}$. Setting $\mathcal{A}_\mathcal{P} := \{A_x \mid x \in \mathcal{P}\}$, we obtain an antichain-covered poset $(\mathcal{Q}, \mathcal{A}_\mathcal{P})$ called the *dual ACP* of $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$. See Figure 1.5 for an example.
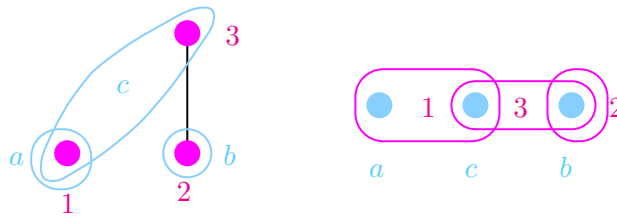
dual ACP

Figure 1.5: Primal and dual ACP. Magenta numbers are elements of $\mathcal{P}$ and blue letters are elements of $\mathcal{Q}$. Both represent the lattice in Figure 1.4

Proposition 1.2.27. *We have* $(\mathit{fing}_{\mathcal{A}_\mathcal{P}}(\mathcal{F}(\mathcal{Q})), \supseteq) \cong (\mathit{fing}_{\mathcal{A}_\mathcal{Q}}(\mathcal{I}(\mathcal{P})), \subseteq)$.

Proof. First observe that $fing_{\mathcal{A}_{\mathcal{P}}}(y) = \uparrow A_y \backslash \bigcup_{A_z \subseteq \uparrow A_y} A_z$. Hence

$$
\begin{aligned}
& (fing_{\mathcal{A}_{\mathcal{P}}}(\mathcal{F}(\mathcal{Q})), \supseteq) \\
= & (\{\uparrow \cup_{y \in F} A_y \mid F \in \mathcal{F}(\mathcal{Q})\}, \supseteq) \\
\cong & (\{\overline{\uparrow \cup_{y \in F} A_y} \mid F \in \mathcal{F}(\mathcal{Q})\}, \subseteq) \\
= & (\{\bigcap_{y \in F} \overline{\uparrow A_y} \mid F \in \mathcal{F}(\mathcal{Q})\}, \subseteq) \\
= & (\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_{\mathcal{Q}}}, \subseteq)
\end{aligned}
$$

The last equality comes from Lemma 1.2.7 Finally, Proposition 1.2.5 says that the $(\lceil \mathcal{I}(\mathcal{P}) \rceil_{\mathcal{A}_{\mathcal{Q}}}, \subseteq) \cong (fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P})), \subseteq)$. $\quad\square$

Dual to Definition 1.2.14 one defines an ACP $(\mathcal{Q}, \mathcal{A}_{\mathcal{P}})$ to be *cogood* if

1. $y \parallel y' \Longrightarrow fing_{\mathcal{A}_{\mathcal{P}}}(\uparrow y) \parallel fing_{\mathcal{A}_{\mathcal{P}}}(\uparrow y')$,
2. $\forall x \in \mathcal{P} \, \exists y \in A_x : fing_{\mathcal{A}_{\mathcal{P}}}(\overline{\downarrow A_x} \cup \{y\}) = fing_{\mathcal{A}_{\mathcal{P}}}(\overline{\downarrow A_x}) \cup \{x\}$.

Indeed, an ACP $(\mathcal{Q}, \mathcal{A}_{\mathcal{P}})$ is cogood if and only if it is the dual of a good ACP. Given a finite lattice $\mathcal{L}$ we define $A_j := \{m \in \mathcal{M}(\mathcal{L}) \mid j \in \downarrow m^+ \backslash \downarrow m\}$ for all $j \in \mathcal{J}(\mathcal{L})$ and let $\mathcal{A}_{\mathcal{J}(\mathcal{L})}$ be their collection. One can then prove that $(\mathcal{M}(\mathcal{L}), \mathcal{A}_{\mathcal{J}(\mathcal{L})})$ is the dual ACP of $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$.

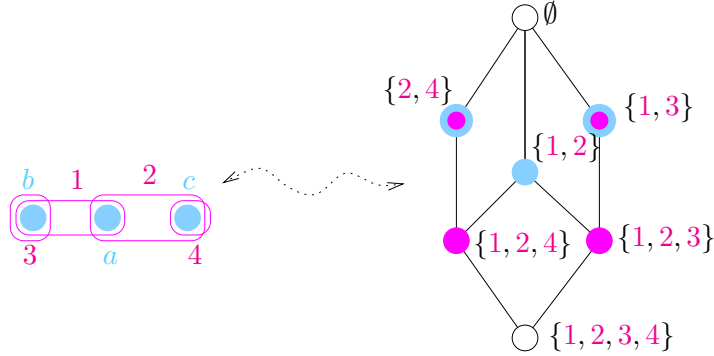Figure 1.6: Representation of a ULD by its cogood ACP. Compare Figure 1.2 for a representation by its good ACP

Question 1.2.28. An example for the representation of a ULD by a cogood ACP is shown in Figure 1.6. Is there a nice characterization of cogood ACPs representing ULDs? It would be enough to characterize the duals of reduced antichain-partitions. By Theorem 1.2.24 this would yield a new characterization of ULDs.

1.3 Hasse Diagrams of Upper Locally Distributive Lattices

In this section we prove a new characterization of ULD lattices in terms of arc-colorings of Hasse diagrams. In many instances where a set of combinatorial objects carries the order structure of a lattice this characterization yields a slick proof of distributivity or upper local distributivity. We have mentioned examples for this in the beginning of this chapter and we will provide a new major application in Section 1.4.

In the proof of our characterization we will establish the equivalence to the original definition of ULD given by Dilworth [31], see Definition 1.1.6. At the end we add a new proof of the known fact that a lattice which is both ULD and LLD is distributive. Graphs, posets and lattices in this section are generally assumed to be *finite* unless specified differently. The following is the class of arc-colorings, which will play the central role in this section:

Definition 1.3.1. Let $D = (V, A)$ be a directed graph and $d \in \mathbb{N}$. An arc-coloring
U-coloring
$c : A \to [d]$ of D is a *U-coloring* if it satisfies the following two rules. For every $x, y, z \in V$ with $y \neq z$ and $(x, y), (x, z) \in A$ one has:

up-proper
- (U$_1$) $c(x, y) \neq c(x, z)$,   (*up-proper*)
- (U$_2$) There is a $w \in V$ and arcs $(y, w), (z, w)$ such that $c(x, y) = c(z, w)$ and
up-complete
$c(x, w) = c(y, w)$, see Figure 1.7.   (*up-complete*)
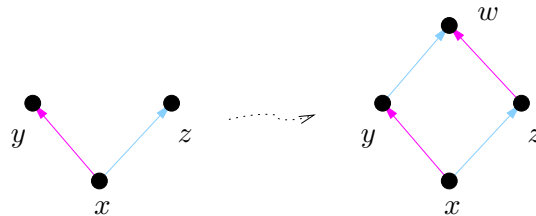
Figure 1.7: The up-completion of U-colorings.

In order to motivate this definition and to present the flavor of its applications think of the vertex set V of D as a set of combinatorial objects and of the arcs as local transformations. We have seen one example of this in the introduction in terms of domino tilings. Even if we will not treat concrete application until the next section, as an example let V be the set of *Eulerian orientations* of a
Eulerian orientation
planar graph G, i.e., orientations such that every vertex has equal in- and outdegree. It is easy to see, that reversing the orientation of a directed cycle in such an orientation preserves the property of being Eulerian. Now choose the local transformations as reversals of directed facial cycles. More precisely, an arc of D corresponds
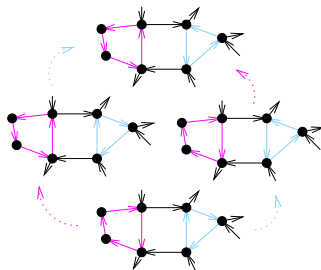
to a pair (D_1, D_2) of Eulerian orientations of G such that D_2 may be obtained from D_1 by reversing the orientation of a *counter-clockwise (ccw)* forward directed facial cycle C. The *natural color* of the arc (D_1, D_2) then is C and it is easy to see that the natural coloring is a U-coloring. In many applications certain natural local transformations lead to such a natural coloring.

<div style="float:left">counter-clockwise (ccw)</div>

<div style="float:right">natural color</div>

For our applications we want that the U-colored digraph D yields the cover-relations of a poset. This way, we obtain order-structure on many kinds of sets of combinatorial objects. So we define a poset $\mathcal{P}$ to be a *U-poset* if the arcs of its Hasse diagram $D_{\mathcal{P}}$ admit a U-coloring. The first main result of this section is that actually *every* acyclic digraph with a U-coloring is the Hasse diagram of a U-poset. Moreover, we prove some properties of U-posets. Therefore define the *Jordan-Dedekind chain condition* of a poset $\mathcal{P}$ as: given any pair of elements all maximal chains between them are of the same length. Even stronger, the $\mathcal{P}$ together with an up-proper coloring of its cover-relations (e.g. with a U-coloring) is said to satisfy the *colored Jordan-Dedekind chain condition* if given any pair of elements the maximal chains between them all use the same multiset of colors. The first main results of the present section then reads:

<div style="float:right">U-poset</div>

<div style="float:right">Jordan-Dedekind chain condition</div>

<div style="float:right">colored Jordan-Dedekind chain condition</div>

Theorem 1.3.2. *An acyclic digraph D admits a U-coloring if and only if $D_{\mathcal{P}}$ is the Hasse diagram of a U-poset $\mathcal{P}$. Moreover, each connected component of a U-poset has a unique maximum and satisfies the colored Jordan-Dedekind chain condition.*

The second main result of this section is that under relatively weak conditions U-posets are ULDs and equivalently isomorphic to cover-preserving join-sublattice of the dominance order. We will indeed be in this case for all our main applications later on.

We define the *dominance order* on $\mathbb{N}^d$ as $x \leq y :\Longleftrightarrow x_i \leq y_i$ for all $i \in [d]$. With this order $\mathbb{N}^d$ forms an infinite distributive lattice with componentwise maximum $\max$ and minimum $\min$ as join and meet. We will only consider finite subposets of $\mathbb{N}^d$. A subposet $\mathcal{P}$ of $\mathcal{Q}$ is *cover-preserving* if $x \prec_{\mathcal{P}} y \Longrightarrow x \prec_{\mathcal{Q}} y$. These definitions allow to state the second main result of this section.

<div style="float:right">dominance order</div>

<div style="float:right">cover-preserving</div>

Theorem 1.3.3. *For a finite poset $\mathcal{L}$ the following are equivalent:*

- *(i) There is an acyclic digraph D with U-coloring c and unique source such that $\mathcal{L} \cong \mathcal{P}_D$, where $\mathcal{P}_D$ is the transitive hull of D,*
- *(ii) there is $d \in \mathbb{N}$ and an order-embedding $\gamma : \mathcal{L} \to \mathbb{N}^d$ such that $\gamma(\mathcal{L})$ is a cover-preserving join-sublattice of $\mathbb{N}^d$,*
- *(iii) $\mathcal{L}$ is an upper locally distributive lattice.*

Moreover, given such $\mathcal{L}$ its U-colorings, the cover-preserving join-sublattice embeddings, and chain-partitions of $\mathcal{M}(\mathcal{L})$ translate into each other via the equivalence.

The following lemma describes the iterated application of the rules of U-colorings. It is the main tool for the proof of Theorem 1.3.2 and the $(i) \Longrightarrow (ii)$ part of Theorem 1.3.3.
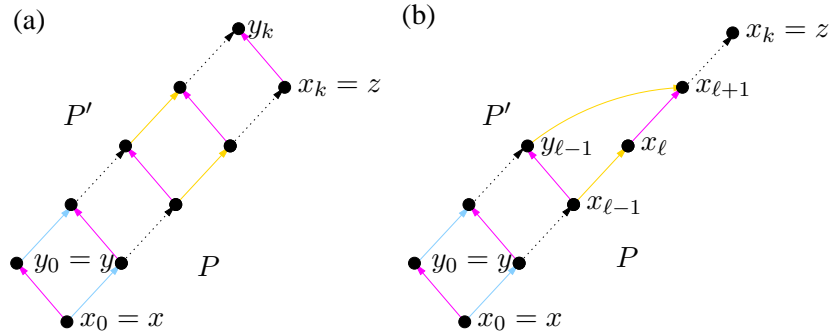
Figure 1.8: Iterated application of U_1 and U_2 in Lemma 1.3.4.

Lemma 1.3.4. *Let $D = (V, A)$ be a digraph with a U-coloring c and $x, y, z \in V$. If (x, y) is an arc and $P = (x = x_0, \ldots, x_k = z)$ a directed path, then there is a path $P' = (y = y_0, \ldots, y_\ell)$ with $\ell \leq k$ such that $(x_i, y_i) \in A$ and $c(x_i, y_i) = c(x, y)$ for $i = 1, .., \ell$. We either have* (a): $\ell = k$ *or* (b): $y_\ell = x_{\ell+1}$, *see Figure 1.8. Moreover, case (b) happens if and only if there is an arc $(x_\ell, x_{\ell+1})$ on P with $c(x_\ell, x_{\ell+1}) = c(x, y)$.*

Proof. We apply rule U_2 recursively to arcs (x_i, y_i) and (x_i, x_{i+1}) to define a vertex $y_{i+1} \in P'$ with arcs (y_i, y_{i+1}) and (x_{i+1}, y_{i+1}) such that $c(x_i, y_i) = c(x_{i+1}, y_{i+1})$. The iteration either ends if $i = k$ (case (a)), or if the two arcs needed for the next application of the rule are the same, i.e., $y_i = x_{i+1}$ (case (b)). In this case $c(x_i, x_{i+1}) = c(x_{i-1}, y_{i-1}) = c(x, y)$, i.e., there is an arc on the path P whose color equals the color of arc (x, y). Rule U_1 implies that case (b) occurs whenever there is an arc on P whose color equals the color of arc (x, y). $\square$

Remark 1.3.5. The proof does not imply that $y_i \neq x_j$ in all cases, as it is suggested by Figure 1.8. An example is given Figure 1.9. From the analysis below it follows that in all *bad* cases D is not acyclic (or infinite).

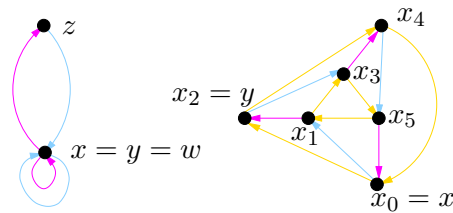Figure 1.9: Two bad things that can happen in U-colored digraphs with directed cycles. The one on the left visualizes that the up-completion of a U-coloring might look different from the one in Figure 1.7. The digraph on the right shows, that the application of Lemma 1.3.4 does not always look as in Figure 1.8. Choose $P = x_0, .., x_5$ and $y = x_2$. We get $P' = (y_i)_{0 \leq i \leq 6}$, where $y_i = x_{i+2 \pmod 6}$.

If $D = (V, A)$ is a connected, acyclic digraph with a U-coloring c, then the transitive closure of D is a finite poset $\mathcal{P}_D$. From the next two propositions it will follow that in this case D is indeed transitively reduced, i.e., it is the Hasse diagram of $\mathcal{P}_D$. Hence, $\mathcal{P}_D$ is a U-poset.

Proposition 1.3.6. *Let $D = (V, A)$ be a connected digraph with a U-coloring c. For every pair $s, t \in V$ there exists a vertex $r \in V$, such that s lies on a directed (s, r)-path and t on a directed (t, r)-path. In particular if D is acyclic, then it has a unique sink $\top$, i.e., a vertex without outgoing arcs.*   sink

Proof. Let $D = (V, A)$ be a connected digraph with a U-coloring c. Let $s, t \in V$. We show that there is a shortest (s, t)-path S that does not traverse a triple of vertices (y, x, z) of the form *sink-source-sink*. By a sink-source-sink triple (y, x, z) we refer to that y, z have no outgoing arcs in S and x has no incoming arcs in S. If we have a path S without such a triple, then S has a unique sink, i.e., a vertex without outgoing arcs in S. This would be r.

The proof is constructive. Let S be any shortest (s, t)-path with sink-source-sink-triple (y, x, z). Let y_0 be the vertex before arriving at x and denote the restriction of S between x an z as $P = (x = x_0, \ldots, x_k = z)$. We apply Lemma 1.3.4 to the arc (x, y_0) and P. Since P was a shortest path we are in case (a) of the lemma. The lemma gives us a path P' from y_0 to y_k and assures that $(z, y_k) \in A$. We have a new shortest path S'. It consists of S until y_0, then P', then the arc (z, y_k) and then the part of S from z to t. Our new path has the sink-source-sink-triple (y, y_0, y_k). The number of arcs connecting the triple (y, y_0, y_k) is less than in (y, x, z). We can continue like that until the sink-source-sink-triple lies on a 2-path as in the precondition for U_2 and we transform it into a single sink. This way we reduce S until we obtain a path with a unique sink.

If D is acyclic, then it has at least one sink. Suppose it has two sinks $\top_1$ and $\top_2$. By what we have proved there is a vertex r such that $\top_1$ lies on a directed $(\top_1, r)$-path and $\top_2$ on a directed $(\top_2, r)$-path. Thus at least one of $\top_1$ and $\top_2$ was no sink – a contradiction. $\qquad\square$

Let P be a directed path in $D = (V, A)$ with U-coloring c. We define the *colorset $c(P)$ of P* as the multi-set of colors used on the arcs of P.   colorset of a path

Proposition 1.3.7. *Let $D = (V, A)$ be an acyclic digraph with U-coloring c and $x, z \in V$. If P, Q are directed (x, z)-paths in D, then $c(P) = c(Q)$.*

Proof. Assume D to be connected otherwise we prove the claim component by component. By Proposition 1.3.6 there is a unique sink $\top$ in D. Denote for any $x \in V$ by $S(x)$ the set of vertices that lie on directed $(x, \top)$-paths. Since D is acyclic if $(x, y) \in A$, then $S(x) \supsetneq S(y)$.

Suppose there is a pair of vertices x, z contradicting the statement of the proposition. Take a counterexample that minimizes $S(x)$. Let P, Q be directed (x, z)-paths and $c(P) \neq c(Q)$. Let y be the successor of x on Q. The arc (x, y) and the path P fulfill the conditions of Lemma 1.3.4.

If we are in case (a) of Lemma 1.3.4, then the path $P' = (y = y_0, \ldots, y_k)$ has the colorset $c(P)$. The path Q' defined by starting at y, then following Q until z and then taking the arc to y_k has the same colorset as Q. This contradicts the minimality in the choice of x, z, because we have constructed (y, y_k)-paths P' and Q' with $c(P') \neq c(Q')$ and $S(y) \subsetneq S(x)$.

If we are in case (b) of Lemma 1.3.4, then the path $P' = (y = y_0, \ldots, y_\ell = x_{\ell+1}, \ldots, x_k = z)$ has the colorset $c(P) \backslash \{c(x, y)\}$. The path Q' starting at y, and then following Q until z has the colorset $c(Q) \backslash \{c(x, y)\}$. This contradicts minimality of x, z, because we have constructed (y, z)-paths P' and Q' with $c(P') \neq c(Q')$ and $S(y) \subsetneq S(x)$. $\qquad\square$

Since the colorset of a directed (x, z)-path in an acyclic D with U-coloring only depends on the end-vertices x, z we also know that all (x, z)-paths have the same length. This implies that D is transitively reduced and means that $\mathcal{P}_D$ fulfills the colored Jordan-Dedekind chain condition. Proposition 1.3.6 yields the existence of a unique maximum per component. We have thus shown Theorem 1.3.2.

Question 1.3.8. Is it true that in every connected U-colored digraph D, there is a set of colors $I \subseteq [d]$, such that $D \backslash c^{-1}(I)$ is connected and directed paths with coinciding start and end vertices have same colorsets? To obtain an *acyclic* connected digraph with that property would be even better, but this is not generally possible, see the example in Figure 1.9.

We will now proceed to prove Theorem 1.3.3 in the form (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i). At the end of every part of the proof we emphasize how U-colorings, cover-preserving join-sublattice embeddings, and chain-partitons of the poset of meet-irreducible translate into each other (Remark 1.3.10, Remark 1.3.13, Remark 1.3.16).

source

For the first part of the proof we will show that every U-colored acyclic digraph D with unique *source*, i.e., a vertex with indegree 0, leads to an order-embedding γ of $\mathcal{P}_D$ into the dominance order on $\mathbb{N}^d$ such that $\gamma(\mathcal{P}_D)$ a cover-preserving join-sublattice of $\mathbb{N}^d$. By Theorem 1.3.2 $\mathcal{P}_D =: \mathcal{L}$ is a U-poset with Hasse diagram $D = D_\mathcal{L}$. So let $\mathcal{L}$ be a finite U-poset with U-coloring c of $D_\mathcal{L}$ and minimum $\mathbf{0}_\mathcal{L}$. In particular, $D_\mathcal{L}$ is connected thus $\mathcal{L}$ has a unique maximum $\mathbf{1}_\mathcal{L}$ by Proposition 1.3.6. A consequence of Proposition 1.3.7 is

colorset of a
vertex

that the *colorset $c(x)$ of a vertex x*, i.e., the colorset of any directed $(\mathbf{0}_\mathcal{L}, x)$-path in $D_\mathcal{L}$ is well-defined. Define a mapping $\gamma : \mathcal{L} \to \mathbb{N}^d$, where d is the number of colors of c, as

characteristic
vector of a
multiset

$\gamma(x) := \chi(c(x))$. Here $\chi \in \mathbb{N}^d$ is the *characteristic vector of a multiset S*, i.e., the entry $\chi(S)_i$ counts how often element i appears in S. In our particular case this means that $\gamma(x)_i$ records how many arcs of color i appear in a $(\mathbf{0}_\mathcal{L}, x)$-path.

Proposition 1.3.9. *The mapping γ is an order-isomorphism from $\mathcal{L}$ to a cover-preserving join-sublattice of the dominance order on $\mathbb{N}^d$.*

Proof. We have to show that $y \leq z \iff \gamma(y) \leq \gamma(z)$, that there is a $w \in \mathcal{L}$ such that $\gamma(w) = \max(\gamma(y), \gamma(z))$, and that $\gamma(y) \prec \gamma(z) \implies \gamma(z) - \gamma(y) = e_i$, where e_i is the ith unit-vector for some $i \in [d]$,

The implication from $y \leq z$ to $\gamma(y) \leq \gamma(z)$ follows from the fact that extending a path requires more colors.

Now we prove that for all $z, y \in \mathcal{L}$ there is a $z, y \leq w \in \mathcal{L}$ with $\gamma(w) = \max(\gamma(y), \gamma(z))$.

For any fixed y we proceed by top-down induction. If a given z is comparable to y, then $w := \mathrm{Max}(z, y)$ and we are done by the first part of the proof. Consider otherwise a maximal x with the property $x \leq z$ and $x \leq y$. By Observation 1.1.1, this exists because $\mathcal{L}$ has a unique minimum. Let (x, y') be the first arc on a (x, y)-path in $D_{\mathcal{L}}$ and let P be an (x, z)-path. Case (b) of Lemma 1.3.4 is impossible because y' would have prevented us from choosing x, we would obtain $x < y' \leq z, y$ in that case. Hence, we are in case (a) and there is a z' covering z such that the arcs (z, z') and (x, y') have the same color i and moreover, P has no arc of color i. Induction implies that there is a $w' \geq z', y$ such that $\gamma(w') = \max(\gamma(z'), \gamma(y))$. Since $\gamma(z') = \gamma(z) + \mathbf{e}_i$ and for the i-th component $\gamma_i(z) = \gamma_i(x) < \gamma_i(y)$ holds we can conclude $\max(\gamma(z'), \gamma(y)) = \max(\gamma(z), \gamma(y))$, i.e., w' may also serve as w. Note that in the case that $y \parallel z$ we have $w > y, z$.

This already implies $\gamma(y) \leq \gamma(z) \implies y \leq z$. If otherwise $y \parallel z$ and $\gamma(y) \leq \gamma(z)$ by the above there would be a $w > y, z$ with $\gamma(w) = \gamma(z)$ contradicting the first part of the proof.

Since an arc (y, z) of the Hasse diagram of $\mathcal{L}$ is colored by precisely one color, say i, we have $y \prec z \implies \gamma(z) - \gamma(y) = e_i \implies \gamma(y) \prec \gamma(z)$.

Since $\gamma(\mathcal{L})$ is a join-closed subposet of $\mathbb{N}^d$ and has a unique minimum $\gamma(\mathbf{0}_{\mathcal{L}})$ it is a join-sublattice of $\mathbb{N}^d$ by Observation 1.1.4. $\qquad\square$

Remark 1.3.10. We have shown, that every U-coloring of an acyclic D with unique source yields an order embedding γ of $\mathcal{P}_D$ as a cover-preserving join-sublattice of $\mathbb{N}^d$.

The next part of the proof of Theorem 1.3.3 is to show that every element of a finite cover-preserving join-sublattice $\mathcal{L}$ of $\mathbb{N}^d$ has a unique minimal representation as a meet of meet-irreducibles.

For every $x \in \mathcal{L} \subseteq \mathbb{N}^d$ let $I(x) := \{i \in [d] \mid x + e_i \in \mathcal{L}\}$ be the set of directions of the arcs emanating from x in the embedding of the Hasse diagram into $\mathbb{N}^d$. With the next lemma we associate a meet-irreducible element with every $i \in I(x)$.

Lemma 1.3.11. *Let $\mathcal{L}$ be a cover-preserving join-sublattice of $\mathbb{N}^d$. For every $i \in I(x)$ there is a unique maximal element $y(i) \in \mathcal{L}$ such that*

$$y(i) \geq x \text{ and } y(i)_i = x_i.$$

The element $y(i)$ is meet-irreducible and $y(i)_j > x_j$ for all $j \in I(x) \backslash \{i\}$.

Proof. Let $i \in I(x)$ and consider the set $S_i(x)$ of all $y \geq x$ with $y_i = x_i$. The (finite) set $S_i(x)$ contains x, and is closed with respect to componentwise maximum hence it contains a unique maximal element $y(i)$. The element $y(i)$ is meet-irreducible, otherwise we could find a successor of y_i in $S_i(x)$.

Since $x + e_j \in S_i(x)$ for $j \in I(x) \backslash \{i\}$ we conclude that $y(i)_j > x_j$. $\qquad\square$

Proposition 1.3.12. *A cover-preserving join-sublattice $\mathcal{L}$ of $\mathbb{N}^d$ is an upper locally distributive lattice.*

Proof. We claim that $M_x = \{y_i : i \in I(x)\}$ is the unique minimal set of meet-irreducibles with $x = \bigwedge M_x$. If $x = \mathbf{1}_{\mathcal{L}}$ the statement is clear since $\bigwedge \emptyset := \mathbf{1}_{\mathcal{L}}$.

First we show $x = \bigwedge M_x$. Since $\mathcal{L}$ is a subposet of $\mathbb{N}^d$ we have $\bigwedge M_x \leq \min(M_x)$, where the latter is the componentwise minimum of all elements of M_x. Since Lemma 1.3.11 tells us $x \leq M_x$ we moreover know $x \leq \bigwedge M_x$. By Lemma 1.3.11 we know that for each $i \in I(x)$ there is a $x \leq y(i) \in M_x$ with $y(i)_i = x_i$. We conclude $x = \min(M_x)$ and consequently $x = \bigwedge M_x$. In particular the meet of $\mathcal{L}$ and $\mathbb{N}^d$ coincide on M_x.

It remains to show that the representation $x = \bigwedge M_x$ is the unique minimal representation of x as meet of meet-irreducibles. Let $i \in I(x)$ and consider a set M of meet-irreducibles with $y(i) \notin M$. It is enough to show that $x \neq \bigwedge M$. If M contains a y with $x \not< y$, then $x \neq \bigwedge M$ is obvious. Consider the set $S_i(x)$ from the proof of Lemma 1.3.11, every element $y \neq y(i)$ in this set is contained in a $(x, y(i))$-path P that contains no arc with direction i. On the other hand $y \prec y + e_i$ is an arc emanating from y with direction i. This implies that y is not meet-irreducible. Hence $M \cap S_i(x) = \emptyset$. All $y > x$ with $y \notin S_i(x)$ satisfy $y \geq x + e_i$. This implies that $x + e_i$ is a lower bound on M, i.e., $x \neq \bigwedge M$. $\square$
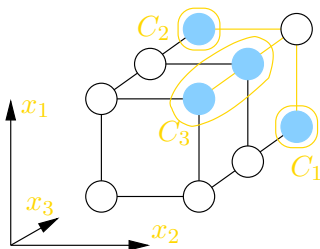

Figure 1.10: An embedded ULD $\mathcal{L}$ and the corresponding chain-partition $\{C_1, C_2, C_3\}$ of $\mathcal{M}(\mathcal{L})$.

Remark 1.3.13. Note that every cover-preserving join-sublattice $\mathcal{L}$ of $\mathbb{N}^d$ yields a d-element chain-partition $\mathcal{C}$ of $\mathcal{M}(\mathcal{L})$ consisting of chains $C_i := \{y \in \mathcal{M}(\mathcal{L}) \mid y + e_i \in \mathcal{L}\}$, see Figure 1.10.

To complete the proof of Theorem 1.3.3 it remains to show that every ULD has a representation as a U-poset, i.e., we have to present a U-coloring of its Hasse diagram. Indeed, we provide a U-coloring of the Hasse-diagram depending on a chain-partition of $\mathcal{M}(\mathcal{L})$.

So let $\mathcal{L}$ be a ULD with chain-partition $\mathcal{C}$ of its poset of meet-irreducibles $\mathcal{M}(\mathcal{L})$. Consider the map $\uparrow x_M := \uparrow x \cap \mathcal{M}(\mathcal{L})$. The definition of meet-irreducible implies that $x = \bigwedge(\uparrow x_M)$ for all x, i.e., the set $\uparrow x_M$ uniquely determines x, by Observation 1.1.5. Moreover, $x \leq y$ if and only if $\uparrow x_M \supseteq \uparrow y_M$, by Lemma 1.2.8.

On the basis of the mappings M_x and $\uparrow x_M$ we will define a U-coloring of the cover relations of $\mathcal{L}$. As colors we use the elements of $\mathcal{C}$.

Lemma 1.3.14. *Let $\mathcal{L}$ be a ULD and $x, y \in \mathcal{L}$. We have $x \prec y$ if and only if $|\uparrow x_M \setminus \uparrow y_M| = 1$.*

Proof. By Lemma 1.2.8 for general finite lattices, an element z with $x < z < y$ satisfies $\uparrow y_M \subsetneq \uparrow z_M \subsetneq \uparrow x_M$ which implies $|\uparrow x_M \setminus \uparrow y_M| \geq 2$.

Let $x < y$ and suppose that $|\uparrow x_M \setminus \uparrow y_M| \geq 2$. Since $\bigwedge M_x < \bigwedge \uparrow y_M$ there has to be some $m \in M_x \setminus \uparrow y_M$. Let $z = \bigwedge(\uparrow x_M - m)$. By Definition 1.1.6 we have $z = \bigwedge(M_x - m) > x$. Since $(\uparrow x_M - m) \supseteq \uparrow y_M$ we have $z \leq y$. Let m' be an element with $m \neq m' \in \uparrow x_M \setminus \uparrow y_M$, it follows that $m' \in \uparrow z_M$ and $m' \notin \uparrow y_M$. Therefore $z \neq y$ and we have shown that $x < z < y$, i.e., the pair x, y is not in a cover relation. $\qquad\square$

Proposition 1.3.15. *Let $\mathcal{L}$ be a ULD with Hasse diagram $D_{\mathcal{L}}$ and $\mathcal{C}$ a chain-partition of $\mathcal{M}(\mathcal{L})$. The mapping $c : A \to \mathcal{C}$ with $c(x, y)$ being the $C \in \mathcal{C}$ which contains $\uparrow x_M \setminus \uparrow y_M$ is a U-coloring of $D_{\mathcal{P}}$.*

Proof. To verify that c is a U-coloring we have to check the two properties U_1 and U_2. First note that $\uparrow x_M \setminus \uparrow y_M \in M_x$ for a $x \prec y$.

We start with U_1: Let $x \prec y(1), y(2)$ be two cover relations. Since $x = y(1) \wedge y(2)$ we have the representation $x = \bigwedge(\uparrow y(1)_M \cup \uparrow y(2)_M)$ of x as the meet of meet-irreducibles, hence, $M_x \subseteq \uparrow y(1)_M \cup \uparrow y(2)_M$. Suppose both covers have the same color. Since M_x is an antichain and the colors correspond to chains both covers must correspond to the same element of M_x, i.e., $\uparrow x_M \setminus \uparrow y(1)_M = \uparrow x_M \setminus \uparrow y(2)_M = m$. Thus, $m \in M_x$ but $m \notin \uparrow y(1)_M \cup \uparrow y(2)_M$ – a contradiction.

It remains to show that the coloring satisfies U_2: Let $x \prec y(1), y(2)$ be two cover relations such that $x \prec y(i)$ has color c_i, i.e., there is $m_i \in C_i$ such that $\uparrow y(i)_M = \uparrow x_M - m_i$. Consider $z = \bigwedge(\uparrow x_M - m_1 - m_2)$. Since z is representable as the meet of elements from $\uparrow y(i)_M$ we know $z \geq y(i)$ for $i = 1, 2$. Since $y(1), y(2)$ both cover x and are incomparable it follows that $z > y(1), y(2)$. From $\uparrow x_M - m_1 - m_2 \subseteq \uparrow z_M \subset \uparrow y(i)_M = \uparrow x_M - m_i$ it follows that $|\uparrow y(i)_M \setminus \uparrow z_M| = 1$. Lemma 1.3.14 implies that z covers each y and the labels of these covers are as required. $\qquad\square$

Remark 1.3.16. We have shown that every chain-partition $\mathcal{C}$ of $\mathcal{M}(\mathcal{L})$ determines a U-coloring of $\mathcal{L}$.

We have now shown the equivalence of the three parts of Theorem 1.3.3. As noted in Remark 1.3.10, Remark 1.3.13, and Remark 1.3.16, U-colorings, cover-preserving join-sublattice embeddings and chain-partitions of the poset of meet-irreducibles translate to each other via the equivalence.

Remark 1.3.17. In order to establish a one-to-one correspondence of U-colorings, cover-preserving join-sublattice embeddings and chain-partitions of the poset of meet-irreducibles along the lines of the proof of Theorem 1.3.3 it is necessary to define adequate isomorphism-classes of these objects. This can be done but we will not go into that detail here.
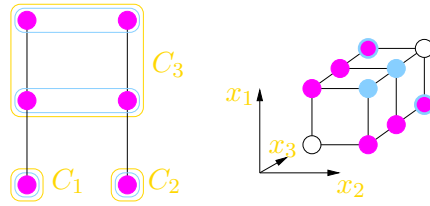
Figure 1.11: On the left: an antichain-partitoned poset with a chain-partition $\{C_1, C_2, C_3\}$ of the index-poset on the antichains ($A \leq A' :\Leftrightarrow\, \uparrow A \geq \uparrow A'$, see Section 1.2). On the right: the corresponding embedded ULD in $\mathbb{N}^3$.

In the light of the representation of ULDs as antichain-partitioned posets (Theorem 1.2.24), Theorem 1.3.3 enables us to state the following generalization of Dilworth's Embedding Theorem for distributive lattices, see Figure 1.11.

Theorem 1.3.18. *Every order-embedding of a finite ULD $\mathcal{L}$ as a cover-preserving join-sublattice of the dominance order on $\mathbb{N}^d$ corresponds to a reduced antichain-partitioned $(\mathcal{P}, \mathcal{A}_\mathcal{Q})$ with a chain-partition of $\mathcal{Q}$ into d chains and viceversa.*

Remark 1.3.19. Since finding a minimum chain-partition of a poset is equivalent to finding a maximum matching in a bipartite graph by [47], Theorem 1.3.18 particularly yields that a join-sublattice embedding of a ULD of minimal dimension may be computed in polynomial time in $|M(L)|$. This result may be deduced from a result of David Eppstein about the lattice dimension of a graph [35].

Before proving a special case of Theorem 1.3.3 for distributive lattices, we continue with some comments about possible generalizations of this section's results.
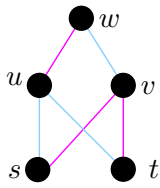
 Let D be a digraph with a U-coloring. We need acyclicity, connectivity and the unique source to conclude that D corresponds to a finite ULD. We feel that among these conditions the *unique source* has a somewhat artificial flavor. Abstaining on this condition it can be shown (along the lines of our proof) that the corresponding poset $\mathcal{P}_D$ has a unique maximum and the property that for all $x \in \mathcal{P}_D$ there is a unique minimal set M_x of meet-irreducibles such that x is a maximal lower bound for M_x. The figure on the right shows a small example, in this case $M_s = M_t = \{u, v\}$. As exemplified by the figure such a poset does not need to have unique joins nor meets at all.

Question 1.3.20. The meet-representability in general U-posets is quite weak. Instead it would be of interest to characterize join-semilattices $\mathcal{P}$, where for all $x \in \mathcal{P}$ there is a unique minimal set M_x of meet-irreducibles such that x is the *unique* maximal lower bound for M_x. Note that this still does not turn $\mathcal{P}$ into a lattice.

Question 1.3.21. Another question arises, when dropping the restriction to finite lattices. A first class of interest would be ULDs with the property that every two elements are connected

by a *finite* path in the Hasse diagram. How can their Hasse diagrams be characterized as class of arc-colored digraphs.

Instead of generalizing Theorem 1.3.3, it is very be useful when specializing it. In many applications of the characterization of ULDs the lattice in question is actually distributive. Such a situation is the topic of the next section.

Theorem 1.3.22. *If an acyclic and connected digraph D admits a U- and an L-coloring then D is a Hasse diagram and $\mathcal{L}_D$ is isomorphic to a cover-preserving sublattice of $\mathbb{N}^d$. In particular, $\mathcal{L}_D$ is a distributive lattice.*

Proof. Let c_U and c_L be a U- and an L-coloring of D, respectively. Consider the coloring $c = c_U \times c_L$. The claim is that c is both a U- and an L-coloring of D. The rule U_1 and its dual L_1 are immediately inherited from the corresponding rules for c_U and c_L.

Consider a subposet $x \prec y(1)$, $x \prec y(2)$, $y(1) \prec z$, $y(2) \prec z$. Proposition 1.3.7 guarantees the colored Jordan-Dedekind chain condition for c. We have that $\{c(x, y(1)), c(y(1), z)\} = \{c(x, y(2)), c(y(2), z)\}$. Together with U_1 and L_1 for c we conclude that $c(x, y(i)) = c(y(j), z)$ for $i \neq j$. This implies rules U_2 and L_2 for c.

Since D is connected, Proposition 1.3.6 applied to $\mathcal{P}_D$ and c yields that $\mathcal{P}_D$ has unique $\mathbf{0}_{\mathcal{P}_D}$ and a unique $\mathbf{1}_{\mathcal{P}_D}$.

With Proposition 1.3.9 we have that c yields an order-embedding $\gamma : \mathcal{P}_D \to \mathbb{N}^d$ which is cover-, meet-, and join-preserving. Hence $\mathcal{P}_D$ is a sublattice of the dominance order on $\mathbb{N}^d$. Since the latter is distributive, so is $\mathcal{P}_D$. $\qquad\square$

In the following we will show one of our main application of U-colorings. We will use Theorem 1.3.22 to show distributivity. For an application of the criterion to ULDs that are not distributive see Section 1.5, where chip-firing games and vector-addition languages will be introduced.

1.4   The Lattice of Tensions

The concept of Δ-*tensions* is a unifying generalization of many known distributive lattices coming from digraphs. The most important special cases will be explained in the subsections at the end of this section. In this section we prove that the Δ-tensions of any digraph carry a distributive lattice structure twice. First we show that Δ-tensions may be seen as a sublattice of the dominance order and thus inherit the distributive lattice structure. Second we will provide local transformations on Δ-tensions, which yield an acyclic and connected digraph with a U- and an L-coloring on the set of Δ-tensions. Applying Theorem 1.3.22 we will then obtain the result again, but with the additional information of how to generate the lattice combinatorially.

Our proof starts with the observation that Δ-tensions are actually affinely equivalent to ordinary tensions. Those form the orthogonal space to the integer flows of a digraph. In the literature the space of integer tensions is also referred to as *cut space*. Tensions are a classical research topic of algebraic graph theory, see [52]. A second step is to reduce ordinary tensions to feasible vertex potentials. Vertex potentials are also referred to as *height functions* in many contexts, e.g., Propp [92]. We will introduce a (classical) bijection between tensions and vertex potentials. It may be seen as *the coboundary operator of a graph*, see [78]. The structural results we obtain for Δ-tensions in this section, will all first be proved on vertex-potentials and then be translated back to Δ-tensions.

For the definition Δ-tensions we need to introduce some standard vocabulary: Directed graphs lead to *oriented arc-sets*. An *oriented cycle* C of a digraph $D = (V, A)$ corresponds to a cycle of the underlying undirected graph together with a direction of traversal. This way C is partitioned in a set of *forward arcs* C^+ and *backward arcs* C^-. We collect the oriented cycles of D in $\mathcal{C}(D)$. Similarly we will view walks and paths as oriented arc-sets. Now we come to the main definition of this section.

oriented arc-set
oriented cycle

forward arcs
backward arcs

Definition 1.4.1. . Let $D = (V, A)$ be a directed multi-graph with upper and lower integral arc capacities $c_u, c_l : A \to \mathbb{Z} \cup \{\pm\infty\}$, i.e., some arcs might have unbounded capacities. Given a number Δ_C for each oriented cycle $C \in \mathcal{C}$ of D we define the set $\mathcal{T}_\Delta(D, c_l, c_u)$ of Δ-*tensions* as the set of vectors $x \in \mathbb{Z}^A$ such that

Δ-tension

(D$_1$) $c_l(a) \leq x(a) \leq c_u(a)$ for all $a \in A$,   (capacity constraints)
(D$_2$) $\Delta_C = \sum_{a \in C^+} x(a) - \sum_{a \in C^-} x(a)$ for all C.   (circular balance conditions)

circular balance

We abbreviate the *circular balance* $\sum_{a \in C^+} x(a) - \sum_{a \in C^-} x(a)$ of a tension x with respect to a cycle C by $\delta(C, x)$.

Remark 1.4.2. In previous work on the subject [41, 42, 43] we have referred to Δ-tensions as Δ-bonds. Also, in [39, 41, 42, 92] instead of *circular balance* the term *circular flow-difference* was used. Since tensions are not flows but orthogonal to flows that name may cause confusion.

Remark 1.4.3. Note that prescribing Δ on a basis of the cycle space $\mathcal{C}$ of D already suffices to determine Δ everywhere.

We will prove a distributive lattice structure on $\mathcal{T}_\Delta(D, c_l, c_u)$. The proof of our result relies on a reduction to *ordinary* tensions and then to *vertex potentials*. For this in the following two lemmas we will reduce the variety of data (D, c_l, c_u, Δ) we need to look at. First note that restricting our attention to connected digraphs does not cause a loss of generality. Given data $(D^1, c_l^1, c_u^1, \Delta^1)$ and $(D^2, c_l^2, c_u^2, \Delta^2)$ there is an obvious extension to a union structure (D, c_l, c_u, Δ) where D is the union of graphs and the c_l, c_u, Δ are concatenations of vectors. Since Δ-tensions factor into a Δ^1- and a Δ^2-tension we have:

Lemma 1.4.4. $\mathcal{T}_\Delta(D, c_l, c_u) \cong \mathcal{T}_{\Delta^1}(D^1, c_l^1, c_u^1) \times \mathcal{T}_{\Delta^2}(D^2, c_l^2, c_u^2).$

The most important case of Δ-tensions is if $\Delta = \mathbf{0}$ is the all-zeroes-vector. In this case we refer to Δ-tensions as *tensions*. For convenience, we will denote the set of tensions as $\mathcal{T}(D, c_\ell, c_u)$. By the circular balance condition tensions without capacity constraints form the orthogonal space to the integer flows of a digraph, also known as *cut space*. Indeed, we can restrict our attention to the case of tensions: tension

Lemma 1.4.5. *Let D be any digraph with arc-capacities c_ℓ, c_u. Given some $x \in \mathcal{T}_\Delta(D, c_\ell, c_u)$ we have*

$$\mathcal{T}_\Delta(D, c_\ell, c_u) \cong \mathcal{T}(D, c_\ell - x, c_u - x).$$

Proof. The map $\varphi : \mathcal{T}_\Delta(D, c_\ell, c_u) \to \mathcal{T}(D, c_\ell - x, c_u - x)$ is defined by $\varphi(y) := y - x$. The image clearly satisfies the capacity constraints and also the circular balance conditions by

$$\delta(C, \varphi(y)) = \sum_{a \in C^+} (y(a) - x(a)) - \sum_{a \in C^-} (y(a) - x(a)) = \delta(C, y) - \delta(C, x) = 0.$$

Indeed, the translation φ is a bijection with inverse $z \mapsto z + x$. $\square$

Given a connected D with capacities c_ℓ, c_u fix an *arbitrary* vertex $v_0 \in V$. Call a vector $\pi \in \mathbb{Z}^V$ a *feasible vertex potential* if $\pi(v_0) = 0$ and $c_\ell(a) \leq \pi(w) - \pi(v) \leq c_u(a)$ for all feasible potential $a = (v, w) \in A$. We collect the set of feasible vertex potentials in $\Pi_{v_0}(D, c_\ell, c_u)$.

Lemma 1.4.6. *Let D be a connected digraph with arc-capacities c_ℓ, c_u and $v_0 \in V$. We have $\mathcal{T}(D, c_\ell, c_u) \cong \Pi_{v_0}(D, c_\ell, c_u)$*

Proof. To every $x \in \mathcal{T}(D, c_\ell, c_u)$ we can associate a feasible vertex potential $\pi_x \in \Pi_{v_0}(D, c_\ell, c_u)$ by setting $\pi_x(v) := \sum_{a \in P^+} x(a) - \sum_{a \in P^-} x(a)$, for a (v_0, v)-path P. Here P^+ and P^- are forward and backward arcs of P, respectively. To see that π_x is well-defined, i.e., independent of the choice of P, take two (v_0, v)-paths P and Q. Their symmetric difference is a union of oriented cycles $C_1, \ldots, C_k$: traverse arcs of $C_i \cap P$ in the

order they appear along P and those in $C_i \cap Q$ in the reverse order of Q. It is straight-forward to get $(\sum_{a \in P^+} x(a) - \sum_{a \in P^-} x(a)) - (\sum_{a \in Q^+} x(a) - \sum_{a \in Q^-} x(a)) = \delta(C_1) + \ldots + \delta(C_k)$. Since the circular balance of x is 0 on all cycles of D this difference is 0 and the map $\pi : \mathcal{T}(D, c_\ell, c_u) \to \Pi_{v_0}(D, c_\ell, c_u)$ is well-defined, i.e., independent of the choice of P.

On the other hand for $\pi \in \Pi_{v_0}(D, c_\ell, c_u)$ define x_π as $x_\pi(a) := \pi(w) - \pi(v)$ for $a = (v, w) \in A$. It is straight-forward to calculate that for every directed walk W with start vertex u and end vertex u' we have $\sum_{a \in W^+} x_\pi(a) - \sum_{a \in W^-} x_\pi(a) = \pi(u') - \pi(u)$. In particular if W is a cycle the sum is 0, i.e., $x_\pi \in \mathcal{T}(D, c_\ell, c_u)$.

To see $\pi_{x_\pi} = \pi$ we compute $\pi_{x_\pi}(v) = \sum_{a \in p^+} x_\pi(a) - \sum_{a \in p^-} x_\pi(a) = \pi(v) - \pi(v_0)$. Since $\pi(v_0) = 0$ by default, this equals $\pi(v)$. On the other hand let $a' = (v, w)$ then $x_{\pi_x}(a') = \pi_x(w) - \pi_x(v)$. Let P be a (v_0, w)-path taking a' as last arc and Q the path without a'. (If this is not possible take P as a (v_0, v)-path.) We compute $\sum_{a \in P^+} x(a) - \sum_{a \in P^-} x(a) - (\sum_{a \in Q^+} x(a) - \sum_{a \in Q^-} x(a)) = x(a')$.

We have shown $\mathcal{T}(D, c_\ell, c_u) \cong \Pi_{v_0}(D, c_\ell, c_u)$. Moreover, the bijections $\pi \mapsto x_\pi$ and $x \mapsto \pi_x$ are inverses of each other. $\qquad \square$

We can summarize the last lemmas as:

Lemma 1.4.7. *For every set of data* (D, c_ℓ, c_u, Δ) *there are* c'_ℓ, c'_u *such that* $\mathcal{T}_\Delta(D, c_\ell, c_u) \cong \Pi_{v_0}(D, c'_\ell, c'_u)$ *for every* $v_0 \in V(D)$.

This lemma is so useful for finding a distributive lattice on the Δ-tensions of a digraph because we can prove:

Theorem 1.4.8. *Let* D *be a digraph with capacities* c_ℓ, c_u. *For every* $v_0 \in V(D)$ *the set* $\Pi_{v_0}(D, c_\ell, c_u)$ *induces a sublattice of the dominance order on* $\mathbb{Z}^V$, *i.e., carries the structure of a distributive lattice.*

Proof. We only have to show that $\Pi_{v_0}(D, c_\ell, c_u)$ is closed with respect to componentwise $\max$ and $\min$. These are meet and join of the dominance order.

Let $\pi_1, \pi_2 \in \Pi_{v_0}(D, c_\ell, c_u)$, i.e., $c_\ell(a) \leq \pi_i(w) - \pi_i(v) \leq c_u(a)$ for all $a = (v, w) \in A$ and $i = 1, 2$. Say $\pi_1(w) \leq \pi_2(w)$ and $\pi_1(v) \geq \pi_2(v)$, then

$$\pi_2(w) - \pi_2(v) \leq \pi_2(w) - \pi_1(v) \leq \pi_1(w) - \pi_1(v).$$

Hence the maximum is feasible on a. The case $\pi_1(w) \geq \pi_2(w)$ and $\pi_1(v) \leq \pi_2(v)$ works similar. Thus $\Pi_{v_0}(D, c_\ell, c_u)$ is $\max$-closed. By an analogous argument it can be shown that $\Pi_{v_0}(D, c_\ell, c_u)$ is $\min$-closed. As a sublattice of a distributive lattice $\Pi_{v_0}(D, c_\ell, c_u)$ then carries a distributive structure. $\qquad \square$

The following is now an easy consequence:

Theorem 1.4.9. *Let* D *be a digraph with capacities* c_ℓ, c_u. *The set* $\mathcal{T}_\Delta(D, c_l, c_u)$ *carries the structure of a distributive lattice.*

Proof. By Lemma 1.4.7 the set $\mathcal{T}_\Delta(D, c_l, c_u)$ is isomorphic to $\Pi_{v_0}(D, c'_\ell, c'_u)$ for $v_0 \in V(D)$ and some capacities c'_ℓ, c'_u. By Theorem 1.4.8 the dominance order on $\Pi_{v_0}(D, c'_\ell, c'_u)$ is a distributive lattice. The set $\mathcal{T}_\Delta(D, c_l, c_u)$ inherits this structure from $\Pi_{v_0}(D, c'_\ell, c'_u)$. □

The rest of the section consists of having a closer look at the Hasse diagram of the distributive lattice on the Δ-tensions. Reducing the input-data we can find a U- and an L-coloring of the Hasse diagram of the lattice, where colors *naturally* correspond to vertices of D. The whole lattice then will be generated by *local vertex pushes*. This is important, because it will in fact unify the *local transformations* that come up in many special cases of tensions.

From now on we shall assume that the data (D, c_l, c_u, Δ) are such that the set of corresponding Δ-tensions is non-empty and finite. Moreover, we want to simplify matters by concentrating on connected graphs and getting rid of *rigid arcs*. These are arcs $a \in A$ with $x(a) = y(a)$ for all pairs x, y of Δ-tensions. rigid arc

Let a be a rigid arc of D. If a is a loop we delete it from the graph. Since a was rigid, restricting all the data to $D \backslash a$ yields a bijection between Δ-tensions. If a is not a loop, then contract a obtaining D/a. The cycles in D/a and the cycles in D are in bijection. Let C/a be the cycle in D/a corresponding to C in D. Define $\Delta'_{C/a} = \Delta_C$ if $a \notin C$ and $\Delta'_{C/a} = \Delta_C - x(a)$ if $a \in C^+$ and $\Delta'_{C/a} = \Delta_C + x(a)$ if $a \in C^-$. These settings yield the bijection that proves

Lemma 1.4.10. $\mathcal{T}_\Delta(D, c_l, c_u) \cong \mathcal{T}_{\Delta'}(D/a, c_l, c_u)$.

The data (D, c_l, c_u, Δ) are *reduced* if D is connected, there is no rigid arc, and reduced data $\mathcal{T}_\Delta(D, c_l, c_u)$ is neither empty nor infinite. Henceforth we will assume that any given set of data is reduced.

As for the proof of Theorem 1.4.9 in the first part of this section, we will reduce Δ-tensions to vertex-potentials. Lemma 1.4.7 establish a one-ton-one-correspondence between reduced data for Δ-tensions and reduced data for vertex potentials, i.e., we are given a finite non-empty set $\Pi_{v_0}(D, c_\ell, c_u)$, without rigid arcs. This is, for every $a \in A(D)$ there are potentials π_1, π_2 with $x_{\pi_1}(a) \neq x_{\pi_2}(a)$.

We will now introduce the local tansfromations of vertex potentials, called push and pop, and show that they yield a connected acyclic digraph with U- and L-coloring on the set of vertex-potentials. We then apply Theorem 1.3.22 and obtain a distributive lattice on that set.

Given $\pi \in \Pi_{v_0}(D, c_\ell, c_u)$ and $v \in V \backslash \{v_0\}$ *pushing v in π* is to move from π to $\pi + e_v$. vertex push Here e_v denotes the vector, which has a 1 in the vth entry and is 0 elsewhere. Pushing v in π is only allowed if $\pi + e_v$ is feasible. The inverse operation of vertex pushing is *vertex popping*. Define D_Π as the directed graph with vertex set $\Pi_{v_0}(D, c_\ell, c_u)$ and arcs of vertex pop the form $(\pi, \pi + e_v)$. The *natural color* $c(\pi, \pi + e_v)$ of an arc $(\pi, \pi + e_v)$ of D_Π is v. natural color

Note that D_Π is a subgraph of the Hasse diagram of the dominance order on $\Pi_{v_0}(D, c_\ell, c_u)$. The latter is a distributive lattice by Theorem 1.4.8. This yields that D_Π is acyclic. Since $\Pi_{v_0}(D, c_\ell, c_u)$ is closed with respect to $\min$ and $\max$, it follows that the natural coloring of D_Π is a U- and an L-coloring. In order to apply Theorem 1.3.22 it only remains to show that D_Π is connected, in other words:

Lemma 1.4.11. *Let (D, c_l, c_u) be reduced data and $\pi_1, \pi_2 \in \Pi_{v_0}(D, c_\ell, c_u)$. There is a sequence of pushes and pops that transforms π_1 into π_2.*

Proof. We proceed by induction on the ℓ_1-distance $\sum_{v \in V} |\pi_1(v) - \pi_2(v)|$ of π_1 and π_2. If this sum is 0 the statement is clearly true.

Otherwise, partition V into the the set $S^- := \{v \in V \mid \pi_1(v) - \pi_2(v) < 0\}$ and its complement S^+. If we can push one vertex $v \in S^-$ in π_1, then we can apply induction on $\pi_1 + e_v$ and π_2. First, observe $S^- \neq \emptyset$, otherwise interchange the roles of π_1 and π_2. Second, note $v_0 \notin S^-$. Suppose no vertex in S^- can be pushed in π_1. This means every
saturated arc
cosaturated arc $v \in S^-$ has a *saturated* incoming arc (w, v), i.e., $\pi_1(v) - \pi_1(w) = c_u$, or a *cosaturated* outgoing arc (v, w), i.e., $c_\ell = \pi_1(w) - \pi_1(v)$. No such arc a can have its other endpoint w in S^+. Otherwise with $\pi_2(w) \leq \pi_1(w)$ and $\pi_2(v) > \pi_1(v)$ we obtain that π_2 is not feasible with respect to a.

Hence the digraph induced by S^- contains a cycle C with saturated backward arcs and cosaturated forward arcs, i.e., $\sum_{a \in C^+} c_\ell(a) - \sum_{a \in C^-} c_u(a) = 0$. Changing any of the values on the arcs would force some others to violate their capacity constraints. Hence C is rigid – a contradiction to (D, c_l, c_u) being reduced. $\qquad\square$

We have proved that D_Π is a connected, acyclic digraph on the set of feasible vertex potentials. Moreover pushes and pops yield a U- and L-coloring of D_Π. Since the data (D, c_ℓ, c_u) are reduced in particular $\Pi_{v_0}(D, c_\ell, c_u)$ is finite. We can apply Theorem 1.3.22 and get:

Theorem 1.4.12. *For reduced data (D, c_ℓ, c_u) the set $\Pi_{v_0}(D, c_\ell, c_u)$ carries the structure of a distributive lattice. The local transformations of vertex pushing and popping in $\Pi_{v_0}(D, c_\ell, c_u)$ correspond to moving upwards and downwards on the arcs of the Hasse diagram of that lattice, respectively.*

forbidden vertex
vertex push
In order to translate our result to the language of Δ-tensions we choose a *forbidden vertex* $v_0 \in V$ and define *pushing a vertex $v \in V \backslash \{v_0\}$ in a Δ-tension x* as moving from x to $x + x_v$, where $x(v)$ is:

$$x_v(a) := \begin{cases} +1 & \text{if } a = (w, v) \\ -1 & \text{if } a = (v, w) \\ 0 & \text{otherwise} \end{cases}$$

vertex pop
Analogously define *popping a vertex v in a Δ-tension x* as moving from x to $x - x_v$. Pushing and popping in tension just is defined in the way that enables us to prove the very analogous theorem to Theorem 1.4.12 in terms of Δ-tensions:

Theorem 1.4.13. *For reduced data (D, c_ℓ, c_u, Δ) the set $\mathcal{T}_\Delta(D, c_l, c_u)$ carries the structure of a distributive lattice. The local transformations of vertex pushing and popping in $\mathcal{T}_\Delta(D, c_l, c_u)$ correspond to moving upwards and downwards on the arcs of the Hasse diagram of that lattice, respectively.*

Proof. Given the set $\mathcal{T}_\Delta(D, c_l, c_u)$ we apply Lemma 1.4.7 to obtain an isomorphic set $\Pi_{v_0}(D, c'_\ell, c'_u)$, which by Theorem 1.4.12 is a distributive lattice with cover-relations corresponding to pushes and pops. In order to prove out theorem we only have to convince ourselves that pushing a vertex in a potential in $\Pi_{v_0}(D, c'_\ell, c'_u)$ corresponds to pushing a vertex in the corresponding Δ-tension in $\mathcal{T}_\Delta(D, c_l, c_u)$

We want to understand a push $(x, x + x_v)$ of Δ-tensions. Lemma 1.4.5 transfers $x + x_v$ to a tension $z + x_v$ of (D, c'_ℓ, c'_u). Lemma 1.4.6 maps this to π_{z+x_v}. Let P be a (v_0, u)-path in D, then by definition of π_{z+x_v} in the proof of Lemma 1.4.6 we have $\pi_{z+x_v}(u) := \sum_{a \in P^+}(z + x_v)(a) - \sum_{a \in P^-}(z + x_v)(a)$, which equals $\pi_z(u) + \pi_{x_v}(u)$. By the definition of x_v it is straight-forward to see that $\pi_{x_v}(u) = \sum_{a \in P^+} x_v(a) - \sum_{a \in P^-} x_v(a)$ is 1 if $u = v$ and 0 otherwise. This, is $\pi_{z+x_v} = \pi_z + e_v$. We have computed that the push $(x, x + x_v)$ of Δ-tensions corresponds to a push $(\pi_z, \pi_z + e_v)$ of potentials. This concludes the proof. $\qquad\square$

Remark 1.4.14. In order to prove the push-connectivity of the distributive lattice of Δ-tensions we reduced the data. Instead of contracting rigid arcs one could push potentials on connected subgraphs induced by rigid arcs. This would then generate the same distributive lattice by local transformations. A special case of this is the generation of the distributive lattice on α-orientations by reversing *essential cycles* in [39].

Remark 1.4.15. It is possible to get rid of all lower arc-capacities. For an arc $a = (v, w)$ add an antiparallel copy $a^- := (w, v)$ with upper arc-capacity $c_u(a^-) := -c_\ell(a)$. The new cycle (a, a^-) gets the Δ value 0. Even if more new cycles emerge from this operation, Δ keeps being defined on a cycle basis, i.e., there is no problem by Remark 1.4.3. Applying this to all arcs one obtains a description with only upper arc-capacities. The most reduced description of tensions would then be of the form $\mathcal{T}(D, c)$, where D is a digraph and c upper arc-capacities.

Question 1.4.16. Lattices of Δ-tensions depend on the choice of a vertex $v_0 \in V$. Choosing another vertex v_1 yields a different lattice on the same set of objects. Is there an easy description of the transformation from $\Pi_{v_0}(D, c_\ell, c_u)$ to $\Pi_{v_1}(D, c_\ell, c_u)$? Understanding this transformation might help when looking for a distributive lattice on Δ-tensions with particular properties.

Question 1.4.17. The generation of a random element from a distributive lattice is a nice application for *coupling from the past* (c.f. Propp and Wilson [94]). The challenge is to find good estimates for the mixing time, see Propp [93]. What if the lattice is a Δ-tension lattice?

We will now continue with several applications of Δ-tensions.

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1.4.1 Applications

In the following three subsections we deal with the three most important special cases of Δ-tensions. We show, how they may be interpreted as Δ-tensions. Thus, as a corollary of Theorem 1.4.13 they carry a distributive lattice structure. Figure 1.12 illustrates this at an example.
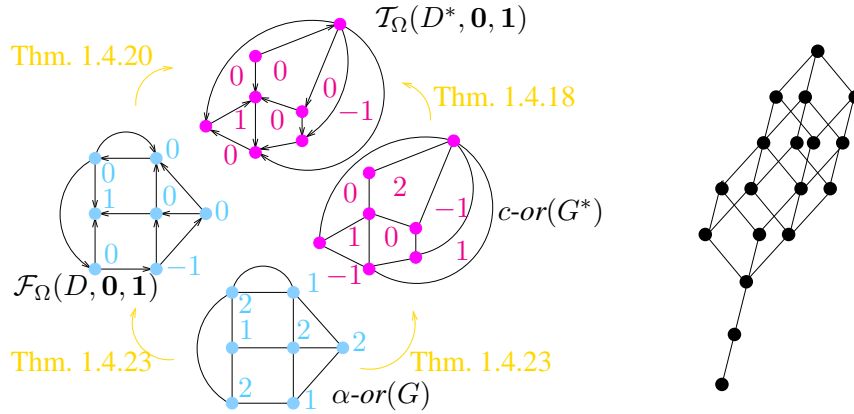

Figure 1.12: The different special cases of Δ-tensions represented by the combinatorial information that encodes them. All the four in the picture are equivalent as we will see in the following subsections. On the right we depict the distributive lattice they carry by Theorem 1.4.13.

1.4.2 The lattice of c-orientations (Propp [92])

circular
flow-difference

c-orientation

Given an orientation $D = (V, A)$ and an oriented cycle C of an undirected graph $G = (V, \mathcal{E})$ we denote by $c_D(C) := |C_D^+| - |C_D^-|$ the *circular flow-difference* of D around C, where C_D^+ is the set of forward arcs of C in D and C_D^- is the set of backward arcs. Given a vector $c \in \mathbb{Z}^{\mathcal{C}}$, which assigns to every oriented cycle C of G an integer $c(C)$, we call an orientation D of G with $c(C) = c_D(C)$ a *c-orientation*. We denote the set of c-orientations of G by c-$or(G)$. The main result in Propp's article [92] is:

Theorem 1.4.18. *Let $G = (V, \mathcal{E})$ be a graph and $c \in \mathbb{Z}^{\mathcal{C}}$. The set c-$or(G)$ of c-orientations of G carries the structure of a distributive lattice.*

Proof. Let $D = (V, A)$ be any orientation of G. Define $\Delta := \frac{1}{2}(c_D - c)$. We interpret $x \in \mathcal{T}_\Delta(D, \mathbf{0}, \mathbf{1})$ as the orientation $D(x)$ of G which arises from D by changing the orientation of $a \in A$ if $x(a) = 1$. For an arc set $A' \subseteq A$ we write $x(A')$ for $\sum_{a \in A'} x(a)$. We calculate:

$$
\begin{aligned}
c_{D(x)}(C) &= |C^+_{D(x)}| - |C^-_{D(x)}| \\
&= |C^+_D| - x(C^+_D) + x(C^-_D) - (|C^-_D| - x(C^-_D) + x(C^+_D)) \\
&= |C^+_D| - |C^-_D| - 2(x(C^+_D) - x(C^-_D)) \\
&= c_D(C) - 2\delta(x,C) = c_D(C) - 2\Delta_C \\
&= c
\end{aligned}
$$

This shows that c-orientations of G correspond bijectively to Δ-tensions in $\mathcal{T}_\Delta(D, \mathbf{0}, \mathbf{1})$. By Theorem 1.4.9 we obtain a distributive lattice structure on the set of c-orientations of G. For an example, see Figure 1.12. □

Remark 1.4.19. The reducedness for Theorem 1.4.13 corresponds to considering c such that all c-orientations are acyclic. The local transformations then correspond to reversals of directed vertex cuts.

For planar graphs c-orientations dualize to α-orientations. Hence Theorem 1.4.18 implies Theorem 1.4.23 of Subsection 1.4.4. This special case has many applications which are collected in Subsection 1.4.4.
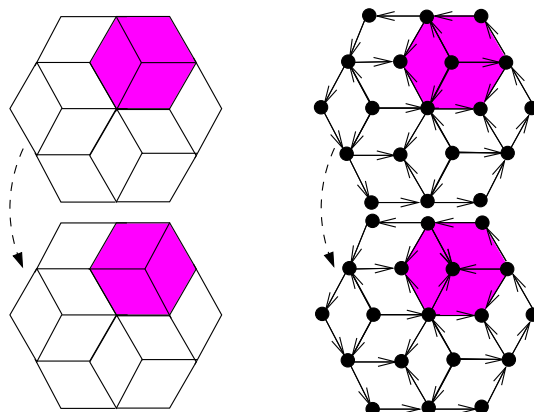

Figure 1.13: Modeling lozenge tilings by c-orientations. Flipping tiles corresponds to reversing directed vertex-cuts.

One application of Theorem 1.4.18 that cannot be obtained using planar α-orientations is a distributive lattice structure on *higher dimensional rhombic tilings*. These objects were introduced and proven to carry a distributive lattice structure in [77]. Since usual lozenge tilings may be seen as sets of piles of cubes in $\mathbb{Z}^3$ (just look at Figure 1.13 and try to give some spacial perspective to what you see), the generalization of higher dimensions are piles of hypercubes in $\mathbb{Z}^d$. We will not prove this here, neither that they carry a distributive lattice structure, nor that they may be modelled as c-orientations of a (generally non-planar) graph. Instead in Figure 1.13 we suggest how to interpret ordinary lozenge tilings as c-orientations.

higher
dimensional
rhombic tilings

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Since this generalizes to non-planar graphs it is essentially different then just dualizing the interpretation as α-orientations in Figure 1.14.

In [67, 68] and independently by Latapy and Magnien in [75] it was proven that c-orientations are indeed universal for the class of distributive lattices, i.e., every distributive lattice may be represented as the c-orientations of a graph. In [67, 68] the set of graphs representing a given lattice was characterized.

1.4.3 The lattice of flows in planar graphs (Khuller, Naor and Klein [66])

Consider a planar digraph $D = (V, A)$, with each arc a having an integer *lower* and *upper* bound on its capacity, denoted $c_\ell(a)$ and $c_u(a)$. For a function $f : A \to \mathbb{Z}$ call
excess $\omega(v, f) := \sum_{a \in \text{in}(v)} f(a) - \sum_{a \in \text{out}(v)} f(a)$ the *excess* at v. The sets $\text{in}(v)$ and $\text{out}(v)$ denote the incoming and outgoing arcs of v, respectively. Given a vector $\Omega \in \mathbb{N}^V$ call f an
Ω-flow Ω-*flow* if $c_\ell(a) \leq f(a) \leq c_u(a)$ for all a and $\Omega_v = \omega(v, f)$ for all $v \in V$. Denote by $\mathcal{F}_\Omega(D, c_\ell, c_u)$ the set of Ω-flows.
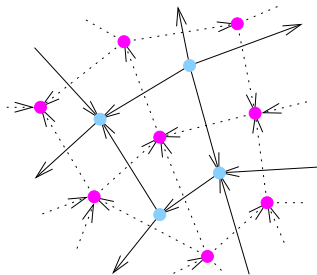
For the proof we need to discuss briefly duality of planar *di*graphs. Given a crossing-free plane embedding of D we look at the planar *dual digraph* D^*. It is an orientation of the planar dual
dual digraph G^* of the underlying undirected graph G of D. Let v be a vertex of G^* corresponding to a facial cycle C_v of the embedding of D. Orient an edge incident to v as outgoing arc of v if the primal arc is forward when traversing C_v in counterclockwise direction. Given values on the arcs of D, e.g. a flow or a tension, we simply transfer them to the corresponding arcs of D^*. Duality translates concepts as displayed in the table below. Note that duality of planar digraphs is not an involution as for undirected graphs, but a map of degree four. This may be seen looking at the last four lines of the table.

D	$\rightsquigarrow$	D^*
vertices	$\rightsquigarrow$	faces
faces	$\rightsquigarrow$	vertices
arcs	$\rightsquigarrow$	arcs
ccw forward arcs of facial cycle	$\rightsquigarrow$	outgoing arcs
outgoing arcs arcs	$\rightsquigarrow$	ccw backward arcs of facial cycle
excess	$\rightsquigarrow$	circular balance
circular balance	$\rightsquigarrow$	-excess

Theorem 1.4.20. *If D is a planar digraph then $\mathcal{F}_\Omega(D, c_\ell, c_u)$ carries the structure of a distributive lattice.*

Proof. So, we consider the dual D^* of D. Since the excess ω at a vertex of D dualizes to the circular balance δ around the corresponding facial cycle of D^*, we have a correspondence between $\mathcal{F}_\Omega(D, c_\ell, c_u)$ and $\mathcal{T}_\Omega(D^*, c_\ell, c_u)$. This yields the distributive lattice structure on Ω-flows of planar graphs via Theorem 1.4.9. For an example, see Figure 1.12. □

Remark 1.4.21. Analogously to the case of Δ-tensions we can assume the data (D, c_ℓ, c_u, Ω) to be reduced. Now the dual operation to vertex pushes is to augment the flow around facial cycles. Instead of choosing a forbidden vertex v_0, which is not allowed to be pushed this time we choose a *forbidden face*. Usually one takes the unbounded face of the plane embedding. Theorem 1.4.13 yields that by flow-augmentation at the remaining facial cycles we can construct the Hasse diagram of a distributive lattice on $\mathcal{F}_\Omega(D, c_\ell, c_u)$.

forbidden face

An application of Theorem 1.4.20, which may not be obtained using c-orientations or α-orientations is the distributive lattice structure on *k-fractional orientations* of planar graphs with prescribed outdegree $\frac{j}{k}$ (Bernardi and Fusy [11]). For the definition of planar k-fractional orientation take a planar graph $G = (V, \mathcal{E})$, where every ordinary edge $e = \{v, w\}$ is replaced by two directed half-edges $h_v(e), h_w(e)$ pointing from their vertex to the middle of e. Additionally we have a map O mapping every half-edges to a value in $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$ such that $O(h_v(e)) + O(h_w(e)) = 1$ for all edges $e = \{v, w\}$. The outdegree of a vertex v is just the sum $\sum_{e \ni v} O(h_v(e))$. We model k-fractional orientations of G with prescribed outdegree $\frac{j}{k}$, as planar Ω-flows of an orientation of G. Let $D = (V, A)$ be any orientation of G. Let $a = (v, w) \in A$ and $e = \{v, w\} \in \mathcal{E}$. Given a k-fractional orientation O define a flow by $f(a) := kO(h_v(e))$. We have a correspondence between k-fractional orientations and integer valued maps from A to $\{0, \ldots, k\}$. The outdegree of O at v is $\frac{j}{k} = (k\text{indeg}(v) - f(\text{in}(v)))/k + f(\text{out}(v))/k$. Equivalently the excess of f at v is $\omega(v, f) = k\text{indeg}(v) - j$. Hence we may model the k-fractional orientations of G with prescribed outdegree $\frac{j}{k}$ as Ω-flows and Theorem 1.4.20 yields a distributive lattice structure on them.

k-fractional orientation

In [12] Bernardi and Fusy show that for certain parameters k-fractional orientations with prescribed outdegree correspond to *Schnyder decompositions* of plane d-angulations of girth d and several equivalent concepts related to planar graphs. All these thus carry a distributive lattice structure, as well.

Khuller, Naor and Klein [66] only consider the special case of Theorem 1.4.20 where $\Omega = \mathbf{0}$. Such Ω-flows without excess are simply called *flows* or *circulations*. We restate their result as clear corollary of Theorem 1.4.20:

flow circulation

Theorem 1.4.22. *Let D be a planar digraph with upper and lower arc capacities c_l and c_u. The set of flows of D within c_l and c_u carries the structure of a distributive lattice.*

1.4.4 Planar orientations with prescribed outdegree (Felsner [39], Ossona de Mendez [88])

α-orientation

Consider a plane graph $G = (V, \mathcal{E})$. Given a map $\alpha : V \to \mathbb{N}$ an orientation $D = (V, A)$ of G is called an *α-orientation* if α records the outdegrees of all vertices, i.e., $\mathrm{outdeg}_D(v) = \alpha(v)$ for all $v \in V$. We denote the set of α-orientations of G by *α-or(G)*. The main result in [39] also obtained in [88] is:

Theorem 1.4.23. *Given a planar graph and a mapping $\alpha : V \to \mathbb{N}$ the set α-or(G) of α-orientations of G carries the structure of a distributive lattice.*

Proof. This may be proven analogously to Theorem 1.4.18, where c-orientations were interpreted as elements of $\mathcal{T}_\Delta(C, \mathbf{0}, \mathbf{1})$. Let D be some orientation of G. We look at $\mathcal{F}_\Omega(D, \mathbf{0}, \mathbf{1})$, where $f(a) = 1$ means reorienting a and $f(a) = 0$, leaving it unchanged. If we set $\omega(v) := \alpha(v) - \mathrm{outdeg}_D(v)$ for all v, then $\mathcal{F}_\Omega(D, \mathbf{0}, \mathbf{1})$ corresponds to the set of α-orientations G. Application of Theorem 1.4.20 yields the distributive lattice structure.

Another way to prove the theorem is to look at the planar dual of G. For a counter-clockwise facial cycle C_v of G^* corresponding to a vertex v of G define $c(C_v) := \deg(v) - 2\alpha(v)$. Now the c-orientations of G^* correspond to the α-orientations of G and we may apply Theorem 1.4.18.

For examples of both constructions, see Figure 1.12. □

In view of Section 1.4 α-orientations appear as a special case of both, planar flows and c-orientations. Nevertheless they already capture a big part of the applications. In the introduction of the thesis, we explained, how domino-tilings may be modelled as α-orientations. Similarly, this can be done for lozenge-tilings, see Figure 1.14
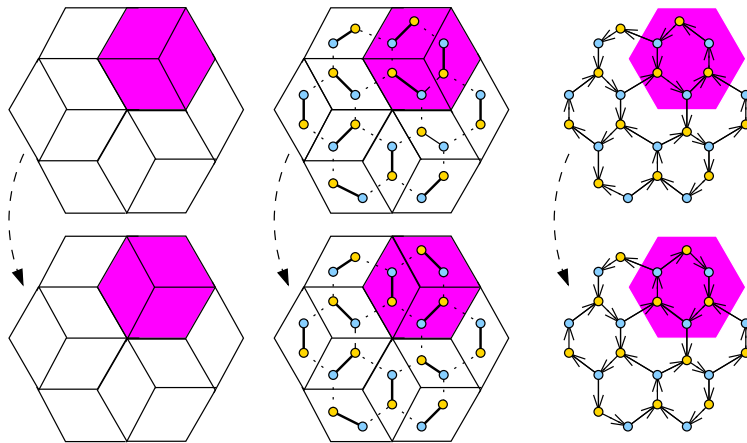

Figure 1.14: Local transformations and generalizations: from lozenge tilings via planar bipartite perfect matchings to α-orientations.

We list some objects which may be modelled as α-orientations on plane graphs and thus carry a distributive lattice structure:

- domino and lozenge tilings of plane regions (Rémila [97] and others based on Thurston [105])
- planar bipartite perfect matchings (Lam and Zhang [73])
- planar bipartite d-factors (Felsner [39], Propp [92])
- planar spanning trees (Gilmer and Litherland [48])
- Schnyder woods of planar triangulations (Brehm [25])
- Eulerian orientations of planar graphs (Felsner [39])

Because of their rich applications in the planar case α-orientations of non-planar graphs are very interesting. Propp [92] comments that to move between the d-factors in *toroidal graphs* it is necessary to operate on non-contractible cycles. This is made more explicit in terms of homology of orientable surfaces in [67], where generalizations of α-orientations to non-planar graphs and oriented matroids are investigated. It remains the difficult:

Question 1.4.24. What is the structure of α-orientations of graphs embedded on an orientable surface different from the plane?

For some application there might be a way "around" non-planar α-orientations. Consider for example of lozenge-tilings, whose lattice structure may be proven without the use of planar α-orientations. They may be modelled directly as c-orientations, as a special case of higher-dimensional rhombic tilings, see Figure 1.13. It would be interesting to find a way around non-planar α-orientations in other special cases of α-orientations. A good candidate may be the set of spanning trees of a graph or bases of a matroid.

1.5 Chip-Firing Games, Vector Addition Languages, and Upper Locally Distributive Lattices

Chip-firing games (CFG) on directed graphs were introduced by Björner and Lovász in [20]. They are a generalization of CFGs on undirected graphs, introduced by Björner, Lovász, and Shor [21]. CFGs have gained a big amount of attention, because of their relations to many areas of mathematics such as algebra, physics, combinatorics, dynamical systems, statistics, algorithms, and computational complexity, see Merino [83] and Goles, Latapy, Magnien, Morvan, and Phan [53] for surveys. The connection to rotor routing and the concept of *sandpile group* are made explicit by Holroyd, Levine, Mészáros, Peres, Propp, and Wilson in [58]. A connection from CFGs to subgraphs, orientations, Tutte polynomial, and embeddings into orientable surfaces is drawn by Bernardi [10]. Moreover, there are connections to tropical geometry, see Baker [8] and Haase, Musiker, and Yu [56]. Here we deal with the fundamental role of CFGs as examples of ULDs or equivalently *antimatroids* (Korte and Lovász [70]), *antimatroids with repetition* (Björner and Ziegler [22]), or *left-hereditary, permutable, locally free languages* (Björner and Lovász [21]). So CFGs are important and popular instances of ULDs but not every ULD arises as a CFG. In the present section we show how CFGs may be interpreted as vector-addition languages. We then characterize those vector-addition languages which yield a ULD in the same way CFGs do. These languages are then called *generalized chip-firing games* (Theorem 1.5.9). The main result of this section is that every ULD may be represented as a *generalized CFG* (Theorem 1.5.10) and that these in turn correspond to finite *intersections* of CFGs (Theorem 1.5.14).

For the definition of CFG let $D = (V, A)$ be a loop-free directed graph without isolated vertices and $\sigma \in \mathbb{N}^V$ a vector called a *chip-configuration*. The number $\sigma(v)$ records the *number of chips* on vertex v in σ. Given a chip-configuration σ a vertex v can be *fired* if it contains at least as many chips as its outdegree and is no sink, i.e., $\sigma(v) \geq \mathrm{outdeg}(v) > 0$. Firing v consists of sending a chip along each of the outgoing arcs of v to their respective end-vertices. The chip-configuration resulting from firing v in a chip-configuration τ is denoted by τ^v. We call a sequence $s = (v_1, \ldots, v_k)$ of vertices of D a *firing-sequence* if v_i can be fired in $((\sigma^{v_1})^{\cdots})^{v_{i-1}}$, for all $i \in [k]$. (Set $\sigma^{v_0} := \sigma$.) We define a directed graph $\mathrm{CFG}(D, \sigma)$ on the set of those chip-configurations τ on D which are *reachable* by a firing-sequence from σ, i.e., $\tau = \sigma^s := ((\sigma^{v_1})^{\cdots})^{v_k}$ for a firing sequence $s = (v_1, \ldots, v_k)$. For two such reachable chip-configurations τ, τ' we define (τ, τ') to be an arc of $\mathrm{CFG}(D, \sigma)$ if $\tau^v = \tau'$ for some vertex $v \in V$. In this case the *natural color* of the arc (τ, τ') is v. Hence, the arcs of $\mathrm{CFG}(D, \sigma)$ are naturally colored with V. We call the digraph $\mathrm{CFG}(D, \sigma)$ together with its natural arc-colring a *chip-firing games (CFG)*.

Remark 1.5.1. Since sinks are not allowed to be fired in a $\mathrm{CFG}(D, \sigma)$ we may actually identify all sinks of a given D to a single super-sink without changing $\mathrm{CFG}(D, \sigma)$. From now on we assume that our digraphs have either no sink or a unique sink.

chip-configuration

vertex firing

firing-sequence

reachable
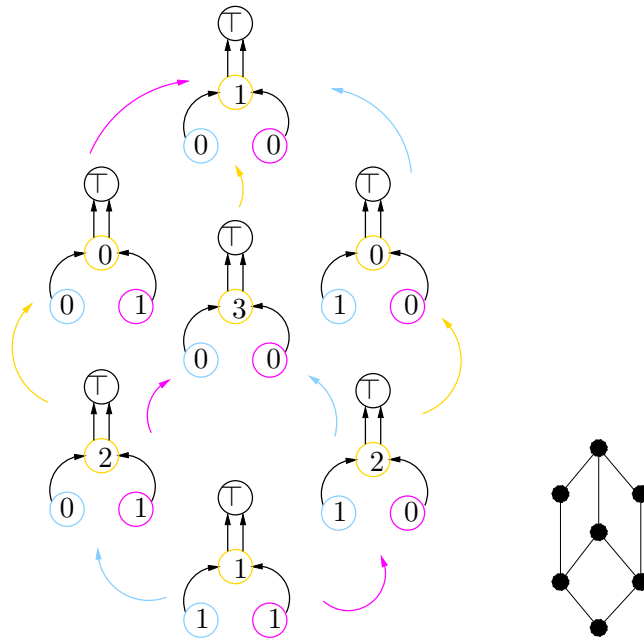
natural color

chip-firing game (CFG)

Figure 1.15: A chip-firing game on a sinky digraph representing the ULD on the right.

It is possible, that a digraph D admits a starting configuration σ, such that there are firing-sequences of infinite length. Take for example a directed cycle, with any positive number of chips distributed arbitrarily on the graph. Here we want to discuss only the finite case. We say that a digraph D is *globally finite* if there is no starting configuration σ that allows firing-sequences of infinite length.

globally finite

Later on, as a special case we obtain the following well known result. It was our original motivation for looking at chip-firing games.

Theorem 1.5.2. *If $D = (V, A)$ is globally finite, then for every $\sigma \in \mathbb{N}^V$ the digraph $CFG(D, \sigma)$ is the Hasse diagram of a ULD.*

In particular we will get that the natural arc-coloring of $CFG(D, \sigma)$ is a U-coloring and that $CFG(D, \sigma)$ is finite and acyclic. Since $CFG(D, \sigma)$ has a unique source σ, Theorem 1.5.2 then can be proven as a direct application of Theorem 1.3.3.

We say that the resulting ULD is *represented* by the CFG. The ULD-properties imply that for a globally finite D in $CFG(D, \sigma)$ there is a unique *stable chip-configuration* $\sigma_\top$, i.e., in $\sigma_\top$ no vertex can be fired. All maximal firing sequences from the starting configuration σ end at $\sigma_\top$. Moreover, by the colored Jordan-Dedekind chain condition all firing sequences between any two configurations fire the same multiset of vertices. Eriksson calls this behavior of a solitary game *strongly convergent* [37].

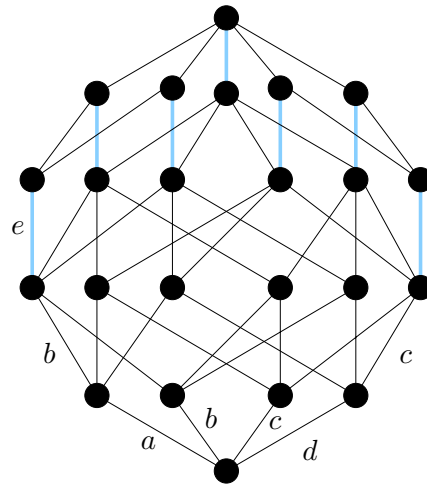stable
chip-configuration

Figure 1.16: A ULD which cannot be represented by a CFG.

In [79] Magnien, Phan, and Vuillon show that every distributive lattice can be represented as a CFG (actually of an undirected graph). Moreover they show that every ULD representable by a CFG is representable by a *simple CFG*, i.e., no vertex appears twice in any firing-sequence. On the other hand not every ULD can be represented as CFG. An example for that (also due to the same group of people) is shown in Figure 1.16. There are two natural question arising here:

Question 1.5.3. Is there a lattice theoretical characterization of ULDs representable by CFGs?

Question 1.5.4. Is there a generalization of CFGs representing the whole class of ULDs?

While we leave Question 1.5.3 unanswered the remaining part of this section presents and analyzes an answer to Question 1.5.4. In the following we will introduce a generalization of CFGs powerful enough to represent the class of ULDs, in other (still undefined) words: every generalized CFG yields a ULD for every starting configuration and conversely every ULD may be represented as a generalized CFG (Theorem 1.5.10). This generalization is still quite close to usual CFGs. More precisely every generalized CFG and therefore every ULD may be represented as an *intersection* of CFGs (Theorem 1.5.14 and Corollary 1.5.15). Moreover our construction still allows important algebraic constructions related to CFGs such as the *sandpile group* [58] and the *sandpile monoid* [6].

To the end of capturing generalized CFGs we define vector-addition languages. Vector addition languages were introduced by Karp and Miller [65]. They are also known as *general Petri nets* (Reisig [96]) and are one of the most popular formal methods for analysis and representation of parallel processes [38]. We will only use them for the very specific reason to define generalized CFGs.

A *vector-addition language* is a language $L(M, \sigma)$ given by an *alphabet* $M \subset \mathbb{R}^d$ and a $\qquad$vector-addition language
starting configuration $\sigma \in \mathbb{R}_{\geq 0}^d$. A word $s = (x_1, \ldots, x_k)$ is in $L(M, \sigma)$ if $x_i \in M$ and

$$\sigma + x_1 + \ldots + x_i \geq \mathbf{0} \text{ for all } 1 \leq i \leq k.$$

For a word $s \in L(M, \sigma)$ denote by $\mathrm{scr}(s)$ its *score*, i.e., the multiset of its letters and $\qquad$score of word
by $\mathrm{cnf}(s) := \sigma + \sum_{x \in \mathrm{scr}(s)} x$ its *configuration*. We define a digraph $D(M, \sigma)$ on the ver- $\qquad$configuration of word
tex set $\mathrm{cnf}(L(M, \sigma))$. Two configurations $\mathrm{cnf}(s), \mathrm{cnf}(t)$ of words in $L(M, \sigma)$ form an arc
$(\mathrm{cnf}(s), \mathrm{cnf}(t))$ of $D(M, \sigma)$ if there is an $x \in M$ such that $\mathrm{cnf}(s) + x = \mathrm{cnf}(t)$. In this
case the *natural color* of the arc $(\mathrm{cnf}(s), \mathrm{cnf}(t))$ is defined x. $\qquad$natural color

We will later on define generalized CFGs in terms of vector-addition languages. So in
order to *generalize* CFGs it would be good if CFGs could be encoded as vector-addition
languages themselves. And indeed, one important feature of CFGs is, that they *can* be
interpreted as vector-addition languages:

Let D be a loopless digraph without isolated vertices. The *Laplacian* of D is a set M of $\qquad$Laplacian
$|V|$ vectors in $\mathbb{Z}^V$, where for every $v \in V$ there is a vector $x(v) \in M$ with:

$$x(v)_w := \begin{cases} |\{a \in A \mid a = (v, w)\}| & \text{if } v \neq w, \\ -\mathsf{outdeg}(v) & \text{otherwise.} \end{cases}$$

If D has a sink, (by Remark 1.5.1 D has either one or none), then we delete the corre-
sponding element of M and corresponding components of the remaining element of M. We
obtain the *reduced Laplacian* M' of D. The pair M', σ encodes the same information as $\qquad$reduced Laplacian
D, σ. It is a classical and easy result that $D(L(M', \sigma))$ models $\mathrm{CFG}(D, \sigma)$, see [20]. More
precisely one has correspondences:

D, σ	$\longleftrightarrow$	M', σ
chip-configuration τ	$\longleftrightarrow$	$x \in \mathbb{Z}^d$
fire v in σ^s	$\longleftrightarrow$	add $x(v)$ to $\mathrm{cnf}(s)$
firing-sequences	$\longleftrightarrow$	words in $L(M', \sigma)$
reachable chip-configurations	$\longleftrightarrow$	elements of $\mathrm{cnf}(L(M', \sigma))$
$\mathrm{CFG}(D, \sigma)$	$\longleftrightarrow$	$D(M', \sigma)$

We want to generalize reduced Laplacians of digraphs, in order to characterize the class
of vector-addition languages which represent a ULD for every starting configuration, i.e.,
the class of alphabets M such that $D(M, \sigma)$ is the Hasse diagram of a ULD for every σ.
As mentioned in the beginning we only want to consider globally finite digraphs, i.e., no
starting configuration allows firing sequences of infinite length. So this is another property
which we want to generalize to vector-addition languages. In the graph case there is an easy
necessary condition fro global finiteness. A digraph D is *sinky* if there is a sink $\top \in V$, i.e., $\qquad$sinky
$\mathsf{outdeg}(\top) = 0$, such that every $v \in V$ lies on a directed $(v, \top)$-path.

Lemma 1.5.5. *Every globally finite digraph is sinky.*

Proof. Suppose D is not sinky, i.e., there is a vertex v in D which does not lie on a directed path to a sink of D. Then D has a non-trivial strong component S, such that no arcs point from S into $D\backslash S$. Let d^+ be the maximal outdegree of the induced subgraph $D[S]$. Define a chip-configuration σ by putting d^+ chips on every vertex of S and 0 elsewhere. Every reachable configuration from σ has at least one vertex with at least as many chips on it than d^+, i.e., $\sigma(v) \geq \mathsf{outdeg}(v)$. Thus, there are infinite firing-sequences and D is not globally finite. $\square$

It will turn out soon in the more general context of vector-addition languages (Proposition 1.5.7), that the converse of Lemma 1.5.5 is true, as well.

In order to find a generalization of *globally finite* to vector-addition languages define a language to be *finite* if it consists of words of finite lengths only. In analogy to the digraph case call an alphabet $M \subseteq \mathbb{R}^d$ *globally finite* if $L(M, \sigma)$ is finite for all σ. An alphabet $M \subseteq \mathbb{R}^d$ is said to be *sinky* if all $x \in M$ satisfy $\sum_{i \in [d]} x_i \leq 0$ and there are *special vectors* $x^* \in M$, satisfying $\sum_{i \in [d]} x_i^* < 0$ or x^* has a private negative coordinate $i \in [d]$, i.e., $x_i^* < 0 = y_i$ for all $y \in M\backslash\{x^*\}$. Now for all $x \in M$ we require a *directed path* from x to some special $x^* \in M$. By directed path from x to x^* we refer to a sequence $(x = x(0), x(1), \ldots, x(k) = x^*)$ in M with the property: for all $j \in [k]$ there is a $i \in [d]$ such that $x(j-1)_i > 0 > x(j)_i$.

finite language

globally finite

sinky

special vector

Lemma 1.5.6. *An alphabet $M \subseteq \mathbb{Z}^d$ is the reduced Laplacian of a sinky digraph if and only if M is sinky, every $x \in M$ has exactly one negative entry, and for every $i \in [d]$ there is exactly one $x \in M$ with $x_i < 0$.*

Proof. "$\Longrightarrow$": Let D be a sinky digraph with sink $\top$ and reduced Laplacian M'. Clearly, M' is for every $i \in [d]$ there is exactly one $x \in M$ with $x_i < 0$ and every $x(v) \in M'$ has exactly one negative entry, $x(v)_v$. Moreover, for every $v \in V$ we have $\sum_i x(v)_i = \mathsf{outdeg}(v) - \{a \in A \mid a = (v, \top)\} \leq 0$. The special vectors x^* of the definition of sinky then correspond to neighbors of $\top$. For any vertex v there is a directed $(v, \top)$-path P in D. Let P' be the (v, x^*)-path obtained from P by deleting $\top$. By definition of the reducd Laplacian P' corresponds to a directed (v, x^*)-path in the sense of the definition of a sinky set of vectors.

"$\Longleftarrow$": For every $i \in [d]$ there is exactly one $x \in M$ with $x_i < 0$, i.e., no two $x, x' \in M$ share a negative entry. Since on the other hand every $x \in M$ has exactly one negative entry M has exactly d elements. Refer to the element of M with $x_i < 0$ as $x(i)$. Construct a digraph D with vertex set $M \cup \{\top\}$. Introduce $x(i)_j$ arcs from $x(i)$ to $x(j)$ for $i \neq j$. Since M is sinky $\sum_j x(i)_j \leq 0$ for all i. In case for some $x^*(i)$ we have $\sum_j x^*(i)_j < 0$, introduce $-\sum_j x^*(i)_j$ arcs from $x^*(i)$ to $\top$. Clearly M is the reduced Laplacian of D. The vertices $x^*(i)$ are special vectors in the sense of the definition of a sinky set of vectors. A directed $(x(i), x^*(j))$-path in the sense of the that definition corresponds to directed $(x(i), \top)$-path of D. $\square$

Another class which by Lemma 1.5.6 clearly generalizes the class of reduced Laplacians is the following: We call a finite set $M \subset \mathbb{R}^d$ *Laplacious* if for every $i \in [d]$ there is at most one $x \in M$ with $x_i < 0$.

Laplacious

So a Laplacious sinky alphabet $M \subseteq \mathbb{Z}^d$ differs from the reduced Laplacian of a sinky digraph D only by the fact that vectors in M may have more than one negative entry (sinky implies that they have at least one), and that there might be $i \in [d]$ with $x_i \geq 0$ for all $x \in M$. Indeed, as in the graphic case (Lemma 1.5.6) sinky is enough to ensure global finiteness for a Laplacious alphabet M.

Proposition 1.5.7. *If $M \subseteq \mathbb{R}^d$ is sinky and Laplacious, then M is globally finite.*

Proof. Let $\sigma \in \mathbb{R}^d$ be any starting configuration. Suppose there is a word $s \in L(M.\sigma)$ with infinite score $\mathrm{scr}(s)$. Let Y be the set of vectors appearing infinitely many times in $\mathrm{scr}(s)$ and $I := \{i \in [d] \mid y_i < 0 \text{ for some } y \in Y\}$. Sinky together implies, that every $x \in M$ has at least one negative entry, i.e., $I \neq \emptyset$. Clearly, there are no special vectors in Y and all $y \in Y$ have $\sum_{i \in I} y_i = 0$. Now let y be a *neighbor* of some $x \notin Y$, i.e., there is a $j \in [d]$ with $y_j > 0 > x_j$. Since x is not in Y and M is Laplacious we have $j \notin I$. Since M is sinky we have $\sum_{i \in I} y_i \leq \sum_{i \in [d]} y_i - y_j < 0$. But for $y \in Y$ we clearly need $\sum_{i \in I} y_i = 0$ in order to apply it infinitely many times – a contradiction. □

As promised, with Lemma 1.5.6 Proposition 1.5.7 proves the backward direction of Lemma 1.5.5, i.e., a sinky digraph is globally finite. The converse of Propostion 1.5.7 is not true, e.g., the alphabet $M = \{\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}\}$ is globally finite and Laplacious but not sinky. As a first theorem we can now characterize the class of vector-addition languages which represent finite ULDs for every starting configuration and therefore keep one of the properties of CFGs being of major interest to us. Therefore we make the following

Definition 1.5.8. Let $M \subseteq \mathbb{R}^d$ and $\sigma \in \mathbb{R}^d$. If M is globally finite and Laplacious, then we call the digraph $D(M, \sigma)$ together with its natural arc-coloring a *generalized CFG* and denote it by $\mathrm{CFG}(M, \sigma)$.

generalized CFG

Theorem 1.5.9. *Let $M \subseteq \mathbb{R}^d$ and $\sigma \in \mathbb{R}^d$. The digraph $D(M, \sigma)$ is the Hasse diagram of a ULD for all σ if and only if $D(M, \sigma)$ is a generalized CFG. Moreover, in that case the natural arc-coloring of $D(M, \sigma)$ is a U-coloring.*

Proof. "$\Longrightarrow$": If M is not globally finite, then there is σ such that $L(M, \sigma)$ has infinitely many elements. For us ULDs are finite, i.e., this cannot happen. So suppose M is not Laplacious. Then there are $x, y \in M$ with $x_i, y_i < 0$ for some $i \in [d]$. Define $\sigma := \max(|x|, |y|)$. Then $D(M, \sigma)$ contains the arcs $(\sigma, x), (\sigma, y)$ with colors x and y. But since $\max(|x_i|, |y_i|) + x_i + y_i < 0$ there is no vertex corresponding to the configuration $\sigma + x + y$. Hence, the natural coloring is not a U-coloring. It violates rule U_2. Indeed, $D(M, \sigma)$ admits no U-coloring at all.

"$\Longleftarrow$": We start by showing, that the natural coloring of $D(M, \sigma)$ is a U-coloring. Whenever there are two outgoing arcs $(\mathrm{cnf}(s), \mathrm{cnf}(s) + x), (\mathrm{cnf}(s), \mathrm{cnf}(s) + y)$ of the same vertex $\mathrm{cnf}(s)$ they clearly are of different colors x and y, i.e., the coloring obeys to rule U_1.

Moreover, since no $x, y \in M$ share negative entries we conclude that $\sigma + \sum_{z \in s} z + x, \sigma + \sum_{z \in s} z + y \geq \mathbf{0}$ implies $\sigma + \sum_{z \in s} z + x + y \geq \mathbf{0}$. This is, if $\mathrm{cnf}(s), \mathrm{cnf}(s) + x, \mathrm{cnf}(s) + y$ are configurations of words in $L(M\sigma)$, then $\mathrm{cnf}(s) + x + y$ is, as well. Hence the natural arc-coloring of $D(M, \sigma)$ is a U-coloring.

If there was a directed cycle in $D(M, \sigma)$, then traversing it forever would correspond to a word of infinite score, contradicting global finiteness.

Suppose there is an infinite number of words in $L(M, \sigma)$, i.e., $D(M, \sigma)$ has infinitely many vertices. We show that $D(M, \sigma)$ has no sinks. Let $\mathrm{cnf}(s)$ a vertex of distance k to σ. If $\mathrm{cnf}(s)$ was the only vertex with that distance to σ, then $\mathrm{cnf}(s)$ cannot be a sink, since by U_1 the set of vertices of distance at most k is finite. So say there is a different $\mathrm{cnf}(t)$ of distance k to σ. Take a directed $(\sigma, \mathrm{cnf}(s))$-path P and a directed $(\sigma, \mathrm{cnf}(t))$-path Q which stay together as long as possible. Let $\mathrm{cnf}(x)$ be the last element they have in common and $\mathrm{cnf}(y)$ be the first element only on Q. We can apply Lemma 1.3.4 to the restriction of P from $\mathrm{cnf}(x)$ to $\mathrm{cnf}(s)$ and the arc $(\mathrm{cnf}(x), \mathrm{cnf}(y))$. We have to be in case (a), because otherwise there would be a path P' staying longer with Q. Hence Lemma 1.3.4 yields that $\mathrm{cnf}(s)$ has an outgoing arc and was no sink. Since $D(M, \sigma)$ has no sinks but is acyclic it must have paths of infinite lengths, i.e, $L(M, \sigma)$ was not globally finite.

We have proven that $D(M, \sigma)$ is a finite acyclic digraph with unique source σ and its natural coloring is a U-coloring. Theorem 1.3.3 gives that $D(M, \sigma)$ is the Hasse diagram of a ULD. $\qquad\square$

Since by Lemma 1.5.6 the reduced Laplacian M' of a sinky directed graph D is sinky and Laplacious and thus globally finite by Proposition 1.5.7, we can apply Theorem 1.5.9 to obtain Theorem 1.5.2, i.e., every CFG represents a ULD.

As mentioned above, in [79] it is shown that every ULD representable by a CFG can be represented by a simple CFG. To generalize this result we call a vector-addition language *simple* if no word contains twice the same letter. A generalized chip-firing game $\mathrm{CFG}(M, \sigma)$ is called simple if $L(M, \sigma)$ is simple. We prove

simple language

Theorem 1.5.10. *Every ULD can be represented by a simple generalized CFG.*

Proof. Let $\mathcal{L}$ be a ULD. We look at the representation of $\mathcal{L}$ as a reduced antichain-partitoned poset $(\mathcal{J}(\mathcal{L}), \mathcal{A}_{\mathcal{M}(\mathcal{L})})$, where $A_m := \{j \in \mathcal{J}(\mathcal{L}) \mid m \in \uparrow j^- \backslash \uparrow j\}$. By Theorem 1.2.24 we have that $\mathcal{L} \cong (\mathit{fing}_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\mathcal{I}(\mathcal{J}(\mathcal{L}))), \subseteq)$.

For every $m \in \mathcal{M}(\mathcal{L})$ we define a vector $x(m)$. These will form our alphabet M. In order to prove our theorem we prove that given a word $s = (x(m_1), \ldots, x(m_k)) \in L(M, \sigma)$ in our simple language and a letter $x(m) \in M$ we have $(s, x(m)) \in L(M, \sigma)$ if and only if $\{m_1, \ldots, m_k, m\}$ is the fingerprint of an ideal of $\mathcal{J}(\mathcal{L})$ and $m \neq m_i$ for $i \in [k]$. This

is, there is a $j \in A_m$ such that the fingerprint $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j \backslash \{j\}) \subseteq \{m_1, \ldots, m_k\}$ and $m \neq m_i$ for $i \in [k]$.

Denote $P(j) := fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\downarrow j \backslash \{j\})$ for every $j \in \mathcal{J}(\mathcal{L})$. The set of coordinates $[d]$ of the vectors of our alphabet M will correspond to elements of

$$\bigcup_{m \in \mathcal{M}(\mathcal{L})} ((\mathsf{X}_{j \in A_m} P(j)) \times \{m\}) \cup \mathcal{M}(\mathcal{L}).$$

For all $i = m' \in \mathcal{M}(\mathcal{L})$ define $\sigma_i = 1$ and $x(m)_i = -1$ if $m = m'$ and $x(m)_i = 0$, otherwise. This guarantees that M is sinky and $L(M, \sigma)$ simple. Let $i \in P(j_1) \times \ldots \times P(j_k) \times \{m\}$ for some $A_m = \{j_1, \ldots, j_k\} \in \mathcal{A}_{\mathcal{M}(\mathcal{L})} \in \mathcal{A}_{\mathcal{M}(\mathcal{L})}$. We set $x(m)_i := -1$ and $\sigma_i := 0$. If $i \in P(j_1) \times \ldots \times \{m'\} \times \ldots \times P(j_k) \times \{m\}$, then we set $x(m')_i := 1$ and otherwise $x(m')_i := 0$.

We have already argued that M is sinky, i.e., by Proposition 1.5.7 it is globally finite. In particular we constructed σ such that $L(M, \sigma)$ is simple. For every $i \in [d]$ there is a unique $x(m)$ with $x(m) < 0$, if $i = m \in \mathcal{M}(\mathcal{L})$, then this is $x(m)$. If $i \in P(j_1) \times \ldots \times P(j_k) \times \{m\}$, then it is also $x(m)$. Thus M is Laplacious and globally finite, i.e., $D(M, \sigma)$ is a simple generalized CFG.

We show that $D(M, \sigma)$ is the Hasse diagram of $\mathcal{L}$.

Let $x(m) \in M$, $j_\ell \in A_m$ and $s = (x(m_1), \ldots, x(m_k)) \in L(M, \sigma)$ such that $P(j_\ell) \subseteq \{m_1, \ldots, m_k\}$ and $x(m) \notin s$ for $i \in [k]$. Since $x(m) \notin s$ we have $\text{cnf}(s)_m = 1$. For every negative entry $x(m)_i < 0$ with $i = (m'_1, \ldots, m'_k, m) \in P(j_1) \times \ldots \times P(j_k) \times \{m\}$ we know that $x(m'_\ell) \in s$ and $A_{m'_\ell} \cap P(j_\ell) \neq \emptyset$. Thus by definition $x(m'_\ell)_i = 1$. We have that $s + x(m) \in \text{cnf}(L(M, \sigma))$.

On the other hand if $x(m) \in s$ then $s + x(m) \notin \text{cnf}(L(M, \sigma))$ by simplicity of $L(M, \sigma)$. Suppose for every $j_\ell \in A_m$ there is an $m'_\ell \in P(j_\ell)$ but $x(m'_\ell) \notin s$. For $i = (m'_1, \ldots, m'_k)$ there is no $x(m') \in s$ with $x(m')_i > 0$, but $x(m)_i = -1$. Thus, $x(m)$ cannot be added to s. $\square$

Remark 1.5.11. Note that the simple generalized CFG constructed in the proof of Theorem 1.5.10 has a very special property. Given a sequence $s = (x(m_1), \ldots, x(m_k))$ in M we have:

$$\sigma + x(m_1) + \ldots + x(m_i) \geq \mathbf{0} \text{ for all } i \in [k] \iff \sigma + x(m_1) + \ldots + x(m_k) \geq \mathbf{0}.$$

This is, $\text{scr}(L(M, \sigma)) \cong \{z \in \{0, 1\}^M \mid Mz \geq -\sigma\}$. The fact that $L(M, \sigma)$ has a description by a system of linear inequalities, i.e., may be seen as the set of integer points of a polyhedron, will be used in the context of feasible polytopes of antimatroids (Subsection 2.3.1 of the next Chapter).

Remark 1.5.12. The dimension d of the space containing M in Theorem 1.5.10 is desired to be small. Clearly, smaller d just yields a more compact representation. In Subsection 2.3.1

of the next Chapter we will relate that parameter indeed to an optimization-problem of Korte and Lovász [71].

If the size of the antichains $A_m := \{j \in \mathcal{J}(\mathcal{L}) \mid m \in \uparrow j^- \backslash \uparrow j\} \in \mathcal{A}_{\mathcal{M}(\mathcal{L})}$ is bounded by k, then the construction in the proof of Theorem 1.5.10 yields a representation of the ULD by $M \subset \mathbb{Z}^d$, where $d \in O(|\mathcal{M}(\mathcal{L})|^{k+1})$.

The size of the vectors in M may be reduced further. Even if it does not cause a change of the asymptotical behavior, we present a method, which is helpful for special ULD-classes: Define the set of antichains $B(j)$ as $\mathrm{Max}\{A_m \in \mathcal{A}_{\mathcal{M}(\mathcal{L})} \mid A_m \subseteq \downarrow j \backslash \{j\}\}$ and $C(j) := \{j' \in \mathcal{J}(\mathcal{L}) \mid j'' \leq j' < j$ for some $j'' \in A_m \in B(j)\}$. In the proof of Theorem 1.5.10 we can set $P(j) := \mathit{fing}_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(C(j))$. This suffices and reduces the size of d in the construction. In the case of the singleton chain partiton, i.e., distributive lattices by Theorem 1.2.1, $C(j)$ would just be the set of cocovers of j.

Question 1.5.13. Theorem 1.5.10 shows representability of ULDs by simple languages. It would be interesting to see whether every Hasse diagram of a ULD together with a U-coloring may arise as a generalized CFG with its natural coloring.

In the following we will present a method to represent a generalized CFG by a finite set of ordinary CFGs. This then yields that every ULD may be represented as an intersection of CFGs. But first let us mention a similar result already in the literature:

colored CFG

In [79] Magnien, Phan, and Vuillon prove that every ULD can be represented as a *colored* or extended CFG. This game is played on a set of loop-free digraphs $D_1, \ldots, D_k$ on the same vertex set each having arcs only of its private color $i \in [k]$ but all sinky with respect to the same vertex $\top$. Every digraph has a chip-configuration $\sigma(i)$ of chips in its private color. Firing a vertex v in the colored CFG means to fire v in all D_i where it has more chips of color i than outgoing arcs in that color. This is, fire it in the sense of a classical CFG in all D_i where it is allowed to be fired. This representation of ULDs does not carry over to vector-addition languages.

Based on the theory of generalized CFGs we will develop a new way to present a ULD by a finite set of CFGs: Also this game is played on a set of loop-free digraphs $D_1, \ldots, D_k$ on the same vertex set each having arcs of its private color $i \in [k]$ and all being sinky with respect to the same vertex $\top$. Given respective chip-configurations $\tau(1), \ldots, \tau(k)$ firing a vertex $v \in V$ is allowed if v can be fired in all the $(D_i, \tau(i))$ (viewed as ordinary CFGs). Firing v then consists in actually doing so. Given starting configurations $\sigma(1), \ldots, \sigma(k)$

intersection of CFGs

the *intersection of the CFGs* is denoted by $\bigcap CFG(D_i, \sigma(i))$. It is the digraph on the set of reachable chip-configurations and there is an arc from $\tau(1), \ldots, \tau(k)$ to $\tau'(1), \ldots, \tau'(k)$ if $\tau(i)^v = \tau'(i)$ for all $i \in [k]$ and some $v \in V$. Thus, as in ordinary CFGs the arcs of $\bigcap CFG(D_i, \sigma(i))$ are naturally colored by V.

Examples for both of the above constructions can be found in Figure 1.17. We can prove the very analogue result to the one of [79].
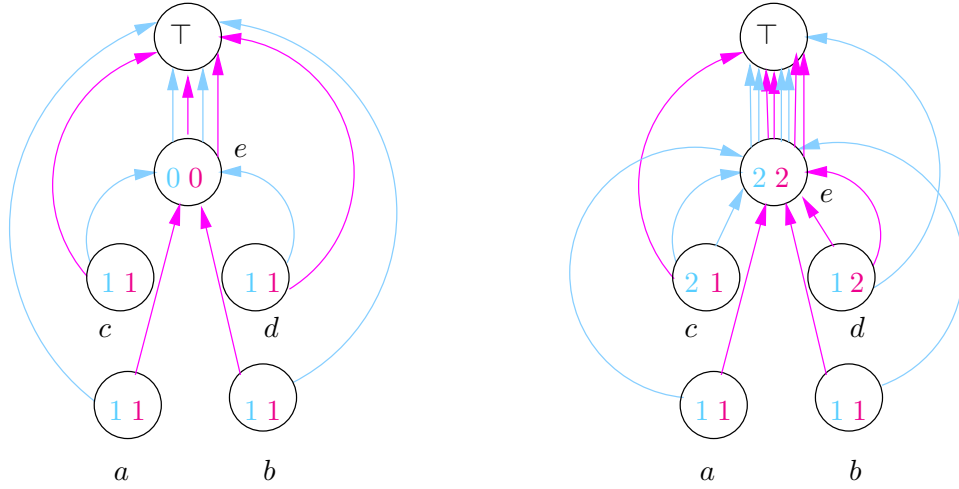
Figure 1.17: The ULD of Figure 1.16 can be represented as bicolored CFG following [79] and as intersection of two CFGs with the depicted starting configurations, respectively.

Theorem 1.5.14. *For every Laplacious, globally finite $M \subseteq \mathbb{Z}^d$ and $\sigma \in \mathbb{Z}^d$ there are sinky $(D_i, \sigma(i))_{i \in [k]}$ such that the digraphs $\mathrm{CFG}(M, \sigma)$ and $\bigcap \mathrm{CFG}(D_i, \sigma(i))$ are isomorphic and have identical natural colorings, and viceversa.*

Proof. For "$\Longleftarrow$" just take the reduced Laplacians $M'(D_1), \ldots, M'(D_k)$ and write them vertically one above the other. We obtain a sinky, Laplacious M and firing a vertex in the intersection of $D_1, \ldots, D_k$ in a chip-configuration $\sigma(1), \ldots, \sigma(k)$ corresponds to adding the corresponding vector in M' to the vector concatenation $(\sigma(1), \ldots, \sigma(k))$.

For "$\Longrightarrow$" Let $M \subseteq \mathbb{Z}^d$ be globally finite and Laplacious. Consider $\mathrm{CFG}(M, \sigma)$ with starting configuration σ. We obtain a representation as intersection of CFGs as in the "$\Longleftarrow$"-direction by partitioning $[d]$ and possibly adding few extra components to such that all alphabets induced by the partition are sinky and have exactly one negative entry per row and per column. By Lemma 1.5.6 they are reduced Laplacians.

So partition $[d] = V_1 \cup \ldots \cup V_k$ such that the set of entries $\{x_i \mid i \in V_j\}$ has at most one negative element for every $j \in [k]$ and $x \in M$ and if for some x and j we have $\sum_{i \in V_j} x_i \geq 0$, then $\{x_i \mid i \in V_j\}$ has no negative entries.

Now if for some x and j we have $\sum_{i \in V_j} x_i \geq 0$ or $\{x_i \mid i \in V_j\}$ has no negative entries, then we add a component ℓ to V_j. Set $x_\ell = -\sum_{i \in V_j} x_i - 1$ and $\sigma_\ell = \sum_{i \in V_j} x_i + r$ and $x'_\ell = 0$ for all $x' \neq x$. Here r is the number of times x appears in a directed path from σ to $\sigma_\top$ in $\mathrm{CFG}(M, \sigma)$. The number r exists because M is globally finite. We have not changed $\mathrm{CFG}(M, \sigma)$ nor its U-coloring. For the new set V'_j we have $\sum_{i \in V'_j} x_i < 0$, i.e., for every x. That is, it is sinky. Also $\{x_i \mid i \in V'_j\}$ has exactly one negative entry for all x. If there was a coordinate $i \in V'_j$ with $x_i \geq 0$ for all $x \in M$ we can clearly delete it without changing the vector-addition language. So because M was Laplacious for every $i \in V'_j$ there is exactly one x with $x_i < 0$ and by Lemma 1.5.6 the alphabet V'_j is a reduced Laplacian.

Applying this to the whole alphabet M we call the new alphabet and starting configuration $\widetilde{M}$ and $\widetilde{\sigma}$, respectively. The restriction $\widetilde{M}$ to any V_i' is the reduced Laplacian of a digraph D_i. Denote by $\widetilde{\sigma}(i)$ the restriction of the starting configuration $\widetilde{\sigma}$ to V_i'. We have $\mathrm{CFG}(M, \sigma) \cong \mathrm{CFG}(\widetilde{M}, \widetilde{\sigma}) \cong \bigcap_{i \in [k]} CFG(D_i, \widetilde{\sigma}(i))$. $\qquad\square$

Theorem 1.5.10 and Theorem 1.5.14 together yield

Corollary 1.5.15. *Every ULD is the intersection of finitely many CFGs and viceversa.*

CFG-dimension

Question 1.5.16. One interesting question arising from Corollary 1.5.15 is to determine the *CFG-dimension* of a given ULD, i.e., the minimum number of CFGs that are needed to represent a given ULD as their intersection. For example the ULD of Figure 1.16 is not representable by a CFG, but by the intersection of two CFGs (shown in the right of Figure 1.17). Hence it has CFG-dimension 2.

Question 1.5.17. The proof of Theorem 1.5.10 relies on the representation of ULDs as antichain-partitoned posets (Theorem 1.2.24). In the light of Theorem 1.2.3, where general lattices are represented by antichain-covered posets it would be an interesting question whether every lattice $\mathcal{L}$ can be represented as a certain type of vector-addition language.

In [67] it has been shown that even every acyclic digraph can be represented as a type of vector-addition language with $M \subseteq \{0, \pm 1\}^d$, but for this representation every arc would is represented by an individual vector. In the case of Hasse diagrams of lattices it would be sensible to ask for a correspondence between M and $\mathcal{M}(\mathcal{L})$.

1.6 Conclusions

In this chapter we have characterized ULDs in terms of colored Hasse diagrams, (multi)set-systems, antichain-covered posets, vector-addition-languages, intersections of chip-firing games, and embeddings into $\mathbb{N}^d$. There are many possible directions of further study. In the following we discuss some of them.

Classes of combinatorial objects

We have provided (upper locally) distributive lattices arising from local transformations on given sets of combinatorial objects. Indeed, the combinatorics encoded in these lattices correspond to chain-partitions of meet-irreducible posets of ULDs or to join-sublattices of $\mathbb{N}^d$ (Theorem 1.3.18).
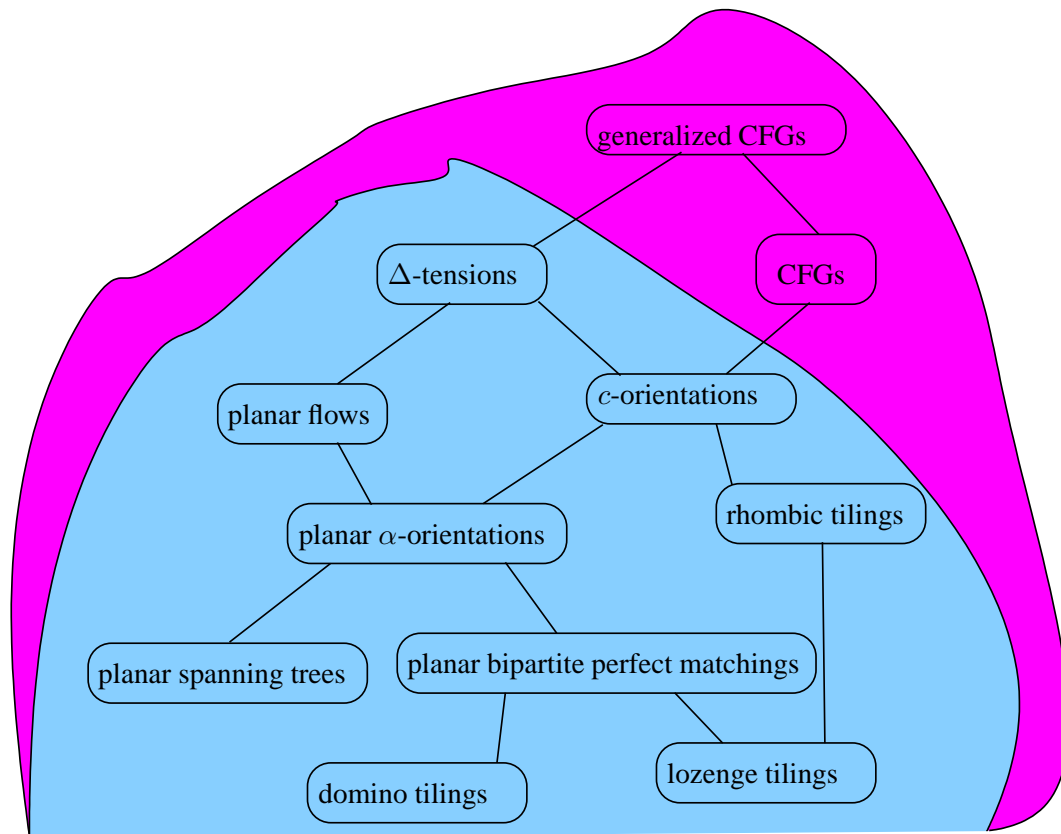
Figure 1.18: What happened in Chapter 1?

In Figure 1.18 we give a retrospective overview. The $\leq$-relation induced by the Hasse diagram in Figure 1.18 stands for the inclusions of the class of embeddings into $\mathbb{N}^d$ induced

by the combinatorial objects in the bubbles. All bubbles in the pink area yield embeddings of distributive lattices. The bubbles in the blue area yield embeddings of upper locally distributive lattices. Already c-orientations suffice to represent all distributive lattices, but even Δ-tensions are not general enough to represent all distributive lattice embeddings. Generalized CFGs represent all upper locally distributive lattices, but may every embedded ULD be represented by a generalized chip-firing game?

Tension Lattices

The generation of a random element from a distributive lattice is a nice application for *coupling from the past* (c.f. Propp and Wilson [94]). The challenge is to find good estimates for the mixing time, see Propp [93]. What if the lattice is a Δ-tension lattice or more restrictively the set of α-orientations of a planar graph? For the latter it would be of particular interest to obtain a lattice theoretic characterization of distributive lattices arising from planar α-orientations.

Lattices of Δ-tensions depend on the choice of a vertex $v_0 \in V$. Choosing another vertex v_1 yields a different lattice on the same set of objects. Is there an easy description of the transformation $\Pi_{v_0}(D, c_\ell, c_u)$ to $\Pi_{v_1}(D, c_\ell, c_u)$? Or is there a natural candidate for choosing v_0 in order to guarantee certain properties of the lattice?

Because of their diverse applications it would be very useful to obtain structural results for α-orientations of graphs embedded on an orientable (or non-orientable) surface different from the plane – or more generally for nonplanar graphs. This seems to be a hard problem. We feel that it is better to start with particular instances of planar α-orientations, such as perfect bipartite matchings or spanning trees. In these cases, it might be possible to generalize their distributive lattice structure to the non-planar case without making the step through α-orientations.

U-Posets Without Global Minimum

The characterization of ULDs in terms of U-posets with global minimum leads to questions about U-posets without global minimum.

We have shown that in U-posets every element x has a unique minimal set of meet-irreducibles having x as lower bound. But x might not be the only lower bound of that set. It would be of interest to characterize join-semilattices $\mathcal{L}$, where for all $x \in \mathcal{L}$ there exists a unique minimal set M_x of meet-irreducibles such that x is the unique maximal lower bound for M_x. Note that this still does not turn $\mathcal{L}$ into a lattice. Also it would be interesting to describe those U-posets, which correspond to join-subsemilattices of $\mathbb{N}^d$. Another point would be to characterize infinite ULDs which arise as join-sublattices of $\mathbb{N}^d$.

In the context of chip-firing games it is natural to allow to *cofire* vertices, i.e., a vertex receives a chip from all its out-neighbors. Versions of this game where negative numbers

of chips are allowed, have been considered under the name of *dollar game* by Baker and
Norine [8]. They are related to tropical geometry [56]. Sticking to the non-negative case, i.e.,
insisting on the non-negativity constraint, it would be interesting to investigate the resulting
class of U-posets. This relates to Question 1.3.20. A resulting topic to analyze would be
vector-addition subtraction languages.

Note that the distributive lattice of Δ-tensions corresponds to finite vector-addition sub-
traction language. The alphabet consisits of the rows of the vertex by arc network matrix of
a digraph.

More Questions Related to Chip-Firing Games

In the last section we put a certain emphasis on generalizing CFGs while "staying close"
to them. So Question 1.5.16 asks for the minimum number of CFGs necessary to repre-
sent a ULD as their intersection. And in particular we are interested in a lattice theoretical
characterization of ULDs representable by CFGs.

Here comes a description of several concepts, which generalize from CFGs to generalized
CFGs and possible research topics related to them. The *sandpile monoid* of a sinky digraph   sandpile monoid
consists of all stable configurations, i.e., such that no vertex can be fired. The "sum" of
two configurations is defined by adding chips of both configurations on corresponding ver-
tices and afterwards fire, until the maximum configuration is reached. Our generalization to
globally finite Laplacious matrices now allows to define the same structure for such matri-
ces. There is a remarkable theory about sandpile monoids [6]. What can we say about the
generalization to vector-addition languages?

In particular the *sandpile group* is an Abelian subgroup of the sandpile monoid, which   sandpile group
has been of vivid interest again quite recently [58, 76]. Its relations to the *critical group* of
a digraph form a strong connection to algebraic graph theory. Now, also this concept may
be generalized to generalized CFGs and we would like to have generalizations of the known
results for the sandpile group in this broader setting.

A last question relates to famous *Frankl's Conjecture* also known as *union-closed sets
conjecture*. It states that every lattice $\mathcal{L}$ contains a join-irreducible j, such that $|{\uparrow}j| \leq |\mathcal{L}|/2$.
The maximal lattice class for which the conjecture is known to be true is the class of *lower
semi-modular lattices* [95] – a class containing distributive, but not upper locally distributive
lattices. We feel that ULDs are the next class to tackle. In particular it would be a challenge
to prove Frankl's Conjecture for chip-firing games.

Similar Results for Other Classes of Lattices

We have characterized ULDs in terms of antichain-partitioned posets. In Theorem 1.2.3
we show that every finite lattice corresponds to an antichain-covered poset. In view of duality

we ask for a characterization of dual ACPs of good antichain-partitioned posets. This would yield a new characterization of ULDs.

Also, what other lattice classes have nice characterizations in terms of their antichain-covered posets? We have characterized cover-preserving join-sublattice embeddings of ULDs into $\mathbb{N}^d$ in terms of chain-partitions of the poset of meet-irreducibles in Theorem 1.3.18. Can this generalization of Dilworth's Embedding Theorem be taken further to other lattice classes? The proof of Theorem 1.5.10 relies on the representation of ULDs as antichain-partitoned posets. In the light of Theorem 1.2.3 it would be an interesting question whether every lattice $\mathcal{L}$ can be represented as a certain type of vector-addition language. Natural-seeming candidates for such generalizations would on the one hand be upper semi-modular lattices and other hand lattices which satisfy the colored Jordan-Dedekind chain condition.

Chapter 2

Polyhedra

In the previous chapter we dealt with different types of lattices and their representations, e.g. colored Hasse diagrams, (multi)set-systems, antichain-covered posets, vector-addition-languages, chip-firing games and embeddings into $\mathbb{N}^d$. In the present chapter we will develop geometric representations of lattices combined with Euclidean convexity. Or turned the other way around we will look at polyhedra in $\mathbb{R}^d$ combined with the dominance order. Looking back, one result of this chapter is that all the (upper locally) distributive lattices arising in the previous chapter may be seen as integral points or even vertex sets of polyhedra which form (join-)sublattices of $\mathbb{R}^d$. Therefore they carry an (upper locally) distributive lattice structure in a natural way. Indeed, this was the starting point for the the study of polyhedra having order-theoretical properties as subsets of the dominance order.

So the classes of polyhedra we look at form (upper locally) distributive lattices in $\mathbb{R}^d$. Thus, they are called *(upper locally) distributive polyhedra*. We will provide characterizations of upper locally distributive polyhedra (ULD-polyhedra) and distributive polyhedra (D-polyhedra) in terms of their representation as intersection of bounded halfspaces ($\mathcal{H}$-*description*). Figure 2.1 suggests how such polyhedra might look like.
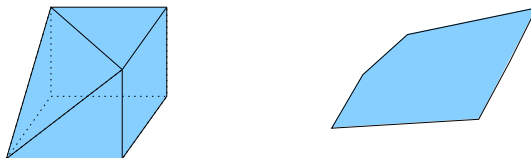
$\mathcal{H}$-description

Figure 2.1: A 3-dimensional ULD-polytope and a 2-dimensional D-polytope.

Generally, the polyhedral point of view allows links to discrete geometry such as linear programming or the theory of face lattices of polytopes. Our characterization of ULD-polyhedra in terms $\mathcal{H}$-descriptions yields a combinatorial model for these polyhedra in terms of chip-firing games. Moreover, we obtain a connection to *feasible polytopes of*

antimatroids. A corollary of our characterization is a short new proof for a characterization of distributive polyhedra.

Aside from being an interesting combination of geometric and order theoretic concepts, distributive polyhedra are a unifying generalization of several distributive lattices which arise from graphs. In fact a distributive polyhedron corresponds to a directed graph with arc-parameters, such that every point in the polyhedron corresponds to a vertex potential on the graph. Alternatively, an edge-based description of the point set can be given. The objects in this model are dual to generalized flows, i.e., dual to flows with gains and losses. Moreover we obtain a connection to oriented bicircular matroids – a class of graph-related matroids of recent interest in combinatorics.

A particular specialization are tensions of digraphs, discussed in Section 1.4. These models can be specialized to yield some cases of distributive lattices that have been studied previously. The contribution here is, that they additionally may be seen as the integer points of a distributive polyhedron.

As another new application of the theory of D-polyhedra we exhibit a distributive lattice structure on generalized flows of *breakeven* planar digraphs.

So this chapter is about polytopes and polyhedra. It relates to the second parts of the paper [69] and presents the content of [43]. It is structured as follows:

In Section 2.1 we introduce those notions of order which we want to combine with convexity. We discuss polyhedra which are join-closed with respect to the dominance order and polyhedra that have meets for every pair of elements. Combining both notions we define upper locally distributive polyhedra and distributive polyhedra. We will not provide a real introduction into the theory of polyhedra, but define new terms "on the fly" whenever we need them. For the basics we refer to [110].

In Section 2.2 we study distributivity and upper local distributivity for affine spaces in $\mathbb{R}^d$. We find that both classes coincide and provide a full characterization. We associate graph-model to distributive affine spaces and characterize their bases. This characterization is a main ingredient for the characterizations of ULD-polyhedra and D-polyhedra.

In Section 2.3 we give a characterization of upper locally distributive polyhedra in terms of their $\mathcal{H}$-description. As main ingredients we characterize polyhedra which are closed under taking the componentwise maximum and polyhedra which have lower bounds for all pairs of points. Moreover, it is shown that ULD-polyhedra can be modelled by chip-firing games. Based on the $\mathcal{H}$-description of ULD-polyhedra we also contribute new insights to a membership problem for feasible polytopes of antimatroids discussed by Korte and Lovász [71].

In Section 2.4 we discuss the important subclass of D-polyhedra and prove a characterization of D-polyhedra in terms of their $\mathcal{H}$-description. This is a corollary of the main the characterization of ULD-polyhedra.

The characterization of D-polyhedra leads to a wide range of combinatorial interpretations. In Subsection 2.4.1 we use the geometric characterization of D-polyhedra to give a combinatorial description in terms of vertex-potentials of arc-parameterized digraphs. Moreover, we provide a family of objects in the arc-space of an arc-parameterized digraph – called *generalized tensions*. They correspond to the points of a distributive polyhedron. Hence they carry a distributive lattice structure and turn out to be the most general distributive lattice obtainable by the "potential approach".

In Subsection 2.4.2 we consider the special case of distributive polyhedra coming from ordinary digraphs (without arc-parameters) as an example which is of fundamental importance. We prove that in this case even the *integral* generalized tensions carry a distributive lattice structure. These integral generalized tensions correspond to the Δ-tensions of a directed graph. Hence we endow those with a polyhedral structure. As was shown in Section 1.4, the distributive lattice on Δ-tensions generalizes an extensive list of distributive lattices related to graphs. Our results imply that these objects correspond to the integral points of integral distributive polyhedra. In particular we obtain that known classes of polytopes, e.g. *order-polytopes* [102] and more generally *polytropes* [63], also called *alcoved polytopes* [74], are distributive and may be modeled by Δ-tensions.

In Subsection 2.4.3 we consider the case of general arc-parameterized digraphs. We give a combinatorial description of the generalized tensions of a parameterized digraph. We show that they are dual to generalized flows – important objects of combinatorial optimization. Moreover, our theory opens a new perspective on *bicircular oriented matroids*. Our main theorem may be seen as a characterization of arc-space objects which carry a distributive lattice structure coming from a D-polyhedron.

Subsection 2.4.4 contains a new application of the theory. We prove a distributive lattice structure on the class of *breakeven* generalized flows of planar digraphs. This can be understood as a generalization of the distributive lattice on integral planar flow obtained in Subsection 1.4.3.

Section 2.5 concludes with final remarks and open problems.

2.1 Polyhedra and Poset Properties

In this section we will introduce those poset properties which, viewed in the dominance order, will be combined with convexity in order to characterize ULD-polyhedra and D-polyhedra. Besides proving the first few lemmas and provide a couple of observations, the idea is to give a first impression of how the combination of order and geometry feels like. As a kind of "preliminaries section" some things might seem unmotivated. The main idea here is to carry over properties and intuitions from the purely combinatorial setting in the first chapter to Euclidean space. We will from now on always regard $\mathbb{R}^d$ together with the dominance order on it, i.e., for $x, y \in \mathbb{R}^d$ we have $x \leq y \iff x_i \leq y_i$ for all $i \in [d]$. One of the most important definitions for this chapter is the following:

ULD-polyhedron

Definition 2.1.1. We define a *ULD-polyhedron* as a polyhedron $\mathsf{P} \subseteq \mathbb{R}^d$ such that for all $x, y \in \mathsf{P}$

U-polyhedron

    1. the componentwise maximum $\max(x, y)$ is in P,   (*U-polyhedron*)

meet-polyhedron

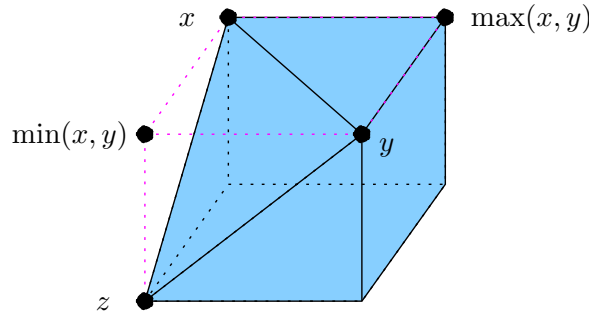   2. there is some $z \in \mathsf{P}$ with $z \leq x, y$.   (*meet-polyhedron*)

Figure 2.2: A 3-dimensional ULD-polytope.

L-polyhedron

    A polyhedron which is closed under $\min$ is called an *L-polyhedron*. Since for every x, y in an L-polyhedron P also $\min(x, y) \in \mathsf{P}$ and $\min(x, y) \leq x, y$ L-polyhedra are meet-polyhedra.

Observation 2.1.2. The property of being a U-polyhedron is invariant under *scaling*, *translation*, *cartesian product* and *intersection*. The same holds for meet-polyhedra with the exception of *intersection*, see Remark 2.3.10 for an example.

Observation 2.1.3. An important observation about the interplay of order and geometry is that the set of elements below x, i.e., $\downarrow x := \{y \in \mathbb{R}^d \mid y \leq x\}$ and dually $\uparrow x$ are convex

apex

cone

polyhedral cones in $\mathbb{R}^d$. Their *apex* is x and they are *generated* by the vectors $-e_1, \ldots, -e_d$ and $e_1, \ldots, e_d$, respectively. Generally the *cone* with apex x generated by a finite set of vectors V is

$$\mathrm{cone}(V, x) := x + \{\sum_{y \in V} \lambda_y y \mid \lambda_y \geq 0\}.$$

This already serves to prove the following basic analogue to the fact that a finite join-semilattice has a unique maximum, see Observation 1.1.1.

Lemma 2.1.4. *Let* P *be a U-polyhedron and* $x \leq y$ *for all* $x \in$ P *and some* $y \in \mathbb{R}^d$. *Then* P *has a unique vertex* $\mathbf{1}_\mathsf{P}$, *such that* $x \leq \mathbf{1}_\mathsf{P}$ *for all* $x \in$ P.

Proof. Denote by $\downarrow y$ the cone $\{x \in \mathbb{R}^d \mid x \leq y\}$. Translate y such that every bounding hyperplane of $\downarrow y$ touches P, i.e., for all $i \in [d]$ there is $x \in$ P such that $x_i = y_i$. The max of all those x is y and it is in P. Thus $y = \mathbf{1}_\mathsf{P} \in$ P. Since $\mathbf{1}_\mathsf{P} \in$ P is the apex of the cone $\downarrow\mathbf{1}_\mathsf{P}$ which contains P, $\mathbf{1}_\mathsf{P}$ is a vertex. □

So one page ago we defined ULD-polyhedra. We want that the point set of a ULD-polyhedron forms a ULD with respect to the dominance order on $\mathbb{R}^d$. The first problem here is that ULDs were defined only on *finite* ground sets (Definition 1.1.6). The figure suggests that in particular for unbounded polyhedra a direct translation of Definition 1.1.6 is difficult. We need to find for every element ℓ a set of meet-irreducibles M_ℓ representing it as its meet, but there is no such element in x_2-direction. On the other hand the example in the figure even is a D-polyhedron (see Definition 2.1.8). Thus it should count as a ULD-polyhedron. This problem can be overcome if we endow the dominance order with points at infinity, i.e., we have to look at $(\mathbb{R} \cup \{\infty\})^d$ and again we take the componentwise ordering where $\mathbb{R} \cup \{\infty\}$ is ordered as usual but endowed with a global maximum ∞. This allows to interpret unboundedness in a way which is compatible with the idea of ULDs. In our example the meet-irreducible representation of ℓ would then be $\{m, (\ell_1, \infty)\}$.
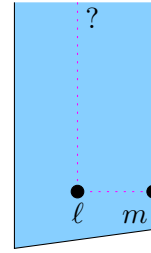
Therefore make the following definition. Let $S \subseteq \mathbb{R}^d$ and $x \in S$. We call $I \subseteq [d]$ a *set of unbounded directions* of x if for every $v \in \mathbb{R}^I$ there is a $w \geq v$ such that $x + w \in S$ and denote by $\mathcal{I}(x)$ their collection. Define

$$\phi(x) := \{y \in (\mathbb{R} \cup \{\infty\})^d \mid y_i = \infty \text{ if } i \in I \text{ and } y_i = x_i \text{ otherwise, for some } I \in \mathcal{I}(x)\}.$$

One can then show the following

Theorem 2.1.5. *A set* $S \subseteq \mathbb{R}^d$ *is a join-sublattice of the dominance order if and only if* $\phi(S)$ *forms a ULD with respect to the dominance order on* $(\mathbb{R} \cup \{\infty\})^d$, *i.e., for every* $y \in \phi(S)$ *there is a unique inclusion-minimal set of meet-irreducible* $M_y \subseteq (\mathbb{R} \cup \{\infty\})^d$ *such that* $y = \bigwedge M_y$.

Note that this makes sense in view of the finite case, where a correspondence between ULDs and join-sublattices of $\mathbb{N}^d$ was established (Theorem 1.3.3). Since here we will deal with polyhedra and points at infinity is something we want to avoid we take the above theorem as a definition, i.e.:

Definition 2.1.6. A subset $S \subseteq \mathbb{R}^d$ is called *Euclidean ULD* if S forms a join-sublattice of
the dominance order.

Now we prove a proposition justifying the name ULD-polyhedra. It can be understood as a Euclidean generalization of the fact that a finite join-semilattice which has meets for all pairs of elements is a lattice (Observation 1.1.4).

Proposition 2.1.7. *A polyhedron* P *is a ULD-polyhedron if and only if* P *endowed with the dominance order forms a join-sublattice of* $\mathbb{R}^d$. *This is, ULD-polyhedra are Euclidean ULDs.*

Proof. By Definition 2.1.6 the "$\Longleftarrow$"-direction is trivial. Let us prove "$\Longrightarrow$":

A U-polytope P forms a join-subsemilattice of the dominance order on $\mathbb{R}^d$. The property of being a meet-polyhedron, i.e., for all $x, y \in$ P there is some $z \in$ P with $z \le x, y$, means that every pair of elements has a meet. In order to show that both properties together imply that P forms a lattice it remains to prove that every pair $x, y \in$ P has a unique meet. The set $Z := \{z \in$ P $\mid z \le x, y\}$ equals $\downarrow x \cap \downarrow y \cap$ P, where all the three are U-polyhedra. Hence by Observation 2.1.2 also Z is a U-polyhedron. Now since $\min(x, y) \ge z$ for $z \in Z$ Lemma 2.1.4 tells us that Z has a unique maximum $\mathbf{1}_Z$ – the meet of x and y. Hence P is a join-sublattice of $\mathbb{R}^d$. $\qquad\square$

As in the case of ordinary ULDs and distributive lattices, a very important subclass of ULD-polyhedra with plenty of nice combinatorial interpretations is the following:

Definition 2.1.8. A polyhedron P $\subseteq \mathbb{R}^d$ is called *distributive* if it is a U-polyhedron and an L-polyhedron. Distributive polyhedra are abbreviated D-polyhedra.

In other words, a polyhedron P is distributive if and only if

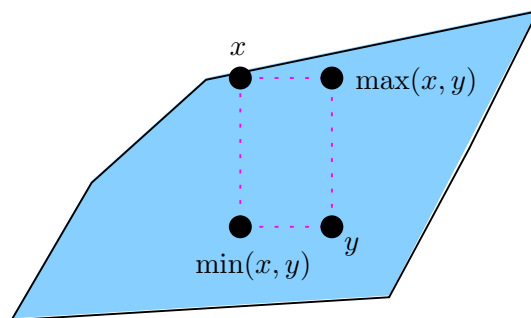$$x, y \in \mathsf{P} \Longrightarrow \min(x, y), \max(x, y) \in \mathsf{P}.$$

Figure 2.3: A 2-dimensional D-polytope.

Since L-polyhedra are meet-polyhedra D-polyhedra are indeed ULD-polyhedra.

Remark 2.1.9. The dominance order is a distributive lattice on $\mathbb{R}^d$. Join and meet in the lattice are given by the componentwise $\max$ and $\min$. Sublattices of distributive lattices are distributive. Since D-polyhedra are exactly those polyhedra, which form sublattices of $\mathbb{R}^d$, they are distributive lattices. This justifies the name *distributive polyhedra*.

Remark 2.1.10. By Birkhoff's Fundamental Theorem of Finite Distributive Lattices [14] every *finite* distributive lattice is isomorphic to a union- and intersection-closed set of finite sets. The characteristic vectors of these sets form the vertices of a distributive polytope – the *order polytope*, see [102] or the figure in the introduction of the thesis for an example. We will explain in Section 2.4.2 that the order polytope is indeed a D-polytope. In this sense, every finite distributive lattice may be represented as the vertex set of an integral distributive polyhedron.

An analogue statement for ULDs is not known, but as we will see in Subsection 2.3.1 every finite ULD may be represented as the set of integer-points of a (not necessarily integral) ULD-polytope.

2.2   Affine Space

In this section we will characterize distributive and upper locally distributive affine space. The proof will take the whole section. Indeed, one of the first things to note will be that both properties are equivalent for affine spaces. The characterization will be an important ingredient for the characterization of distributive and upper locally distributive polyhedra. We will see the first link to the theory of arc-parameterized digraphs and an interesting description of distributive space in terms of basis.

An *affine space* $\mathsf{S} \subseteq \mathbb{R}^n$ is the translation of a linear space S' by some vector x, i.e.,   affine space $\mathsf{S} := \{y \in \mathbb{R}^n \mid y = s + x \text{ for some } s \in \mathsf{S}'\}$.

Remark 2.2.1. Since by Observation 2.1.2 the class of U-polyhedra is closed under translation, we actually will only consider linear spaces in this section. At the end we resume the results of this section in terms of general affine spaces, see Theorem 2.2.11.

As announced, the first easy and basic result of this section is

Proposition 2.2.2. *A linear space is a U-polyhedron if and only if it is an L-polyhedron if and only if it is a D-polyhedron.*

Proof. Since $\min(x, y) = x + y - \max(x, y)$ the result follows.   □

So in the following we will characterize linear distributive space. We display this characterization already, even if several terms in the statement will be defined only in the course of the proof.

Theorem 2.2.3. *For a linear subspace $\mathsf{S} \subseteq \mathbb{R}^n$ the following are equivalent:*

(i) S *is distributive.*

(ii) S *has a non-negative disjoint basis B.*

(iii) S $= \{p \in \mathbb{R}^n \mid N_\Lambda^T p = \mathbf{0}\}$, *where N_Λ is the generalized network-matrix of an arc-parameterized digraph D_Λ.*

The structure of the proof is to show "(i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i)". So we start by showing the "(i)$\Longrightarrow$(ii)"-part of Theorem 2.2.3. Therefore we need to define NND basis. For a vector $x \in \mathbb{R}^n$ we call $\underline{x} := \{i \in [n] \mid x_i \neq 0\}$ the *support* of x. Set $p(x) := \max(\mathbf{0}, x)$ and $n(x) := -\min(\mathbf{0}, x)$. Call a set of vectors $B \subseteq \mathbb{R}^n$ *non-negative disjoint (NND)* if the elements of B are componentwise non-negative and have pairwise disjoint supports. Note that an NND set of non-zero vectors is linearly independent. Moreover, we have the following useful extension-property.

Lemma 2.2.4. *Let $I \cup \{x\} \subset \mathbb{R}^n$ be linearly independent, then $I \cup \{p(x)\}$ or $I \cup \{n(x)\}$ is linearly independent.*

Proof. Suppose there are linear combinations $p(x) = \sum_{b \in I} \mu_b b$ and $n(x) = \sum_{b \in I} \nu_b b$, then $x = \sum_{b \in I} (\mu_b - \nu_b) x_b$, which proves that $I \cup \{x\}$ is linearly dependent – a contradiction. $\square$

And indeed:

Proposition 2.2.5. *Every linear distributive* S $\subseteq \mathbb{R}^n$ *has a non-negative disjoint basis B.*

Proof. Let S be distributive and $I \subset$ S an NND set of support-minimal non-zero vectors. If I is not a basis of S, then there is $x \in$ S such that:

(1) $I \cup \{x\}$ is linearly independent,

(2) $\exists_{i \in [n]} : x_i > 0$,

(3) $\underline{x}$ is minimal among the vectors with (1) and (2).

Claim: $I \cup \{x\}$ is NND.

If x is not non-negative, then $p(x)$ and $n(x)$ are non-negative. Also since S is distributive both are contained in S and have smaller support than x. By Lemma 2.2.4 one of $I \cup \{p(x)\}$ and $I \cup \{n(x)\}$ is linearly independent – a contradiction to the support-minimality of x.

If there is $b \in I$ such that $\underline{x} \cap \underline{b} \neq \emptyset$, then choose $\mu \in \mathbb{R}$ such that for some coordinate $j \in \underline{x} \cap \underline{b}$ we have $x_j = \mu b_j$. We distinguish two cases.

If $\underline{x} \subseteq \underline{b}$, then $\emptyset \neq \underline{\mu b - x} \subsetneq \underline{b}$ contradicts the support-minimality in the choice of $b \in I$.

If $\underline{x} \not\subseteq \underline{b}$, then since $I \cup \{\mu b - x\}$ is linearly independent one of $I \cup \{p(\mu b - x)\}$ and $I \cup \{n(\mu b - x)\}$ is linearly independent by Lemma 2.2.4. By the choice of μ we have $\underline{p(\mu b - x)} \subsetneq \underline{b}$ and $\underline{n(\mu b - x)} \subsetneq \underline{x}$ and obtain a contradiction to the support-minimality in the choice of b or x, respectively. $\square$

We have proven the part "(i)$\Longrightarrow$(ii)" of Theorem 2.2.3. But the NND basis of a distributive space even basically unique:

Proposition 2.2.6. *An NND basis is unique up to scaling.*

Proof. Suppose $S \subseteq \mathbb{R}^n$ has NND bases B and B'. Suppose there are $b \in B$ and $b' \in B'$ such that $\emptyset \neq \underline{b} \cap \underline{b'} \neq \underline{b'}, \underline{b}$. By Proposition 2.2.5 we have $\min(b, b') \in S$ but $\underline{\min(b, b')}$ is strictly contained in the supports of b and b'. Since B and B' are NND the vector $\overline{\min(b, b')}$ can neither be linearly combined by B nor by B'.

Suppose that there are $b \in B$ and $b' \in B'$ such that $\underline{b} \subsetneq \underline{b'}$. By the above we then know, that $\underline{b'}$ is a disjoint union of supports of several vectors in B. But this contradicts that B and B' generate the same space.

By the same reason we have $\bigcup_{b \in B} \underline{b} = \bigcup_{b' \in B'} \underline{b'}$. Together we know that the supports of vectors in B and B' induce the same partition of that set. Since B and B' are NND, the vectors $b \in B$ and $b' \in B'$ with $\underline{b} = \underline{b'}$ must be scalar multiples of each other. $\qquad\square$

The next step is to prove part "(ii)$\Longrightarrow$(iii)" of Theorem 2.2.3. We want to define a class of network matrices of arc-parameterized digraphs such that for every linear space S which has a NND basis there is a network matrix N_Λ in the class such that $S = \{p \in \mathbb{R}^n \mid N_\Lambda^T p = \mathbf{0}\}$.

An *arc-parameterized digraph* is a triple $D_\Lambda = (V, A, \Lambda)$, where $D = (V, A)$ is a directed multi-graph, i.e., D may have loops, parallel, and anti-parallel arcs. We call D the *underlying digraph* of D_Λ. Moreover, for convenience we set $V = [n]$ and $|A| = m$. Now Λ is a non-negative vector in $\mathbb{R}^m_{\geq \mathbf{0}}$ with and entry λ_a for every $a \in A$. It has the property that $\lambda_a = 0$ implies that a is a loop. For emphasis we repeat: All arc-parameters λ_a are non-negative. *[arc-parameterized digraph]* *[underlying digraph]*

Given an arc-parameterized digraph D_Λ we define its *generalized network-matrix* to be the matrix $N_\Lambda \in \mathbb{R}^{n \times m}$ with a column $z_a := e_j - \lambda_a e_i$ for every arc $a = (i, j)$ with parameter λ_a. Here e_k denotes the kth *unit-vector* in $\mathbb{R}^n$, i.e, e_k has a 1 in the kth entry and is 0 elsewhere. Note that if a is a loop, then this produces a column z_a with at most one non-zero entry, which can be negative or positive depending on λ_a. The column z_a has only zero-entries if and only if a is a loop and $\lambda_a = 1$. *[generalized network-matrix]* *[unit-vector]*

Observation 2.2.7. A matrix is a generalized network matrix if and only if each column contains at most one positive entry and at most one negative entry. Positive entries sharing a column with a negative entry are 1.

Proposition 2.2.8. *Let $S \subseteq \mathbb{R}^n$ be a linear subspace which has a NND basis B. There is a generalized network-matrix N_Λ such that $S = \{p \in \mathbb{R}^n \mid N_\Lambda^T p = \mathbf{0}\}$. Moreover, N_Λ can be chosen, such that the underlying digraph D of D_Λ is a union of a forest and loops with arc-parameter 0 at isolated vertices.*

Proof. We construct an arc-parameterized digraph D_Λ, such that the columns of its generalized network-matrix N_Λ form a basis of the orthogonal complement of S (with respect to the standard scalar product).

For every $b \in B$ choose some directed spanning tree on $\underline{b}$. For every $i \notin \bigcup_{b \in B} \underline{b}$ insert a loop $a = (i, i)$. To an arc $a = (i, j)$ with $i, j \in \underline{b}$ we associate the arc parameter $\lambda_a := b_j / b_i > 0$. For loops we set $\lambda_a := 0$. Collect the λ_a of all the arcs in a vector $\Lambda \in \mathbb{R}^m_{\geq 0}$. The resulting arc-parameterized digraph D_Λ is a disjoint union of loops and a forest – as claimed in the proposition's statement.

Denote by $\mathrm{col}(N_\Lambda)$ the set of column-vectors of N_Λ. If $b \in B$ and $z_a \in \mathrm{col}(N_\Lambda)$, then either $\underline{b} \cap \underline{z_a} = \emptyset$ or $\langle b, z_a \rangle = b_j - \lambda_a b_i = b_j - (b_j/b_i) b_i = 0$ for $a = (i, j)$. Therefore, $\mathrm{col}(N_\Lambda)$ is orthogonal to S. The underlying digraph of D_Λ consists of trees and loops only, and $\lambda_a \neq 1$ for loops a. Thus, $\mathrm{col}(N_\Lambda)$ is linearly independent. To conclude that $\mathrm{col}(N_\Lambda)$ generates $\mathsf{S}^\perp$ in $\mathbb{R}^n$ we calculate:

$$
\begin{aligned}
& |B| + |\mathrm{col}(N_\lambda)| \\
= \quad & |B| + \sum_{b \in B}(|\underline{b}| - 1) + |[n] \setminus \bigcup_{b \in B} \underline{b}| \\
= \quad & \sum_{b \in B} |\underline{b}| + n - |\bigcup_{b \in B} \underline{b}|.
\end{aligned}
$$

Since the supports in B are mutually disjoint this equals n. Thus the dimension of the span of $\mathrm{col}(N_\Lambda)$ is $n - |B|$, the dimension of $\mathsf{S}^\perp$. Since $\mathrm{col}(N_\Lambda) \subseteq \mathsf{S}^\perp$ both spaces coincide. $\qquad\square$

For the proof of Theorem 2.2.3 it remains to show "(iii)$\Longrightarrow$(i)".

Proposition 2.2.9. *Let N_Λ be a generalized network-matrix. The linear space $\mathsf{S} = \{p \in \mathbb{R}^n \mid N_\Lambda^T p = \mathbf{0}\}$ is distributive.*

hyperplane

Proof. Note that $\{p \in \mathbb{R}^n \mid N_\Lambda^T p = \mathbf{0}\}$ is the intersection of *hyperplanes* $\mathsf{H}_z := \{x \in \mathbb{R}^n \mid \langle z, x \rangle = 0\}$, where the $z \in \mathrm{col}(N_\Lambda)$.

We want to prove that H_z is distributive. By Proposition 2.2.2 it is enough to show, that H_z is max-closed. By Observation 2.2.7 z has at most one positive entry z_j and one negative entry z_i. Let $x, y \in \mathsf{H}_z$. If $x_j \leq y_j$ and $x_i \leq y_i$ or $x_j \geq y_j$ and $x_i \geq y_i$, then clearly $\max(x, y) \in \mathsf{H}_z$.

So say $x_j < y_j$ and $x_i > y_i$. We have $0 = x_j z_j + x_i z_i \geq y_j z_j + x_i z_i \geq y_j z_j + y_i z_i = 0$. But $y_j z_j + x_i z_i = \langle z, \max(x, y) \rangle$, i.e., $\max(x, y) \in \mathsf{H}_z$. The case $x_j > y_j$ and $x_i < y_i$ is symmetric.

Thus, S is the intersection of distributive spaces and is itself distributive by Observation 2.1.2. $\qquad\square$

We have proved Theorem 2.2.3. Note that together with Proposition 2.2.8 we actually showed that the generalized network matrix representing a distributive linear space, may be assumed to come from a union of a forest and loops. We restate:

Theorem 2.2.10. *Every linear distributive space* $S \subseteq \mathbb{R}^n$ *may be represented as* $\{p \in \mathbb{R}^n \mid N_\Lambda^T p = \mathbf{0}\}$, *where the underlying digraph* D *of* D_Λ *is the union of a forest and loops at isolated vertices with arc-parameter* 0.

As remarked at the beginning of the section all the involved properties in the above results are invariant under translations. In particular we define a basis of an affine space as a basis of the linear space obtained by translation onto the origin. We resume the results of the section in terms of affine spaces.

Theorem 2.2.11. *For an affine subspace* $S \subseteq \mathbb{R}^n$ *the following are equivalent:*

(i) S *is distributive.*

(ii) S *has a non-negative disjoint basis* B.

(iii) $S = \{p \in \mathbb{R}^n \mid N_\Lambda^T p = c\}$, *where* N_Λ *is the generalized network-matrix of an arc-parameterized digraph* D_Λ. *Moreover, the underlying digraph* D *may be assumed to be the union of a forest and loops with arc parameter* 0 *at isolated vertices.*

We close this section with a lemma, which will prove usefull in the upcoming sections. We show how the representation by equalities in Theorem 2.2.11 may be replaced by an inequality-description, while maintaining a generalized network matrix.

Lemma 2.2.12. *A distributive affine space* $S \subseteq \mathbb{R}^n$ *may be represented as* $\{p \in \mathbb{R}^n \mid N_\Lambda^T p \leq c\}$ *and* $\{p \in \mathbb{R}^n \mid \widetilde{N}_{\widetilde{\Lambda}}^T p \geq \widetilde{c}\}$, *where* N_Λ *and* $\widetilde{N}_{\widetilde{\Lambda}}$ *are generalized network-matrices.*

Proof. It is standard to replace a description by linear equalities by inequalities. In order to obtain again a generalized network matrix, we have to scale such that all positive entries equal 1, by Observation 2.2.7. More precisely:

Let $S = \{p \in \mathbb{R}^n \mid N_{\Lambda'}'^T p = c'\}$ be the representation of S guaranteed by Theorem 2.2.11. We scale the rows of negative copies $-N_{\Lambda'}'^T$ and $-c'$ such that all positive entries of $-N_{\Lambda'}'^T$ become 1. Denote the new generalized network matrix as $N_{\Lambda''}''$ and the new capacity-vector as c''. We obtain a generalized network matrix $N_\Lambda = (N_{\Lambda'}', N_{\Lambda''}'')$ and a vector $c = (c'^T, c''^T)^T$ such that $S = \{p \in \mathbb{R}^n \mid N_\Lambda^T p \leq c\}$.

To obtain a description as $\{p \in \mathbb{R}^n \mid \widetilde{N}_{\widetilde{\Lambda}}^T p \geq \widetilde{c}\}$, just scale every column m_a of N_Λ and the corresponding entry c_a of c by a negative number μ such that the positive entry of μm_a is 1. $\qquad\square$

2.3 Upper Locally Distributive Polyhedra

In the following we will characterize U-polyhedra and meet-polyhedra. By Definition 2.1.1 we can then combine both characterizations to obtain a characterization of ULD-polyhedra.

We start with U-polyhedra not yet ULD-polyhedra, c.f. Definition 2.1.1. We will use the Representation Theorem for Polyhedra [110], which says, that every polyhedron is representable as the intersection of its affine hull with all facet-defining halfspaces. Hence the following lemmas will describe properties of the single ingredients of the Representation Theorem for Polyhedra in the case of U-polyhedra.

affine hull

Define the *affine hull* of a point set $S \subseteq \mathbb{R}^d$ as the minimal affine space containing S. The affine hull may as well be defined as:

$$\mathrm{aff}(S) := \{\sum_{p \in S'} \lambda_p p \mid S' \subseteq S \text{ is finite and } \sum_{p \in S'} \lambda_p = 1\}.$$

Lemma 2.3.1. *The affine hull* $\mathrm{aff}(\mathsf{P})$ *of a U-polyhedron* P *is a U-polyhedron.*

Proof. Let $x, y \in \mathrm{aff}(\mathsf{P})$. Scale P to P' such that $x, y \in \mathsf{P}' \subseteq \mathrm{aff}(\mathsf{P})$. Since by Observation 2.1.2 scaling preserves max-closedness we have $\max(x, y) \in \mathsf{P}' \subseteq \mathrm{aff}(\mathsf{P})$. $\square$

Lemma 2.3.2. *The orthogonal projection* $\mathsf{P}' \subseteq \mathbb{R}^I$ *of a U-polyhedron* $\mathsf{P} \subseteq \mathbb{R}^d$ *to a subset of* $I \subseteq [d]$ *of coordinates is a U-polyhedron.*

Proof. Let $x', y' \in \mathsf{P}'$ be projections of points $x, y \in \mathsf{P}$ with $x_i = x_i'$ and $y_i = y_i'$ for all $i \in I$. Now $\max(x, y) \in \mathsf{P}$ and its projection equals $\max(x', y')$. $\square$

affine hyperplane
induced halfspace

Given an *affine hyperplane* $\mathsf{H} = \{x \in \mathbb{R}^d \mid \langle x, z \rangle = c\}$ we denote by $\mathsf{H}^{\geq} = \{x \in \mathbb{R}^d \mid \langle x, z \rangle \geq c\}$ and $\mathsf{H}^{\leq} = \{x \in \mathbb{R}^d \mid \langle x, z \rangle \leq c\}$ the *halfspaces induced* by H.

Lemma 2.3.3. *A halfspace* $\mathsf{H}^{\geq} = \{x \in \mathbb{R}^d \mid \langle z, x \rangle \geq c\}$ *is a U-polyhedron if and only if* z *has a most one negative entry. Moreover,* $\mathsf{H}^{\geq}$ *is* max*-closed if and only if for all* x, y *on the hyperplane* H *we have* $\max(x, y) \in \mathsf{H}^{\geq}$.

Proof. By translation invariance we may assume $c := 0$.

"$\Longleftarrow$": Let z be a vector with unique negative entry z_i and let $x, y \in \mathsf{H}^{\geq}$. If $x_i \leq y_i$, then $\sum z_j \max(x_j, y_j) \geq \sum z_j y_j \geq 0$. Hence $\max(x, y) \in \mathsf{H}^{\geq}$. If $z \geq \mathbf{0}$, then the statement follows directly.

"$\Longrightarrow$": Suppose on the other hand there are two entries $z_i, z_j < 0$. Then $x := e_i/z_i - e_j/z_j$ and $y := e_j/z_j - e_i/z_i$ are in $\mathsf{H}^{\geq}$. But $\max(x, y) = -e_i/x_i - e_j/x_j$ is certainly not in $\mathsf{H}^{\geq}$. Hence $\mathsf{H}^{\geq}$ is not a U-polyhedron.

To see the last part of the statement, suppose there were $x, y \in \mathsf{H}^{\geq}$ and $\max(x, y) \notin \mathsf{H}^{\geq}$. Consider the line segments $[x, \max(x, y)]$ and $[y, \max(x, y)]$. For every $x' \in [x, \max(x, y)]$ and $y' \in [y, \max(x, y)]$ we have $\max(x', y') = \max(x, y)$. Both line segments intersect H, i.e., in particular there are $x', y' \in H$ with $\max(x', y') \notin \mathsf{H}^{\geq}$. $\square$

For a polyhedron P we define $F \subseteq P$ to be a *face* if there is a hyperplane H_F such that P is contained in the induced halfspace $H_F^{\geq}$ and $F = P \cap H_F$. The halfspace $H_F^{\geq}$ is called *face-defining*. In particular, a face of a polyhedron is a polyhedron.

The *dimension* of a polyhedron P is defined as the dimension of $\mathrm{aff}(P)$. A face F of P is called *facet* if F has dimension one less than P. A polyhedron $P \subseteq \mathbb{R}^d$ is called *full-dimensional* if $\mathrm{aff}(P) = \mathbb{R}^d$. Facets of full-dimensional polyhedra have unique facet-defining halfspaces.

Lemma 2.3.4. *Let P be a U-polyhedron. Every facet of P has a facet-defining halfspace which is a U-polyhedron.*

Proof. Let $S := \mathrm{aff}(P)$, which by Lemma 2.3.1 is also a U-polyhedron. By Theorem 2.2.11 there is an NND basis $B = \{b_1, \ldots, b_k\}$ for S. Choose coordinates $I = \{i_1, \ldots, i_k\}$ with $i_j \in \underline{b_j}$ for all $j \in [k]$. Every point in S and therefore every point in P is determined by its I-coordinates. So in order to describe P we project P onto its I-coordinates. The new polyhedron $P' \in \mathbb{R}^I$ is a U-polyhedron by Lemma 2.3.2 and it is full-dimensional.

Now let $H'^{\geq}_{F'}$ be the unique facet-defining halfspace of a facet F' of P'. Suppose $H'^{\geq}_{F'}$ is not $\max$-closed, i.e., by Lemma 2.3.3 there are $x, y \in H'_{F'}$ and $\max(x, y) \notin H'^{\geq}_{F'}$. We can scale P' to P'' such that the scaled facet F'' contains x, y. Then $\max(x, y) \notin P''$. Hence by Observation 2.1.2 also P' was not a U-polyhedron – a contradiction. Thus, all the facet-defining halfspaces of P' are U-polyhedra.

Taking the cartesian product of the facet-defining halfspaces $H'^{\geq}_{F'}$ of P' with $\mathbb{R}^{[d]\setminus I}$ one obtains a complete set of facet-defining halfspaces for P. Since $\mathbb{R}^{[d]\setminus I}$ and $H'^{\geq}_{F'}$ are $\max$-closed also their cartesian product is, by Observation 2.1.2. We have obtained a complete set of $\max$-closed facet-defining halfspaces for P. $\square$

Remark 2.3.5. Considering the middle paragraph of the proof of Lemma 2.3.4 for $\min$- and $\max$-closedness simultaneously, one obtains, that every facet of a D-polyhedron has a defining halfspace which is distributive. But we will see that for D-polyhedra we actually have that the faces are D-polyhedra themselves, see Lemma 2.4.1.

We now have all the ingredients to characterize U-polyhedra in terms of their $\mathcal{H}$ description. Analogously to alphabets in Section 1.5 a matrix M is called *Laplacious* if and only if M has at most one negative entry per row.

Theorem 2.3.6. *A polyhedron $P \subseteq \mathbb{R}^d$ is a U-polyhedron if and only if there is Laplacious matrix M such that $P = \{x \in \mathbb{R}^d \mid Mx \geq c\}$, for some c.*

Proof. "$\Longleftarrow$": If M is of the claimed form, then P is the intersection of halfspaces, which are U-polyhedra by Lemma 2.3.3. Since by Observation 2.1.2 intersection preserves the property of being a U-polyhedron, also P is a U-polyhedron.

"$\Longrightarrow$": By the Representation Theorem for Polyhedra [110] we can write

$$P = \big(\bigcap_{F \text{ facet}} H_F^{\geq} \big) \cap \mathrm{aff}(P).$$

By Lemma 2.3.1 also $\mathrm{aff}(\mathsf{P})$ a U-polyhedron. Since $\mathrm{aff}(\mathsf{P})$ is an affine space by Lemma 2.2.12 we may represent it as $\mathrm{aff}(\mathsf{P}) = \{p \in \mathbb{R}^n \mid N(\mathsf{P})_{\Lambda(\mathsf{P})}^T p \geq c(\mathsf{P})\}$, where $N(\mathsf{P})_{\Lambda(\mathsf{P})}$ is a generalized network-matrix.

By Lemma 2.3.4 the facet-defining halfspaces $\mathsf{H}_\mathsf{F}^{\geqq}$ of P may be chosen as U-polyhedra. Hence, by Lemma 2.3.3 each of them may be represented by a (single-row) Laplacious matrix $z(\mathsf{F})$, i.e., $\mathsf{H}_\mathsf{F}^{\geqq} = \{x \in \mathbb{R}^d \mid \langle x, z(\mathsf{F})\rangle \geq c(\mathsf{F})\}$. Putting all of the row-vectors $z(\mathsf{F})$ vertically on $N(\mathsf{P})_{\Lambda(\mathsf{P})}^T$ and the $c(\mathsf{F})$ on $c(\mathsf{P})$, we obtain a description of P of the desired form. $\qquad\square$

Remark 2.3.7. Equivalently one proves, that a polyhedron P is a L-polyhedron if and only if there is matrix M with at most one *positive* entry per row such that $\mathsf{P} = \{x \in \mathbb{R}^d \mid Mx \geq c\}$, for some c. This is, $-M$ is Laplacious.

Remark 2.3.8. Full-dimensional polyhedra have a unique irredundant description as intersection of bounded halfspaces. Thus, if we insist that the inequalities are of the form $\geq$, then any irredundant M describing a full-dimensional U-polyhedron must be Laplacious.

We have characterized U-polyhedra. ULD-polyhedra were defined as being U-polyhedra and meet-polyhedra at the same time. Thus, in order to characterize ULD-polyhedra, we are left with the task to characterize meet-polyhedra. So let $\mathsf{P} \subseteq \mathbb{R}^d$ be a polyhedron. We call a translated coordinate hyperplane $\mathsf{H}_i(c) := \{x \in \mathbb{R}^d \mid \langle e_i, x\rangle = c\}$ a *lower bound* of P if $\mathsf{P} \subseteq \mathsf{H}_i^{\geqq}(c)$ and $\mathsf{P} \cap \mathsf{H}_i(c) \neq \emptyset$. The polyhedron $\mathsf{Q}(\mathsf{P}) := \bigcap_{\mathsf{H} \text{ lower bound of } \mathsf{P}} \mathsf{H} \cap \mathsf{P}$ arising as the intersection of all lower bounds of P with P is called the *min-polyhedron* of P.

lower bound of polyhedron

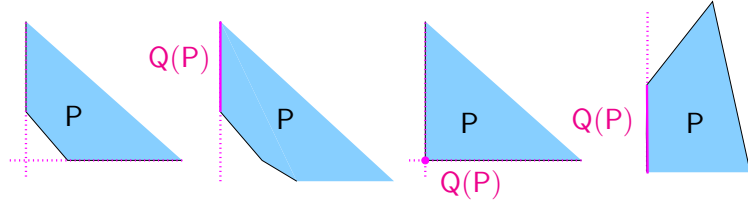min-polyhedron

Figure 2.4: Some polyhedra and their min-polyhedra. The lower bounds are dotted. The polyhedron on the left has an empty min-polyhedron. Only the two right-most polyhedra are meet-polyhedra.

Proposition 2.3.9. *A polyhedron P is a meet-polyhedron if and only if its min-polyhedron $\mathsf{Q}(\mathsf{P})$ is not empty and has the same lower bounds as P. In particular for the bounded case we have that a polytope P is a meet-polytope if and only if it has an element z such that $\mathsf{P} \subset \uparrow z$.*

Proof. "$\Longrightarrow$": Let P be a meet-polyhedron, i.e., for all $x, y \in \mathsf{P}$ there is a $z' \in \mathsf{P}$ with $z' \leq \min(x, y)$. Choose one $x(i) \in \mathsf{H}_i(c)$ from each lower bound of P. Let z' be an element of P below $\min(x(1), \ldots, x(k))$. Clearly, z' is in $\mathsf{Q}(\mathsf{P})$, i.e., the latter is not empty. Let j be such that $\mathsf{H}_j(c)$ is not a lower bound of P for any c, i.e., the j-coordinates of points in P are

not bounded from below. For every $y \in \mathsf{P}$ there is a point $z \leq \min(z', y)$ in P. Since z' is in the intersection of all lower bounds and $z \leq z'$, also z is. Thus $z \in \mathsf{Q}(\mathsf{P})$ and $\mathsf{H}_j(c)$ is not a lower bound for $\mathsf{Q}(\mathsf{P})$ either.

"$\Longleftarrow$": Let $x, y \in \mathsf{P}$. Since $\mathsf{Q}(\mathsf{P}) \neq \emptyset$ there is a $z' \in \mathsf{Q}(\mathsf{P})$, which is a less or equal than x and y on all entries which have a lower bound. Since $\mathsf{Q}(\mathsf{P})$ has not more lower bounds than P, there is a $z \in \mathsf{Q}(\mathsf{P})$ with $z \leq z'$ which is also less or equal x, y on all other entries, i.e., $z \leq x, y$. $\qquad \square$

Remark 2.3.10. As noted in Observation 2.1.2 the class of meet-polyhedra is not closed under intersection. This particularly explains, why we cannot expect a characterization of their $\mathcal{H}$-descriptions only in terms of the describing matrix. The following example, shows that this problem occurs even for U-polytopes. Consider $\mathsf{P}_1 = \{x \in \mathbb{R}^3 \mid \mathbf{0} \leq x \leq \mathbf{1}; x_1 - x_2 + x_3 \geq 0\}$ and $\mathsf{P}_2 = \{x \in \mathbb{R}^3 \mid \mathbf{0} \leq x \leq \mathbf{1}; x_1 - x_2 + x_3 \geq \varepsilon\}$ for some small $\varepsilon > 0$. Both are defined by the same Laplacious matrix, i.e., are U-polytopes by Theorem 2.3.6. Moreover, $\mathbf{0} \in \mathsf{P}_1$ and $\uparrow\mathbf{0} \supset \mathsf{P}_1$. Hence, P_1 is a meet-polytope by Proposition 2.3.9, i.e., it is a ULD-polytope. Indeed, P_1 is the ULD-polytope depicted in Figure 2.2. On the other hand $(\varepsilon, 0, 0), (0, 0, \varepsilon) \in \mathsf{P}_2$ but neither their minimum $\mathbf{0}$ nor any point below it is contained in P_2. Hence P_2 is no meet-polytope. Since both are intersections of up to translation the same meet-closed halfspaces the property of meet-closedness is not intersection-closed.

Plugging Proposition 2.3.9 and Theorem 2.3.6 together we can finally characterize ULD-polyhedra.

Theorem 2.3.11. *A polyhedron* P *is a ULD-polyhedron if and only if*

1. P *is representable as*

$$\{x \in \mathbb{R}^d \mid \begin{pmatrix} I & 0 \\ M_1 & M_2 \end{pmatrix} x \geq \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\},$$

   where I is the identity matrix and (M_1, M_2) is Laplacious.
2. *There is $y < \mathbf{0}$ such that $M_2 y \geq \max(\mathbf{0}, c_2 - M_1 c_1)$.*

Proof. "$\Longleftarrow$": The matrix in the statement clearly is Laplacious, hence P is a U-polyhedron by Theorem 2.3.6. Take $y < \mathbf{0}$ such that $M_2 y \geq \max(\mathbf{0}, c_2 - M_1 c_1)$. Let $(I0)$ be the "upper half" of the matrix in the statement of the theorem. Since $(I0)(c_1, y) = c_1$ and $M_2 y \geq c_2 - M_1 c_1$, the vector (c_1, y) is in P. Since $M_2 y \geq \mathbf{0}$ also $(c_1, \lambda y) \in \mathsf{P}$ for all $\lambda > 1$. Since $y < \mathbf{0}$ and λy gets arbitrary large negative entries the lower bounds of P are exactly $\mathsf{H}_i(c)$ where i is a column-index of M_1 and c the ith entry of c_1. Since $(c_1, \lambda y)$ is in their intersection we have $(c_1, \lambda y) \in \mathsf{Q}(\mathsf{P})$. Since this is true for all $\lambda > 1$ the min-polyhedron $\mathsf{Q}(\mathsf{P})$ does not have more lower bounds than P. By Proposition 2.3.9 P is a meet-polyhedron.

"$\Longrightarrow$": Since P is a U-polyhedron it may be represented by a Laplacious matrix M and a vector c_2. To obtain the special representation claimed in the theorem we partition the

column-indeces of M into C_1, C_2 – those corresponding to lower bounds and those who do not. This is, there is a $c(i)$ such that $H_i(c(i))$ is a lower bound of P if and only if $i \in C_1$. By M_1 and M_2 we denote the submatrices of M induced by the partition. We may lay a row e_i on top of (M_1, M_2) and add a top-entry $c(i)$ to c for every $i \in C_1$, without changing P. We end up with an identity I matrix on top of M_1 and a vector c_1 on c_2 as desired. Now $Q(P)$ consists of those elements of P with $(I0)x = c_1$. Since P is a meet-polyhedron by Proposition 2.3.9 $Q(P)$ has no more lower bounds than P. So there is a $(x, y) \in Q(P)$ with $(x, y) \in \mathbb{R}^{C_1 \times C_2}$, $y < \mathbf{0}$ and $(x, \lambda y) \in Q(P)$ for all $\lambda > 0$. Clearly, $M_2 y \geq c_2 - M_1 x \geq c_2 - M_1 c_1$. Since $c_2 - M_1 c_1 \leq M_2 \lambda y = \lambda M_2 y$ for all $\lambda > 0$ we have $M_2 y \geq \mathbf{0}$. $\square$

Note that the min-polyhedron of a meet-*polytope* P consists of a single point. Since the case of polytopes is of most interest to us, we restate this special case of Theorem 2.3.11 as an individual theorem:

Theorem 2.3.12. *A polytope* P *is a ULD-polytop if and only if it is representable as*

$$P = \{x \in \mathbb{R}^d \mid Mx \geq c, x \geq z\}$$

where M is Laplacious and $Mz \geq c$. In particular, z is a vertex of P.

Remark 2.3.13. In Section 1.5 we prove that every generalized CFG may be represented by the intersection of ordinary CFGs. We replace the Laplacious alphabet M in the description of a generalized CFG by a set of vertically attached reduced Laplacians M_i' of digraphs D_i. Every x in such a reduced Laplacian M_i' has exactly one negative entry. This change of representation does not affect the particular generalized chip-firing game (Theorem 1.5.14). Analogously to that, one can prove that a Laplacious matrix $\tilde{M}$ representing a ULD-polyhedron may be replaced by a set of vertically attached Laplacious matrices $\tilde{M}_i$ each of which has exactly one negative entry per *column*, and the sum of the entries of any column is less or equal to 0. In contrast to the situation of Section 1.5 the matrices $\tilde{M}_i$ may have non-integer-entries. In order to interpret such a $\tilde{M}_i$ combinatorially we can define chip-firing games on digraphs D_i with arcs of real, positive *volume* instead of arc-multiplicity. This is, every arc $a \in A(D_i)$ has a volume $\nu_a > 0$. Now, firing a vertex v of D_i consits in sending ν_a chips from v along each outgoing arc a to the corresponding neighbor. Now, $\tilde{M}_i$ is the reduced Laplacian of D_i. In this sense, every ULD-polyhedron may be represented as the intersection of CFG-polyhedra.

2.3.1 Feasible Polytopes of Antimatroids

In the following we will discuss a link to a problem of Korte and Lovász [71].

Definition 2.3.14. An *antimatroid* $\mathcal{N} = (E, \mathcal{F})$ is a pair of a finite ground set E and a set $\mathcal{F} \subseteq 2^E$ of *feasible* subsets of E, which satisfy the following three properties:

1. The empty set is in $\mathcal{F}$.

2. The system $\mathcal{F}$ is union-closed.

3. Every $F \in \mathcal{F} \backslash \emptyset$ contains an element e such that $F \backslash \{e\} \in \mathcal{F}$.

Antimatroids were introduced by Edelman [34] and Jamison-Waldner [61]. The axioms defining antimatroids as set systems have some similarity to those of matroids. Matroids can be defined by an exchange axiom (e.g., the basis exchange, or independent set exchange axioms). Korte [70] proved the following theorem, whose proof we will sketch as an application of some of the ULD-characterizations of Chapter 1.

Theorem 2.3.15. *Antimatroids correspond to ULDs.*

Proof. The order $\mathcal{L}_\mathcal{N} := (\mathcal{F}, \subseteq)$ on the feasible sets of an antimatroid is a ULD. Indeed we can color the arcs of its Hasse diagram of the form $(F, F \cup \{e\})$ by e and obtain a U-coloring by property 2. in Definition 2.3.14. The first together with the third property yield that the Hasse diagram has a unique source $\emptyset$. Since the diagram is clearly acyclic Theorem 1.3.3 yields the claim.

On the other hand it is easy to see, that every ULD $\mathcal{L}$ may be represented as the inclusion order on the feasible sets of an antimatroid. We use the ACP-construction from Section 1.2. The ground set for our antimatroid consists of the meet-irreducibles $\mathcal{M}(\mathcal{L})$ of $\mathcal{L}$. For every $m \in \mathcal{M}(\mathcal{L})$ set $A_m := \{j \in \mathcal{J}(\mathcal{L}) \mid m \in \Uparrow j^- \backslash \Uparrow j\}$ and denote by $\mathcal{A}_{\mathcal{M}(\mathcal{L})} := \{A_m \mid m \in \mathcal{M}(\mathcal{L})\}$ their collection. The feasible sets of the antimatroid are now given by $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\mathcal{I}(\mathcal{J}(\mathcal{L})))$. We have $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\emptyset) = \emptyset$ and by Proposition 1.2.5 the system $fing_{\mathcal{A}_{\mathcal{M}(\mathcal{L})}}(\mathcal{I}(\mathcal{J}(\mathcal{L})))$ is union-closed. By Theorem 1.2.24 we have that $\mathcal{A}_{\mathcal{M}(\mathcal{L})}$ is an antichain-partition. This implies the third part of Definition 2.3.14. $\qquad\square$

The *feasible polytope* $\mathsf{P}_\mathcal{N}$ of an antimatroid $\mathcal{N} = (E, \mathcal{F})$ is defined as the *convex hull* of the characteristic vectors of its feasible sets, i.e., $\text{conv}(\{\chi(F) \in \{0,1\}^E \mid F \in \mathcal{F}\})$. The convex hull of a finite set of vectors V is:

$$\text{conv}(V) := \{\sum_{y \in V} \lambda_y y \mid \sum_{y \in V} \lambda_y = 1 \text{ and } \lambda_y \geq 0\}.$$

feasible polytope
convex hull

In [71] the *membership-problem* for feasible polytopes is discussed, i.e., given $x \in \mathbb{R}^E$ decide whether $x \in \mathsf{P}_\mathcal{N}$. The input-size is $|E|$ and the difficulty is, that the number of vertices of $\mathsf{P}_\mathcal{N}$ is generally exponential in $|E|$. In [71] it is shown that for some classes of antimatroids the membership-problem is in **P**, whereas for other classes it is **NP**-hard.

In the following we describe an attempt to find an $\mathcal{H}$-description of $\mathsf{P}_\mathcal{N}$, the size of the description and the time to construct it are certainly an upper bound for the time-complexity of the membership-problem of $\mathsf{P}_\mathcal{N}$. Since an antimatroid $\mathcal{N}$ corresponds to a ULD $\mathcal{L}_\mathcal{N}$ we may apply Theorem 1.5.10 and represent $\mathcal{L}_\mathcal{N}$ as a vector-addition language. This is, we produce a Laplacious matrix $M \in \mathbb{Z}^{d \times E}$ and a vector $\sigma \in \mathbb{Z}^d_{\geq \mathbf{0}}$ such that $F \in \mathcal{F}$ if and only if there is an ordering of $F = \{e(1), \dots, e(k)\}$ with $\sigma + x(e(1)) + \dots + x(e(i)) \geq \mathbf{0}$

for all $i \in [k]$. If the representation of the antimatroid as antichain-partitioned poset may be obtained in polynomial time the proof of Theorem 1.5.10 constructive M in polynomial time in d. Now, by Remark 1.5.11 the produced M has actually a stronger property which in terms of antimatroids reads $\{\chi(F) \mid F \in \mathcal{F}\} = \{z \in \{0,1\}^E \mid Mz \geq -\sigma\} =: \mathsf{P}_{\mathcal{L}}$.

Hence the polytope $\mathsf{P}_{\mathcal{L}}$ contains $\mathsf{P}_{\mathcal{N}}$ and $\mathsf{P}_{\mathcal{L}} \cap \{0,1\}^E = \mathsf{P}_{\mathcal{N}} \cap \{0,1\}^E$. Since both polytopes lie inside the $(0,1)$-hypercube and $\mathsf{P}_{\mathcal{N}}$ is a $(0/1)$-polytope all vertices of $\mathsf{P}_{\mathcal{N}}$ are vertices of $\mathsf{P}_{\mathcal{L}}$. The other vertices of $\mathsf{P}_{\mathcal{L}}$ are no $(0,1)$-vectors. The obvious question is:

Question 2.3.16. Under which circumstances is $\mathsf{P}_{\mathcal{L}} = \mathsf{P}_{\mathcal{N}}$?

In advance: this is not always the case. But for a moment suppose $\mathsf{P}_{\mathcal{L}} = \mathsf{P}_{\mathcal{N}}$. If the number d of rows of the matrix M is polynomial in $|E|$, then we can answer the membership problem in polynomial time. In particular, if the size of the antichains in the representation of $L_{\mathcal{N}}$ is bounded by a constant, then d is polynomial in $|E|$ by Remark 1.5.12. We will now discuss a certain class of antimatroids $\mathcal{N}$, for which the membership-problem has been shown to be **NP**-hard in [71]. We will show, that the size of the antichains in the representation of $\mathcal{L}_{\mathcal{N}}$ is bounded by 2. Hence, membership testing for $\mathsf{P}_{\mathcal{L}}$ works in polynomial time. Thus, for this class of antimatroids we have $\mathsf{P}_{\mathcal{L}} \neq \mathsf{P}_{\mathcal{N}}$, unless **P=NP**.

point-line search antimatroid

The *point-line search antimatroid* $\mathcal{N}(G)$ of an undirected graph $G = (V, \mathcal{E})$ is defined as follows. The groundset consists of $V \cup \mathcal{E}$ and a subset F is feasible if and only if for every edge $\{u, v\} \in F$ at least one of its ends is also in F. In [71] it was shown that the membership-problem for $\mathsf{P}_{\mathcal{N}(G)}$ is **NP**-hard. On the other hand there is a representation as antichain-partitioned poset $(\mathcal{P}, \mathcal{A}_{\mathcal{Q}})$ of $\mathcal{L}_{\mathcal{N}}$ with antichain size bounded by 2:

The bipartite poset $\mathcal{P}$ has a lower half V_1 with an element v_1 for every vertex v of G. The upper half V_2 contains degree many copies $\{v_2(1), \ldots, v_2(\deg(v))\}$ of every vertex v of G. The order relation is defined by $v_1 \leq w_2(i)$ if and only if $v = w$. To define the antichain-partition $\mathcal{A}_{\mathcal{Q}}$ take the singleton-partition on V_1 and for every edge $\{v, w\} \in \mathcal{E}$ lay an antichain $A(v, w)$ over still uncovered copies of v and w, respectively, in V_2. It is easy to check that the feasible sets of $\mathcal{N}$ coincide with $fing_{\mathcal{A}_{\mathcal{Q}}}(\mathcal{I}(\mathcal{P}))$ and that $A_{\mathcal{Q}}$ is a reduced antichain-partition, see Theorem 1.2.24.

Despite the fact, that the constructed polytope $\mathsf{P}_{\mathcal{L}}$ containing a given feasible polytope $\mathsf{P}_{\mathcal{N}}$, does not generally coincide with the latter, we have enriched the knowledge concering the membership-problem. The inequalities derived from Theorem 1.2.24 are new ingredients to the study of feasible polytopes of antimatroids.

Since M is Laplacious, $M\mathbf{0} \geq -\sigma$ and $\mathsf{P}_{\mathcal{L}} \geq \mathbf{0}$ by Theorem 2.3.12 $\mathsf{P}_{\mathcal{L}}$ is a ULD-polytope. So, we have constructed a ULD-polytope containing a feasible polytope of an antimatroid, but this might not be the smallest one. Order-polytopes are exactly those full-dimensional $(0/1)$-polytopes which are distributive one might hope for a generalization of that statement. It is easy to see, that $(0/1)$-polytopes which are ULD-polytopes are feasible polytopes of antimatroids but the converse is not clear.
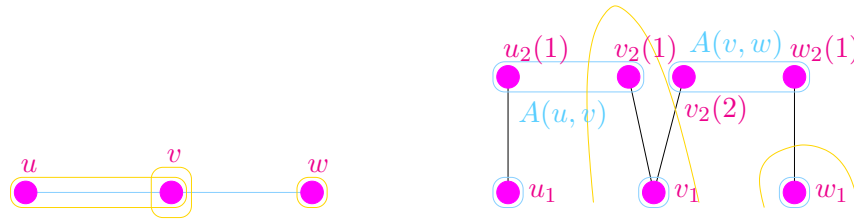
Figure 2.5: A graph and the antichain-partitioned poset corresponding to its point-line search antimatroid. We have marked a feasible set on the left and the corresponding maximal ideal of the ACP on the right.

Question 2.3.17. Are feasible polytopes of antimatroids ULD-polytopes?

For polytopes which are not $(0/1)$-polytopes it is easy to find examples P where the vertices of P form a ULD with respect to the dominance-order, but the whole polytope is not a ULD-polytope. Cyclic polytopes even give a class of examples for this in terms of distributive lattices and D-polytopes. In contrast to $\mathcal{H}$-description by $\mathcal{V}$-*description* of a polyhedron P we refer to a set of points $V = U \cup W \subseteq \mathbb{R}^d$ such that $\mathsf{P} = \text{conv}(U) + \text{cone}(V)$. The "+" denotes the *Minkowski sum*, which consists of the pointwise sums of both objects. We are far away from any answer to the following generalization of Question 2.3.17.

$\mathcal{V}$-description

Minkowski sum

Question 2.3.18. Is there a characterization of ULD-polyhedra in terms of their $\mathcal{V}$-description?

2.4 Distributive Polyhedra

For the rest of the chapter we will fully concentrate on the class of distributive polyhedra. D-polyhedra will turn out to have very nice connections to several combinatorial objects such as arc-parameterized digraphs, generalized flows, and bicircular oriented matroids. In particular we will see, how the discrete distributive lattices discussed in the previous chapter turn out to be the sets of integer points of particularly nice integral distributive polytopes.

For a start, as we did for ULD-polyhedra we want to find a geometric characterization of distributive polyhedra. D-polyhedra are exactly those polyhedra, which are U-polyhedra and L-polyhedra at the same time, see Definition 2.1.8. Hence in principal we obtain an $\mathcal{H}$-description as a consequence of Theorem 2.3.6 and its dual counterpart in terms of L-polyhedra. But in order to promote a special property which D-polyhedra have and ULD-polyhedra do not share, we use a little different approach: faces of D-polyhedra are D-polyhedra. We use this to obtain a new proof of the characterization of D-polyhedra.

Lemma 2.4.1. *Faces of D-polyhedra are D-polyhedra.*

Proof. Let P be a D-polyhedron such that $\mathsf{P} \subseteq \mathsf{H}^{\leq} = \{x \in \mathbb{R}^n \mid \langle z, x \rangle \leq c\}$ and let $\mathsf{F} = \mathsf{P} \cap \mathsf{H}$ be a face. Suppose that there are $x, y \in \mathsf{F}$ such that $\max(x, y) \notin \mathsf{F}$. Since $\max(x, y) \in \mathsf{P}$ this means $\langle z, \max(x, y) \rangle < c$. Since $2c = \langle z, x + y \rangle = \langle z, \max(x, y) \rangle + \langle z, \min(x, y) \rangle$ this implies $\min(x, y) \notin \mathsf{P}$ – a contradiction. $\square$

We have gathered enough instruments to characterize:

Theorem 2.4.2. *A polyhedron $\mathsf{P} \subseteq \mathbb{R}^n$ is a D-polyhedron if and only if*

$$\mathsf{P} = \{x \in \mathbb{R}^n \mid N_\Lambda^T x \leq c\}$$

for some generalized network-matrix N_Λ and $c \in \mathbb{R}^m$.

Proof. "$\Longleftarrow$": If P may be represented as claimed, then it is a U-polyhedron by Theorem 2.3.6 and an L-polyhedron by Remark 2.3.7.

"$\Longrightarrow$": By Lemma 2.4.1 every face F of P is distributive. Lemma 2.3.1 ensures that $\operatorname{aff}(\mathsf{F})$ is distributive. Theorem 2.2.11 yields $\operatorname{aff}(\mathsf{F}) = \{x \in \mathbb{R}^n \mid N(\mathsf{F})_{\Lambda(\mathsf{F})}^T x = c(\mathsf{F})\}$ for a generalized network-matrix $N(\mathsf{F})_{\Lambda(\mathsf{F})}$. In particular this holds for $\operatorname{aff}(\mathsf{P})$, which we will actually represent as $\{x \in \mathbb{R}^n \mid N(\mathsf{P})_{\Lambda(\mathsf{P})}^T x \leq c(\mathsf{P})\}$, using Lemma 2.2.12. For a facet F choose a column z_F of $N(\mathsf{F})_{\Lambda(\mathsf{F})}$ such that $\mathsf{H}_\mathsf{F}^{\leq} := \{x \in \mathbb{R}^n \mid \langle z_\mathsf{F}, x \rangle \leq c_\mathsf{F}\}$ is a facet-defining halfspace for F. Since z_F is a column of a generalized network matrix it is a generalized network matrix itself and by Lemma 2.3.3 we have that $\mathsf{H}_\mathsf{F}^{\leq}$ is distributive.

By the above chain of arguments we can transform the representation given by the Representation Theorem for Polyhedra [110]:

$$\mathsf{P} = (\bigcap_{\mathsf{F} \text{ facet}} \mathsf{H}_\mathsf{F}^{\leq}) \cap \operatorname{aff}(\mathsf{P}) = (\bigcap_{\mathsf{F} \text{ facet}} \{x \in \mathbb{R}^n \mid \langle z_\mathsf{F}, x \rangle \leq c_\mathsf{F}\}) \cap \{x \in \mathbb{R}^n \mid N(\mathsf{P})_{\Lambda(\mathsf{P})}^T x \leq c(\mathsf{P})\}.$$

Here the single matrices involved are generalized network-matrices. Glueing all these matrices horizontally together one obtains a single generalized network-matrix N_Λ and a vector c such that $\mathsf{P} = \{x \in \mathbb{R}^n \mid N_\Lambda^T x \le c\}$. $\qquad\square$

Remark 2.4.3. By Proposition 2.2.8 it follows that the system $N_\Lambda^T x \leqq c$ with equality-*and* inequality-constraints defines a D-polyhedron whenever N_Λ is a generalized network-matrix.

Remark 2.4.4. Generalized network matrices are not the only matrices that can be used to represent D-polyhedra. Scaling columns of N_λ and entries of c simultaneously preserves the polyhedron but may destroy the property of the matrix being a generalized network matrix. If the polyhedron is full-dimensional, the $\mathcal{H}$-description is unique up to that scaling-operation. Hence, for full-dimensional D-polyhedra there is no more ambiguity than scaling in our characterization.

There may, however, be representations of different type if the polyhedron is not full-dimensional. Consider e.g., the D-polyhedron consisting of all scalar multiples of $(1, 1, 1, 1)$ in $\mathbb{R}^4$, it can be described by the six inequalities $\sum_{i \in X} x_i - \sum_{i \notin X} x_i \le 0$, for X a 2-subset of $\{1, 2, 3, 4\}$.

2.4.1 Towards a Combinatorial Model

After the geometrical characterization of D-polyhedra the rest of this section is devoted to understand the combinatorial meaning of distributive polyhedra. We have shown that a D-polyhedron P is completely described by an arc-parameterized digraph D_Λ and an arc-capacity vector $c \in \mathbb{R}^m$. This characterization suggests to consider the points of P as 'graph objects'. A *potential* for D_Λ is a vector $p \in \mathbb{R}^n$, which assigns a real number p_i to each vertex i of D_Λ, such that the inequality $p_j - \lambda_a p_i \le c_a$ holds for every arc $a = (i, j)$ of D_Λ. The points of the D-polyhedron $\mathsf{P}(D_\Lambda, c) := \{p \in \mathbb{R}^n \mid N_\Lambda^T p \le c\}$ are exactly the potentials of D_Λ.

potential
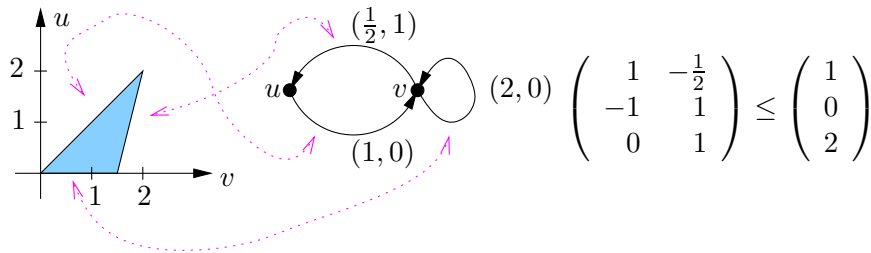
Figure 2.6: D-polytope represented by an arc parameterized D_Λ and transposed generalized network-matrix N_Λ^T with capacities c. The arcs correspond to the defining inequalities. A tuple at arc a stands for (c_a, λ_a).

Theorem 2.4.2 then can be rewritten in terms of vertex potentials

Theorem 2.4.5. *A polyhedron is distributive if and only if it is the set of potentials of an arc-parameterized digraph D_Λ.*

Interestingly there is a second class of graph objects associated with the points of a D-polyhedron. While potentials are weights on vertices, this second class consists of elements of the arc-space of D_Λ. We define $\mathsf{T}(D_\Lambda)$ to be the space $\mathrm{Im}(N_\Lambda^T)$. In the spirit of the terminology of *generalized flows*, c.f. [3], we call the elements of $\mathsf{T}(D_\Lambda)$ the *generalized tensions* of D_Λ. Given a D-polyhedron $\mathsf{P}(D_\Lambda, c)$ with capacity constraints c we look at

$$\mathsf{T}(D_\Lambda, c) := \{x \in \mathbb{R}^m \mid x \leq c \text{ and } x \in \mathrm{Im}(N_\Lambda^T)\} = N_\Lambda^T \mathsf{P}(D_\Lambda, c).$$

The elements of $\mathsf{T}(D_\Lambda, c)$ are then called generalized tensions *within the capacity constraints c.*

Theorem 2.4.6. *Let D_Λ be an arc-parameterized digraph with capacities $c \in \mathbb{R}^m$. The set $\mathsf{T}(D_\Lambda, c)$ is a polyhedron and affinely isomorphic to a D-polyhedron P'. Here P' can be obtained from $\mathsf{P} = \mathsf{P}(D_\Lambda, c)$ by intersecting P with some hyperplanes of type $\mathsf{H}_i = \{x \mid x_i = 0\}$. In particular $\mathsf{T}(D_\Lambda, c)$ inherits the structure of a distributive lattice by the bijection to P'.*

Proof. Since $\mathsf{T}(D_\Lambda, c) = N_\Lambda^T \mathsf{P}$ for the D-polyhedron P of feasible vertex-potentials of D_Λ, and N_Λ^T is a linear map, the set of generalized tensions is a polyhedron.

If N_Λ^T is bijective on P, then the set $\mathsf{T}(D_\Lambda, c)$ inherits the distributive lattice structure from P. This is not always the case. In the rest of the proof we show that we always find a D-polyhedron $\mathsf{P}' \subseteq \mathsf{P}$ such that N_Λ^T is a bijection from P' to $\mathsf{T}(D_\Lambda, c)$.

From Theorem 2.2.11 we know that $\mathrm{Ker}(N_\Lambda^T)$ is a distributive space and that there is an NND basis B of $\mathrm{Ker}(N_\Lambda^T)$. For every $b \in B$ fix an arbitrary element $i(b) \in \underline{b}$. Denote the set of these elements by $I(B)$. Define $\mathsf{S} := \mathrm{span}(\{e_i \in \mathbb{R}^n \mid i \in [n] \backslash I(B)\})$.

1. S is distributive:
   By definition S has an NND basis, i.e., is distributive by Proposition 2.2.11.
2. $\mathsf{T}(D_\Lambda) = N_\Lambda^T \mathsf{S}$:
   Since $\mathsf{T}(D_\Lambda) = \mathrm{Im}(N_\Lambda^T) \supseteq N_\Lambda^T \mathsf{S}$ it suffices to show "$\subseteq$". So let $N_\Lambda^T p = x \in \mathsf{T}(D_\Lambda)$. Define $p' := p - \sum_{b \in B}(\frac{p_{i(b)}}{b_i} b)$. Since $\sum_{b \in B}(p_{i(b)} b) \in \mathrm{Ker}(N_\Lambda^T)$ we have $N_\Lambda^T p' = x$. Moreover $p'_i = 0$ for all $i \in I(B)$, i.e., $p' \in \mathsf{S}$.
3. $N_\Lambda^T : \mathsf{S} \hookrightarrow \mathsf{T}(D_\Lambda)$ is injective:
   Suppose there are $p, p' \in \mathsf{S}$ such that $N_\Lambda^T p = N_\Lambda^T p'$. Then $p - p' \in \mathrm{Ker}(N_\Lambda^T) \cap \mathsf{S}$. But by the definition of S this intersection is trivial, i.e., $p = p'$.

We have shown that N_Λ^T is an isomorphism from S to $\mathsf{T}(D_\Lambda)$ and that S is distributive. Thus $\mathsf{P}' := \mathsf{P} \cap \mathsf{S}$ is a D-polyhedron such that the linear map defined by the matrix N_Λ^T is a bijection from P' to $\mathsf{T}(D_\Lambda, c)$. $\qquad\square$

(margin note: generalized tensions)

The intersection of P with H_i can be modeled by adding a loop $a = (i, i)$ with capacity $c_a = 0$ to the digraph. Hence, with Remark 2.4.3 the preceding theorem says that for every $T(D_\Lambda, c)$ we can add some loops to yield a graph $D'_{\Lambda'}$ and capacities c' such that

$$T(D_\Lambda, c) = T(D'_{\Lambda'}, c') \cong P(D'_{\Lambda'}, c') = P'.$$

In the following we will always assume that generalized tensions $T(D_\Lambda, c)$ have the property that $T(D_\Lambda, c) \cong P(D_\Lambda, c)$. In this case we call (D_Λ, c) *reduced*.

<div style="text-align: right">reduced</div>

Note that $T(D_\Lambda, c)$ can be far from being a D-polyhedron, but it inherits the distributive lattice structure via an isomorphism from a D-polyhedron. In the following we investigate generalized tensions, i.e., the elements of $T(D_\Lambda)$, as objects in their own right. Since $T(D_\Lambda) = \text{Im}(N_\Lambda^T) = \text{Ker}(N_\Lambda)^\perp$ we can make the following fundamental:

Observation 2.4.7. A vector $x \in \mathbb{R}^m$ is a generalized tensions of (D_Λ) if and only if $\langle x, f \rangle = 0$ for all $f \in \text{Ker}(N_\Lambda)$.

Thus, understanding the elements of $\text{Ker}(N_\Lambda)$ as objects in the arc space of D_Λ is vital to our analysis. This will provide the link to flows and generalized flows of directed graphs. In Section 2.4.2 we review the case of *ordinary* tensions, which leads to a description closely related to the definition of Δ-tensions. In Section 2.4.3 we then are able to describe the generalized tensions of D_Λ as capacity-respecting arc values, which satisfy a *generalized circular balance condition* around elements of $\text{Ker}(N_\Lambda)$, see Theorem 2.4.18.

2.4.2 Tensions and Alcoved Polytopes

In this section we present a special case of distributive polytopes with particularly nice properties with respect to integrality constraints and many applications in graph theory. In fact, the results presented here are those which gave first rise to the idea of considering distributive polyhedra in general. In a sense the results of this subsection are a very special version of what we will obtain as a generalization in the subsections afterwards.One particular property of the polytopes in this section is their behavior with respect to integrality constraints. This is something which will not carry over to the general case.

We look at the case where $D_\Lambda = (V, A)$ is an arc-parameterized digraph with $\Lambda \in \{0, 1\}^m$,. More precisely $\lambda_a = 0$ if and only if a is a loop and $\lambda_a = 1$ otherwise. In this case N_Λ is the *network-matrix* N of the underlying digraph D, i.e., $N \in \mathbb{R}^{n \times m}$ consists of columns $e_j - e_i$ for every non-loop arc $a = (i, j)$ and e_i for a loop $a = (i, i)$. Thus, we identify D_Λ with D and forget about Λ.

<div style="text-align: right">network-matrix</div>

By Observation 2.4.7 in order to understand the generalized tensions of D we have to analyze $\text{Ker}(N)$. This is a classical subject of algebraic graph theory [52]: The elements of $\text{Ker}(N) =: F(D)$ are the *flows* of D, i.e., those real arc values $f \in \mathbb{R}^m$ which respect

<div style="text-align: right">flow</div>

flow-conservation at every vertex of D. Each support-minimal element $f \in \mathsf{F}(D)$ is a scalar multiple of the *signed characteristic vector* $\overrightarrow{\chi}(C)$ of an oriented cycle $C \in \mathcal{C}(D)$ of D. We define $\overrightarrow{\chi}(C)$ as the *sign vector* $\overrightarrow{\chi}(C) \in \{+1, -1, 0\}^A$ associated to C, where $\overrightarrow{\chi}(C)_a = 1$ if $a \in C^+$, $\overrightarrow{\chi}(C)_a = -1$ if $a \in C^-$ and $\overrightarrow{\chi}(C)_a = 0$ otherwise.

<div style="margin-left:0">sign vector</div>

<div style="float:right">signed
characteristic
vector</div>

The set $\mathsf{T}(D)$ of generalized tensions of D consists of those $x \in \mathbb{R}^m$ with $\langle x, f \rangle = 0$ for all flows f. This is equivalent to $\langle x, \overrightarrow{\chi}(C) \rangle = 0$ for all $C \in \mathcal{C}$ – the circular balance conditions for tensions. So in this particular case of $\Lambda \in \{0, 1\}^m$ we may define the generalized tensions $\mathsf{T}(D)_{\leq c}$ very analogously to ordinary tensions (Definition 1.4.1) as those $x \in \mathbb{R}^m$ such that

(D'$_1$) $x(a) \leq c(a)$ for all $a \in A$. (capacity constraints)

(D'$_2$) $\mathbf{0} = \sum_{a \in C^+} x(a) - \sum_{a \in C^-} x(a)$ for all C. (circular balance conditions)

The only difference here is, that we have no lower arc-capacities and that we do not restrict to the set to *integer* vectors. We refer to the generalized tensions in this special case as *real tensions* of D within c.

<div style="margin-left:0">real tension</div>

Theorem 2.4.6 yields a distributive lattice structure on the set of real tensions $\mathsf{T}(D, c)$ by identifying it via affine equivalence with a distributive polytope $\mathsf{P}(D, c)$.

The particular form of Λ allows us to show a distributive lattice structure on the *integral tensions* $\mathcal{T}(D, c) := \mathcal{T}_{\mathbf{0}}(D, -\infty, c) = \mathsf{T}(D, c) \cap \mathbb{Z}^m$ with upper arc-capacities c of D, as originally defined in Definition 1.4.1. Just set Δ to the all-zeroes vector and let all arcs have unbounded lower capacity. To the end of proving a distributive lattice structure on integral tensions we first make the following:

<div style="margin-left:0">tension</div>

Observation 2.4.8. The intersection of a D-polytope $\mathsf{P} \subseteq \mathbb{R}^n$ and any other (particularly finite) distributive sublattice $\mathcal{L}$ of $\mathbb{R}^n$ yields a distributive lattice $\mathsf{P} \cap \mathcal{L}$.

So if $\mathsf{P} \subseteq \mathbb{R}^n$ is a D-polyhedron, then $\mathsf{P} \cap \mathbb{Z}^n$ is a distributive lattice. Since by Theorem 2.4.6 we can assume N^T to be bijective on $\mathsf{P}(D, c)$ we obtain a distributive lattice structure on $N^T(\mathsf{P}(D, c) \cap \mathbb{Z}^n)$. However, what we want is a distributive lattice on integral tensions, i.e., on $\mathcal{T}(D, c) = \mathsf{T}(D, c) \cap \mathbb{Z}^m = (N^T \mathsf{P}(D, c)) \cap \mathbb{Z}^m$. Luckily, N is a totally unimodular matrix, which yields $(N^T \mathsf{P}(D, c)) \cap \mathbb{Z}^m = N^T(\mathsf{P}(D, c) \cap \mathbb{Z}^n)$, see [99]. We obtain:

Theorem 2.4.9. *The set of integral tensions $\mathcal{T}(D, c)$ is affinely isomorphic to $\mathsf{P}(D, c) \cap \mathbb{Z}^n$. Thus, $\mathcal{T}(D, c)$ carries a distributive lattice structure and is the set of integer points of a polyhedron.*

Indeed, the result obtained is even more general. In Section 1.4 we have shown that the set of Δ-tensions of a digraph with lower and upper arc-capacities is isomorphic to the set of integral tensions (Lemma 1.4.5) of a digraph with only upper arc-capacities (Remark 1.4.15), the isomorphism is just a translation in $\mathbb{R}^m$. We obtain a version of the first main result of Section 1.4 (Theorem 1.4.9) enhanced with a statement about convexity.

Theorem 2.4.10. *Let D be a digraph with capacities c_ℓ, c_u. The set $\mathcal{T}_\Delta(D, c_l, c_u)$ carries the structure of a distributive lattice and is the set of integer points of a polyhedron.*

As a reminder we restate a list of formerly constructed distributive lattices, which may be modelled as Δ-tensions and by Theorem 2.4.10 form sets of integer points of polyhedra:

- domino and lozenge tilings of a plane region (Rémila [97] and others based on Thurston [105])
- planar spanning trees (Gilmer and Litherland [48])
- planar bipartite perfect matchings (Lam and Zhang [73])
- planar bipartite d-factors (Felsner [39], Propp [92])
- Schnyder woods of a planar triangulations (Brehm [25])
- Eulerian orientations of a planar graph (Felsner [39])
- α-orientations of a planar graph (Felsner [39], Ossona de Mendez [88])
- k-fractional orientations with prescribed outdegree of a planar graph (Bernardi and Fusy [11])
- Schnyder decompositions of a plane d-angulations of girth d (Bernardi and Fusy [12])
- circular integer flows of a planar graph (Khuller, Naor and Klein [66])
- higher dimensional rhombic tilings (Linde, Moore, and Nordahl [77])
- c-orientations of a graph (Propp [92])

From a polytopal point of view, the distributive polyhedra corresponding to tensions of digraphs form pretty nice and special classes:

Given a poset $\mathcal{P}$ its *order polytope* is defined as the convex hull of the characteristic vectors of the ideals $I \in \mathcal{I}(\mathcal{P})$ of a the poset. Order polytopes encode many poset properties and enhance them with a notion of geometry. In [102] Stanley provides a characterization of order polytopes in terms of their $\mathcal{H}$-description. It is easy to see, that the $\mathcal{H}$-description in fact coincides with the one of those distributive polytopes $\mathsf{P}(D_\Lambda, c)$ with parameters $\Lambda, c \in \{0, 1\}^m$. The underlying digraph D is the isomorphic to the Hasse diagram of $\mathcal{P}$ and the tensions of D correspond to the ideals of $\mathcal{P}$. Indeed, one can prove that a $(0/1)$-*polytope* P (all vertices of P are $(0/1)$-vectors) is a D-polytope if and only if it is an order-polytope. Is an analogue statement true for $(0/1)$-polytopes that are ULD-polytopes and feasible polytopes of antimatroids? This relates to Question 2.3.17.

<div style="text-align:right">order polytope</div>

<div style="text-align:right">$(0/1)$-polytope</div>

If we broaden the set of parameters from order polytopes slightly to $\Lambda \in \{0, 1\}^m$ and $c \in \mathbb{Z}^m$ we obtain the more general class of *alcoved polytopes*. Alcoved polytopes have proven to model a big variety of combinatorial objects [74]. We just contributed Δ-tensions and all their special instances to it. Moreover, it has been shown in [63] that alcoved polytopes coincide with *polytropes*, which are of importance in the study of *tropical* convexity.

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Theorem 2.4.2 tells us that alcoved polytopes are distributive. Moreover, their integer points correspond to vertex potentials of digraphs within some capacity constraints. Theorem 2.4.9 characterizes the integer point sets of alcoved polytopes in terms of the arc-space, i.e., P is an alcoved polytope if and only if $N^T P \cap \mathbb{Z}^m$ corresponds to the integral tensions of directed graph.

In the next section we characterize those real-valued subsets of the arc space of parameterized digraphs, which can be proven to carry a distributive lattice structure by the above method as *generalized* Δ-tensions. The generalization of Theorem 2.4.10 to generalized Δ-tensions is stated in Theorem 2.4.20.

2.4.3 General Parameters

In this subsection we will develop a full characterization of generalized tensions. We will make connections to generalized flows and to bicircular oriented matroids. So we look at the case of general tensions of an arc-parameterized digraph D_Λ. The aim of this section is to describe $T(D_\Lambda, c)$ as the orthogonal complement of $\text{Ker}(N_\Lambda)$ within the capacity bounds given by c. For $f \in \mathbb{R}^m$ and $j \in V$ we define the *excess* of f at j as

excess

$$\omega(j, f) := \Big(\sum_{a=(i,j)} f_a \Big) - \Big(\sum_{a=(j,k)} \lambda_a f_a \Big).$$

Since $f \in \text{Ker}(N_\Lambda)$ means $\omega(j, f) = 0$ for all $j \in V$ we think of f as an edge-valuation satisfying a *generalized flow-conservation*. We call the elements of $\text{Ker}(N_\Lambda)$ the *generalized flows* of D_Λ.

generalized flows

Generalized flows were introduced by Dantzig [29] in the sixties and there has been much interest in related algorithmic problems. For surveys on the work, see [3, 106]. The most efficient algorithms known today have been proposed in [44].

We will denote $F(D_\Lambda) := \text{Ker}(N_\Lambda)$ and call it the *generalized flow space*. Let $C(D_\Lambda)$ be the set of support-minimal vectors of $F(D_\Lambda) \setminus \{\mathbf{0}\}$, i.e., $f \in C(D_\Lambda)$ if and only if $\underline{g} \subseteq \underline{f}$ implies $\underline{g} = \underline{f}$ for all $g \in F(D_\Lambda) \setminus \{\mathbf{0}\}$. The elements of $C(D_\Lambda)$ will be called *generalized cycles*. Since the support-minimal vectors $C(D_\Lambda)$ span the entire space $F(D_\Lambda)$ the generalized tensions of D_Λ are already determined by being orthogonal to $C(D_\Lambda)$, i.e., to all generalized cycles.

generalized flow space

generalized cycles

In the following we answer the question what generalized cycles look like as subgraphs of D_Λ. After some definitions and technical lemmas we give a combinatorial characterization of generalized cycles, see Theorem 2.4.17, which makes a link to bicircular oriented matroids. Later this leads to a description of generalized tensions in the spirit of the definition of ordinary tensions in Section 1.4

For an *oriented arc-set S of D_Λ* define its *multiplier* as

oriented arc-set multiplier

$$\lambda(S) := \prod_{a \in S} \lambda_a^{\overrightarrow{\chi}(S)_a},$$

where $\overrightarrow{\chi}(S)_a = \pm 1$ depending on the orientation of a in S.

An oriented cycle C in the underlying digraph D will be called *lossy* if $\lambda(C) < 1$, and *gainy* if $\lambda(C) > 1$, and *breakeven* if $\lambda(C) = 1$. A *bicycle* is an oriented arc set that can be written as $C \cup P \cup C'$ with a gainy cycle C, a lossy cycle C' and a (possibly trivial) simple oriented path P from C to C'; moreover, the intersection of C and C' is a (possibly empty) interval of both. Moreover, C and C' are equally oriented on this interval. See Figure 2.7 for the three typical examples. More precisely, every bicycle is isomorphic to a subdivison of one the graphs in the figure. We denote the set of bicycles and breakeven cycles of D_Λ by $\mathcal{B}(D_\Lambda)$. lossy
gainy
breakeven
bicycle
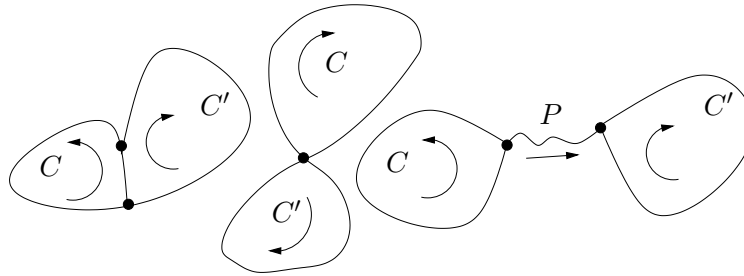
Figure 2.7: Bicycles with $P = \emptyset$ and $P \neq \emptyset$.

Recall that for $x \in \mathbb{R}^m$ the support was defined as $\underline{x} := \{i \in [m] \mid x_i \neq 0\}$. Generally, a *signed set* $X = (X^+, X^-)$ is a pair of disjoint sets of positive and negative elements X^+ and X^-, respectively. The *support* of X is $\underline{X} := X^+ \cup X^-$. For $i \in \underline{X}$ we write $X_i = \pm 1$ if $i \in X^\pm$, respectively. If $i \notin \underline{X}$, then we denote $X_i = 0$. signed set
support

Note that this is a direct generalization of oriented arc-sets and their forward and backward arcs and their signed characteristic vector. We define the *signed support* of a vector x as the signed set with support $\underline{x}$ and $\underset{\rightarrow}{\underline{x}}^+ := \{i \in \underline{x} \mid x_i > 0\}$ and $\underset{\rightarrow}{\underline{x}}^- := \{i \in \underline{x} \mid x_i < 0\}$. signed support

Remark 2.4.11. Note that $\underset{\rightarrow}{\mathsf{C}}(D_\Lambda)$ is exactly the set of signed circuits of the *oriented matroid* induced by the matrix N_Λ, see [19]. In Theorem 2.4.17 we prove $\mathcal{B}(D_\Lambda) = \underset{\rightarrow}{\mathsf{C}}(D_\Lambda)$. Hence we provide a description of the circuits of the matroid based only on the arc-parameterized digraph D_Λ. It turns out that oriented matroids arising as $\mathcal{B}(D_\Lambda)$ are oriented versions of a combination of a classical *cycle matroid* and a *bicircular matroid*. The latter were introduced in the seventies [81, 101]. Active research in the field can be found in [49, 50, 82]. We feel that oriented matroids of generalized network matrices are worth further investigation.

In order to understand generalized flows, in the following lemmas we will determine how flow is transformed when transported trough an arc-parameterized digraph. Let

$W = (a(0), \ldots, a(k))$ be a walk in D, i.e., W may repeat vertices and arcs. We abuse notation and identify W with its *signed support* $\underrightarrow{W}$, which is defined as the signed support of the signed characteristic vector of W, i.e., $\underrightarrow{W} := \overrightarrow{\chi}(W)$. Even more, we write W_i and $W_{a(i)}$ for the same sign, namely the orientation of the arc $a(i)$ in W. Note that cycles and bicycles can be regarded to be walks; these will turn out to be the most interesting cases in our context. A vector $f \subseteq \mathbb{R}^m$ is an *inner flow* of W if $\underrightarrow{f} = \pm \underrightarrow{W}$ and f satisfies the generalized flow conservation law between consecutive arcs of W.

Lemma 2.4.12. *Let $W = (a(0), \ldots, a(k))$ be a walk in D_Λ and f an inner flow of W. Then*

$$f_{a(k)} = K\lambda(W)^{-1}f_{a(0)}$$

where the 'correction term' K is given by $K = W_0 W_k \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(k)}^{\min(0,W_k)}$. In particular the space of inner flows of W is one-dimensional.

Proof. We proceed by induction on k. If $k = 0$, then

$$
\begin{aligned}
& W_0 W_0 \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(0)}^{\min(0,W_0)} \lambda(W)^{-1} f_{a(0)} \\
=\ & \lambda_{a(0)}^{W_0} \lambda(W)^{-1} f_{a(0)} \\
=\ & \lambda_{a(0)}^{W_0} \lambda_{a(0)}^{-W_0} f_{a(0)} \\
=\ & f_{a(0)}.
\end{aligned}
$$

If $k = 1$, then our walk consisting of two arcs has a middle vertex, say i. Since f is an inner flow $\omega(i, f) = 0$. This can be rewritten as $W_0 \lambda_{a(0)}^{-\min(0,W_0)} f_{a(0)} = W_1 \lambda_{a(1)}^{\max(0,W_1)} f_{a(1)}$. Now we can transform

$$
\begin{aligned}
& W_0 W_1 \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(1)}^{\min(0,W_1)} \lambda(W)^{-1} f_{a(0)} \\
=\ & W_0 \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(0)}^{-W_0} f_{a(0)} W_1 \lambda_{a(1)}^{\min(0,W_1)} \lambda_{a(1)}^{-W_1} \\
=\ & W_1 \lambda_{a(1)}^{\max(0,W_1)} f_{a(1)} W_1 \lambda_{a(1)}^{\min(0,W_1)} \lambda_{a(1)}^{-W_1} \\
=\ & f_{a(1)}.
\end{aligned}
$$

If $k > 1$, then we can look at two overlapping walks $W' = (a(0), \ldots, a(\ell))$ and $W'' = (a(\ell), \ldots, a(k))$. Clearly f restricted to W' and W'' respectively satisfies the preconditions for the induction hypothesis. By applying the induction hypothesis to W'' and W' we obtain

$$
\begin{aligned}
f_{a(k)} &= W_\ell W_k \lambda_{a(\ell)}^{\max(0,W_\ell)} \lambda_{a(k)}^{\min(0,W_k)} \lambda(W'')^{-1} f_{a(\ell)} \quad \text{and} \\
f_{a(\ell)} &= W_0 W_\ell \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(\ell)}^{\min(0,W_\ell)} \lambda(W')^{-1} f_{a(\ell)}.
\end{aligned}
$$

Substitute the second formula into the first and observe that $W_\ell W_\ell = 1$, and that from the product of four terms $\lambda_{a(\ell)}$ with different exponents the single $\lambda_{a(\ell)}^{-W_\ell}$ needed for $\lambda(W)^{-1}$ remains. This proves the claimed formula for $f_{a(k)}$. $\qquad \square$

Lemma 2.4.13. *Let $P = (a(0), \ldots, a(k))$ be a simple path from v to v' in D_Λ. If f is an inner flow of P with $\underrightarrow{f} = \underrightarrow{P}$, then $\omega(v, f) < 0$ and $\omega(v', f) > 0$.*

Proof. By definition $\omega(v, f) = -P_0 \lambda_{a(0)}^{\max(0, P_0)} f_{a(0)}$. Since $\lambda_{a(0)} > 0$ and $\underrightarrow{f_{a(0)}} = P_0$ we conclude $\omega(v, f) < 0$. For the second inequality we use Lemma 2.4.12:

$$
\begin{aligned}
\omega(v', f) &= P_k \lambda_{a(k)}^{-\min(0, P_k)} f_{a(k)} \\
&= P_k \lambda_{a(k)}^{-\min(0, P_k)} P_0 P_k \lambda_{a(0)}^{\max(0, P_0)} \lambda_{a(k)}^{\min(0, P_k)} \lambda(P)^{-1} f_{a(0)} \\
&= P_0 \lambda_{a(0)}^{\max(0, P_0)} \lambda(P)^{-1} f_{a(0)}.
\end{aligned}
$$

Since $\lambda_{a(0)}, \lambda(P)^{-1} > 0$ and $\underrightarrow{f_{a(0)}} = \underrightarrow{P_{a(0)}} = P_0$ we conclude $\omega(v', f) > 0$. $\qquad\square$

Lemma 2.4.14. *Let $C = (a(0), \ldots, a(k))$ be a cycle in D_Λ and f an inner flow of C with $\underrightarrow{f} = \underrightarrow{C}$. Then the excess $\omega(v, f)$ at the initial vertex v satisfies $\underrightarrow{\omega(v, f)} = \underrightarrow{1 - \lambda(C)}$.*

Proof. Reusing the computations from Lemma 2.4.12 we obtain

$$
\begin{aligned}
\omega(v, f) &= C_k \lambda_{a(k)}^{-\min(0, C_k)} f_{a(k)} - C_0 \lambda_{a(0)}^{\max(0, C_0)} f_{a(0)} \\
&= C_0 \lambda_{a(0)}^{\max(0, C_0)} \lambda(C)^{-1} f_{a(0)} - C_0 \lambda_{a(0)}^{\max(0, C_0)} f_{a(0)} \\
&= C_0 \lambda_{a(0)}^{\max(0, C_0)} f_{a(0)} (\lambda(C)^{-1} - 1).
\end{aligned}
$$

Since $\lambda_{a(0)} > 0$ and $\underrightarrow{f_{a(0)}} = C_0$ we conclude $\underrightarrow{\omega(v, f)} = \underrightarrow{\lambda(C)^{-1} - 1}$. Finally observe that $\underrightarrow{\lambda(C)^{-1} - 1} = \underrightarrow{1 - \lambda(C)}$. $\qquad\square$

Theorem 2.4.15. *Given a bicycle or breakeven cycle H of D_Λ, the set of flows f with $\underrightarrow{f} = \pm \underrightarrow{H}$ is a 1-dimensional subspace of $\mathsf{F}(D_\Lambda)$.*

Proof. Given $H \in \mathcal{B}(D_\Lambda)$ we want to characterize those $f \in \mathsf{F}(D_\Lambda)$ with $\underrightarrow{f} = \pm \underrightarrow{H}$. Lemma 2.4.12 implies that the dimension of the inner flows of H is at most one. Hence, it is enough to identify a single nontrivial flow on H.

If $H = C \in \mathcal{B}(D_\Lambda)$ is a breakeven cycle, which traverses the arcs $(a(0), \ldots, a(k))$ starting and ending at vertex v, then by Lemma 2.4.14 we have $\underrightarrow{\omega(v, f)} = \underrightarrow{1 - \lambda(C)}$. Since C is breakeven $\lambda(C) = 1$, this implies generalized flow-conservation in v. Since by definition generalized flow-conservation holds for all other vertices we may conclude that f is a generalized flow, i.e., a nontrivial flow on H.

Let $H \in \mathcal{B}(D_\Lambda)$ be a bicycle which traverses the arcs $(a(0), \ldots, a(k))$ such that $C = (a(0), \ldots, a(i))$, $P = (a(i + 1), \ldots, a(j - 1))$ and $C' = (a(j), \ldots, a(k))$. Let v and v' be the initial vertices of C and C', respectively.

Consider the case where P is non-trivial. We construct $f \in \mathsf{F}(D_\Lambda)$ with $\underrightarrow{f} = \underrightarrow{H}$. First take any inner flow f_C of C with $\underrightarrow{f_C} = \underrightarrow{C}$. Since C is gainy Lemma 2.4.14 implies a

positive excess at v. Let f_P be an inner flow of P with $\underrightarrow{f_P} = \underrightarrow{P}$. Lemma 2.4.13 ensures $\omega(v, f_P) < 0$. By scaling f_P with a positive scalar we can achieve $\omega(v, f_C + f_P) = 0$. From Lemma 2.4.13 we know that $f_C + f_P$ has positive excess at v'. Since C' is lossy any inner flow $f_{C'}$ of C' has negative excess at v' (Lemma 2.4.14). Hence we can scale $f_{C'}$ to achieve $\omega(v', f_{C'} + f_P) = 0$. Together we have obtained a generalized flow $f := f_C + f_P + f_{C'}$, i.e., a nontrivial flow on H.

If P is empty, then v and v' coincide. As in the above construction we can scale flows on C and C' such that $\omega(v, f) = 0$ holds for $f := f_C + f_{C'}$, i.e., f is a generalized flow. If C and C' share an interval, then the sign vectors of C and C' coincide on this interval. From $\underrightarrow{f_C} = \underrightarrow{C}$ and $\underrightarrow{f'_C} = \underrightarrow{C'}$ it follows that $\underrightarrow{f} = \underrightarrow{C \cup C'} = \underrightarrow{H}$. Hence f is a flow on H. $\qquad\square$

As a last lemma for the description of generalized cycles we need:

Lemma 2.4.16. *A bicycle does not contain a breakeven cycle.*

Proof. The cycles C and C' of a bicycle $H = C \cup P \cup C'$ are not breakeven. If H contains an additional cycle $\widetilde{C}$, then the support of $\widetilde{C}$ must equal the symmetric difference of supports of C and C'. Let $x := \lambda(C \setminus C')$, $y := \lambda(C \cap C')$, and $z := \lambda(C' \setminus C)$, where orientations are taken according to C and C', respectively. We have $xy = \lambda(C) > 1 > \lambda(C') = zy$. Hence $\lambda(\widetilde{C}) = (zx^{-1})^{\pm 1}$, but $zx^{-1} = zy(xy)^{-1} < 1$. That is, $\widetilde{C}$ cannot be breakeven. $\qquad\square$

Theorem 2.4.17. *For an arc-parameterized digraph D_Λ the set of the supports of generalized cycles, i.e., of support-minimal flows, coincides with the set of bicycles and breakeven cycles. Stated more formally: $\underrightarrow{\mathsf{C}}(D_\Lambda) = \mathcal{B}(D_\Lambda)$.*

Proof. By Theorem 2.4.15 every $H \in \mathcal{B}(D_\Lambda)$ admits a generalized flow f. To see support-minimality of f, assume that $H \in \mathcal{B}(D_\Lambda)$ has a strict subset S which is support-minimal admitting a generalized flow. Clearly S cannot have vertices of degree 1 to admit a flow and must be connected to be support-minimal. Since $S \subset H \in \mathcal{B}(D_\Lambda)$ this implies that S is a cycle. Lemma 2.4.14 ensures that S must be a breakeven cycle. If H was a breakeven cycle itself, then it cannot strictly contain S. Otherwise if $H = C \cup P \cup C'$ is a bicycle then by Lemma 2.4.16 it contains no breakeven cycle.

For the converse consider any $S \in \underrightarrow{\mathsf{C}}(D_\Lambda)$, i.e., the signed support of some flow f. We claim that $\underline{S} := \underrightarrow{f}$ contains a breakeven cycle or a bicycle. If it contains a breakeven cycle, then we are done. So we assume that it does not. Under this assumption it follows that there are two cycles C_1, C_2 in a connected component of $\underline{S}$. If C_1 and C_2 intersect in at most one vertex, then, since reorientation corresponds to inverting the multiplier, we can choose the orientations for these cycles such that $\lambda(C_1) > 1$ and $\lambda(C_2) < 1$. If $C_1 \cap C_2 = \emptyset$, then let P be an oriented path from C_1 to C_2. Now $C_1 \cup P \cup C_2$ is a bicycle contained in $\underline{S}$. The final case is that C_1 and C_2 share several vertices. Let B be an interval of C_2 over C_1, i.e., a consecutive piece of C_2 that intersects C_1 in its two endpoints v and w only. The union of C_1 and B is a theta-graph, i.e., it consists of three disjoint path B_1, B_2, B_3 joining v and w, see Figure 2.8. Let the three paths be oriented as shown in the figure, i.e., not all in the same direction, and let $C = B_1 \cup B_2$ and $C' = B_2 \cup B_3$. If $C \cup C'$ is not a bicycle,

then the cycles are either both gainy or both lossy. Assume that they are both gainy, i.e., $\lambda(C) > 1$ and $\lambda(C') > 1$. Consider the cycles $E = B_1 \cup B_3^{-1}$ and $E' = B_1^{-1} \cup B_3$, since $\lambda(E) = \lambda(B_1)\lambda(B_3)^{-1} = \lambda(E')^{-1}$ it follows that either E or E' is a lossy cycle. The orientation of E is consistent with C and the orientation of E' is consistent with C'. Hence either $C \cup E$ or $C' \cup E'$ is a bicycle contained in $\underline{S}$. This contradicts the support-minimality of f. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$
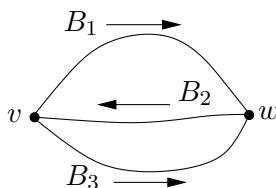
Figure 2.8: A theta graph and an orientation of the three paths.

We are now ready to obtain a characterization of generalized tensions which clearly generalizes the one of tensions in Section 1.4. For $H \in \mathcal{B}(D_\Lambda)$ we define $f(H)$ as the unique $f \in \mathsf{C}(D)$ with $\underline{f(H)} = \underline{H}$ and $\|f(H)\| = 1$. Let $x \in \mathbb{R}^m$ and $H \in \mathcal{B}(D_\Lambda)$. Denote by $\delta(H, x) := \langle x, f(\overrightarrow{H}) \rangle$ the *bicircular balance* of x on H.   bicircular balance

Theorem 2.4.18. *Let D_Λ be an arc-parameterized digraph and $x, c \in \mathbb{R}^m$. Then $x \in \mathsf{T}(D_\Lambda)_{\leq c}$ if and only if*

(1) $x_a \leq c_a$ *for all $a \in A$.*   (capacity constraints)

(2) $\delta(H, x) = 0$ *for all $H \in \mathcal{B}(D_\Lambda)$.*   (bicircular balance conditions)

The theorem helps to explain the name *generalized tensions*: usually a tension is a vector $x \in \mathbb{Z}^A$ such that for every cycle C its signed characteristic vectors is orthogonal to x, i.e., $\langle x, \overrightarrow{\chi}(C) \rangle = 0$. In our context the role of cycles is played by generalized cycles, i.e., by generalized flows f with $\underrightarrow{f} = \underrightarrow{H}$ for some $H \in \mathcal{B}(D_\Lambda)$.

Remark 2.4.19. The appearance of Theorem 2.4.18 is based on the analysis of the set $\mathcal{B}(D_\Lambda)$. The combinatorial description of $\mathcal{B}(D_\Lambda)$ in terms of oriented matroids (see Remark 2.4.11) is what sheds a particularly interesting light on generalized tensions. While we have already characterized and understood the circuits of our oriented matroids, finding a combinatorial characterization of the signed supports of support-minimal tensions corresponds to a characterization of the cocircuits of the oriented matroid and enrich the theory.

We want to make the statement of the Theorem 2.4.18 more general and for aesthetical reasons we want to make it resemble the first main theorem about Δ-tensions in Section 1.4 (Theorem 1.4.9). For the case of making the analogy more apparent we will use Theorem 2.4.18 as a definition and the definition of generalized tensions as a theorem. So let D_Λ be an arc-parameterized digraph with upper and lower arc capacities $c_u, c_\ell \in \mathbb{R}^m$, respectively, and a number Δ_H for each $H \in \mathcal{B}(D_\Lambda)$. A vector $x \in \mathbb{R}^m$ is called a *generalized Δ-tension* if   generalized Δ-tension

(B_1) $c_\ell(a) \leq x(a) \leq c_u(a)$ for all $a \in A$. (capacity constraints)

(B_2) $\delta(H, x) = \Delta_H$ for all $H \in \mathcal{B}(D_\Lambda)$. (bicircular Δ-balance conditions)

Denote by $\mathsf{T}_\Delta(D_\Lambda, c)$ the set of generalized Δ-tensions of D_Λ, note the similarity to the definition of Δ-tensions (Definition 1.4.1). Arguments as in Section 1.4 yield that $\mathsf{T}_\Delta(D_\Lambda, c)$ is just a translation in the arc-space of some $\mathsf{T}(D'_{\Lambda'}, c')$. Thus we obtain the real-valued generalization of Theorem 2.4.9 as a corollary of Theorem 2.4.6.

Theorem 2.4.20. *Let D_Λ be an arc-parameterized digraph with capacities $c \in \mathbb{R}^m$ and $\Delta \in \mathbb{R}^{\mathcal{B}(D_\Lambda)}$. The set $\mathsf{T}_\Delta(D_\Lambda, c)$ of generalized Δ-tensions carries the structure of a distributive lattice and forms a polyhedron.*

2.4.4 Planar Generalized Flow

As an application of the theory of generalized tensions in this subsection we prove a distributive lattice structure on certain classes of generalized flows of planar digraphs. This can be understood as a generalization of the results in Subsection 1.4.3. The *dual digraph D^** of a crossing-free embedding of a 2-connected planar digraph D in the sphere is an orientation of the planar dual G^* of the underlying graph G of D: Orient an edge $\{v, w\}$ of G^* from v to w if it appears as a forward arc in the clockwise facial cycle of D dual to w. Call an arc-parameterized digraph D_Λ *breakeven* if all its cycles are breakeven.

Theorem 2.4.21. *Let D_Λ be a planar breakeven digraph. There is an arc parameterization Λ^* of the dual D^* of D such that $\mathsf{F}(D_\Lambda) \cong \mathsf{T}(D^*_{\Lambda^*})$. More precisely, there is a vector $\sigma \in \mathbb{R}^m$ with positive components such that f is a generalized flow of D_Λ if and only if $x = S(\sigma)f$ is a generalized tension of $D^*_{\Lambda^*}$, where $S(\sigma)$ denotes the diagonal matrix with entries from σ.*

Proof. Let $C_1, \ldots, C_{n^*}$ be the list of clockwise oriented facial cycles of D. For each C_i let f_i be a generalized flow with $\underrightarrow{f_i} = \underrightarrow{C_i}$; since C_i is breakeven such an f_i exists by Lemma 2.4.14. Collect the flows f_i as rows of a matrix M. Columns of M correspond to edges of D and due to our selection of cycles each column contains exactly two non-zero entries. The orientation of the facial cycles and the sign condition implies that each column has a positive and a negative entry. For the column of arc a let $\mu_a > 0$ and $\nu_a < 0$ be the positive and negative entry. Define $\sigma_a := \mu_a^{-1} > 0$ and note that scaling the column of a with σ_a yields entries 1 and $-\lambda_a^* = \nu_a \mu_a^{-1} < 0$ in this column. Therefore, $N_{\Lambda^*} := MS(\sigma)$ is a generalized network matrix. The construction implies that the underlying digraph of N_{Λ^*} is just the dual D^* of D.

Let $f \in \mathsf{F}(D_\Lambda)$ be a flow. Then f can be expressed as linear combination of generalized cycles. Since D_Λ is breakeven we know that the support of every generalized cycle is a simple cycle. The facial cycles generate the cycle space of D. Moreover, if C is a simple cycle and f_C is a flow with $\underrightarrow{f_C} = \underrightarrow{C}$, then f_C can be expressed as a linear combination of

the flows f_i, $i = 1, \dots, n^*$. This implies that the rows of M are spanning for $\mathsf{F}(D_\Lambda)$, i.e., for every f there is a $q \in \mathbb{R}^{n^*}$ such that $f = M^T q$. In other words $\mathsf{F}(D_\Lambda) = M^T \mathbb{R}^{n^*}$.

A vector x is a tension for N_{Λ^*} if and only if x is in the row space of N_{Λ^*}, i.e., there is a potential $p \in \mathbb{R}^{n^*}$ with $x = N_{\Lambda^*}^T p$. In other words

$$\mathsf{T}(D_{\Lambda^*}^*) = N_{\Lambda^*}^T \mathbb{R}^{n^*} = (MS(\sigma))^T \mathbb{R}^{n^*} = S(\sigma) M^T \mathbb{R}^{n^*} = S(\sigma) \mathsf{F}(D_\Lambda).$$

$\square$

Corollary 2.4.22. *Let D_Λ be a planar breakeven digraph and $c \in \mathbb{R}^m$. The set $\mathsf{F}(D_\Lambda)_{\leq c}$ carries the structure of a distributive lattice.*

Proof. The matrix $S(\sigma)$ is an isomorphism between $\mathsf{F}(D_\Lambda)$ and $\mathsf{T}(D_{\Lambda^*}^*)$. Since σ is positive we obtain $\mathsf{F}(D_\Lambda)_{\leq c} = S(\sigma) \mathsf{T}(D_{\Lambda^*}^*, \leq S(\sigma)c$. Theorem 2.4.20 implies a distributive lattice structure on $\mathsf{T}(D_{\Lambda^*}^*)_{\leq S(\sigma)c}$ which can be pushed to $\mathsf{F}(D_\Lambda)_{\leq c}$. $\square$

In fact Theorem 2.4.20 even allows us to obtain a distributive lattice structure for planar generalized flows of breakeven digraphs with an arbitrarily prescribed excess at every vertex.

The reader may worry about the existence of non-trivial arc-parameterizations Λ of a digraph D such that D_Λ is breakeven. Here is a nice construction for such parameterizations. Let D be arbitrary and $x \in \mathbb{R}^m$ be a **0**-tension of D, i.e., $\delta(C, x) := \sum_{a \in C^+} x_a - \sum_{a \in C^-} x_a = 0$ for all oriented cycles C. Consider $\lambda = \exp(x)$ and note that $\lambda_a \geq 0$ for all arcs a and that $\lambda(C) = \left(\prod_{a \in C^+} \lambda_a \right) \left(\prod_{a \in C^-} \lambda_a \right)^{-1} = \exp(\delta(C, x)) = 1$ for all oriented cycles C. Hence weighting the arcs of D with λ yields a breakeven arc-parameterization of D. This construction is universal in the sense that application of the logarithm to a breakeven parameterization yields a **0**-tension.

2.5 Conclusions

Old and New

In the present chapter we have obtained a distributive lattice representation for the set of real-valued generalized Δ-tensions of an arc-parameterized digraph. The proof is based on the bijection between generalized tensions and vertex-potentials. This way the tensions inherit the lattice structure based on componentwise max and min from the vertex-potentials. Consequently we obtain a distributive lattice on the generalized tensions.

In Section 1.4 of the previous chapter we obtained the distributive lattice structure on integral Δ-tensions, by showing that we can build the cover-graph of a distributive lattice by local vertex-push-operations and reach every Δ-tension this way. This qualitatively different distributive lattice representation was possible because we could assume the digraph to be *reduced* in a certain way.

Question 2.5.1. Is there a way to reduce an arc-parameterized digraph such that the distributive lattice on its generalized tensions can be constructed locally by *pushing* vertices?
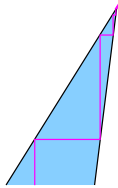
A reduction to D_Λ such that $\mathsf{P}(D_\Lambda, c)$ is full-dimensional, seem to be sufficient, when we ask for pure push-connectivity of the space of generalized tensions. If we additionally require that every pair of generalized tensions is connected via a *finite* sequence of vertex-pushes, we will have to seriously restrict the set of arc-parameterized digraphs. See the figure for a bad example. An interesting class of D-polyhedra which if push-connected is also finitely push-connected are D-polyhedra coming from breakeven arc-parameterized digraphs.

Order Theory

There is a natural *finite* (upper locally) distributive lattice associated to a (upper locally) distributive polytope P. Start from the vertices of P and consider the closure of this set under join and in the distributive case also meet. Let $\mathcal{L}(\mathsf{P})$ be the resulting (upper locally) distributive *vertex lattice* of P. It would be interesting to know what information regarding P is already contained in $\mathcal{L}(\mathsf{P})$.

Question 2.5.2. What do the generalized tensions associated to the elements of $\mathcal{L}(\mathsf{P})$ look like for a distributive P? In particular some special generalized tensions of $\mathcal{L}(\mathsf{P})$ including join-irreducible, minimal and maximal elements are of interest.

Another question arises, when viewing the results of this section as generalizations of results related to the order polytope. Feasible polytopes of antimatroids may be seen as order polytopes of antichain-partitioned posets. In [102] Stanley describes the *chain-polytope* as a polytope closely related to the order polytope.
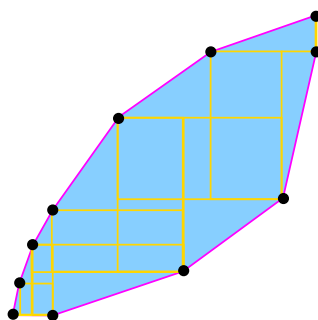
Figure 2.9: The vertex lattice of a distributive polytope.

Question 2.5.3. Is there a generalization of the chain-polytope to antichain-partitioned posets?

An answer to this question may contribute new insights to the study of feasible polytopes of antimatroids. Similarly, the integer points of alcoved polytopes form a distributive lattice as shown in Section 2.4.2. By Dilworth's Embedding Theorem for Distributive Lattices such embedded lattices correspond to chain-partitions of posets. In the case of alcoved polytopes the corresponding chain-partitons may actually be characterized as so-called *plissée partitions*.

Question 2.5.4. Is there a generalization of the chain-polytope to chain-partitioned posets?

An affirmative result into this direction would enrich the theory of alcoved polytopes.

Geometry

We have derived an $\mathcal{H}$-description of D-polyhedra and ULD-polyhedra. The fact that the set of vertices is closed under $\min$ and $\max$ does not imply that the polytope is distributive. The vertices of a *cyclic polytope* form a chain in the dominance order. It can be checked that already the 3-dimensional cyclic polytope on 4 vertices $C_3(4) \subseteq \mathbb{R}^3$ is no D-polytope. More explicitely, a $\mathcal{H}$-description of $C_4(8)$ violating Theorem 2.4.2 may be found in [110].

Question 2.5.5. What does a $\mathcal{V}$-description of (upper locally) distributive polytopes look like? (This again asks for a special set of elements of the vertex lattice $\mathcal{L}(\mathsf{P})$.)

On the other hand if a full-dimensional *0/1-polytope* has $\min$- and $\max$-closed vertex-set, then it is an *order-polytope* and consequently distributive. If a full-dimensional *0/1-polytope* has only $\max$-closed vertex-set, then it is the feasible polytope of an antimatroid.

Question 2.5.6. Are feasible polytopes of an antimatroids ULD-polytopes?

We have been working with sublattices of the dominance order on $\mathbb{R}^d$. More generally one could look at a *Riesz space*, *lattice-ordered vector space* or *vector lattice* on $\mathbb{R}^d$. A Riesz space is always a distributive lattice and (in the finite dimensional case) it may be characterized by the full-dimensional cone $\uparrow x$ of any element of the space [107], see Observation 2.1.3.

Question 2.5.7. What generalizations of ULD- and D-polytopes can be obtained, when looking at general Riesz spaces instead of the dominance order?

Matroid Theory

In Section 2.4.3 we have related arc-parameterized digraphs to *bicircular oriented matroids*, see Remark 2.4.11 and Remark 2.4.19. We have characterized the circuits of the mixed bicircular oriented matroids, we are dealing with. Still we miss a combinatorial description of the signed supports of minimal generalized tensions.

Question 2.5.8. What does the cocircuit of the mixed bicircular oriented matroids related to generalized tensions look like?

Apart from these bicircular oriented matroids the face lattice of a D-polyhedron is a geometric lattice, hence it encodes a simple matroid, see [89]. Moreover faces of D-polyhedra are D-polyhedra (Lemma 2.4.1). In the spirit of [109] it would be interesting to determine the subgraphs of D_Λ, which correspond to faces of $\mathsf{P}(D_\Lambda, c)$.

Question 2.5.9. What is the relation between these two matroids? What do face lattices of D-polyhedra look like?

Optimization

There has been a considerable amount of research concerned with algorithms for generalized flows, see [3] for references. As far as we know it has never been taken into account that the LP-dual problem of a min-cost generalized flow is an optimization problem on a D-polyhedron. We feel that it might be fruitful to look at this connection. A special case is given by generalized flows of planar breakeven digraphs, where the flow-polyhedron is affinely isomorphic to a distributive polyhedron (Corollary 2.4.22).

In particular, it would be interesting to understand the integral points of a D-polyhedron, which by Observation 2.4.8 form a distributive lattice. Related to this and to [42] is the following:

Question 2.5.10. Find conditions on Λ and c such that the set of integral generalized tensions for these parameters forms a distributive lattice.

Another question related to optimization issues, is the membership-problem discussed in Subsection 2.3.1.

Question 2.5.11. Our methods enriched the set of inequalities known of the $\mathcal{H}$-description of the feasible polytope of an antimatroid. Does this help to find more antimatroids for which the membership-problem may be efficiently solved?

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Chapter 3

Cocircuit Graphs of Uniform Oriented Matroids

The notion of oriented matroids (OMs) is a combinatorial abstraction of linear subspaces of the Euclidean space $\mathbb{R}^d$. The theory of OMs has applications and connections to many areas, including combinatorics, discrete and computational geometry, optimization, and graph theory; see e.g. Björner et al. [19]. OMs have several different representations. The translation from one into another representation are of practical interest; the present chapter discusses graph representations of OMs, focussing on algorithms and their complexity, and extends the work of Cordovil, Fukuda, and Guedes de Oliveira [28], Babson, Finschi, and Fukuda [7], and Montellano-Ballesteros and Strausz [85].
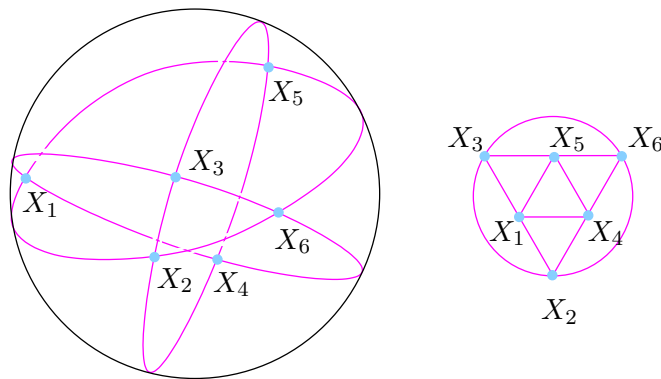
Figure 3.1: A simple spherical pseudoline-arrangement and its cocircuit graph.

OMs may be represented by systems of signed sets (see Definition 3.1.1 for the uniform case). The Topological Representation Theorem of Folkman and Lawrence [45] says that every oriented matroid can be represented as a finite *arrangement of pseudospheres* in $\mathbb{R}^d$. The signed set representation then is derived from the cells of the resulting cell complex.

The cocircuit graph is the 1-skeleton of such an arrangement of pseudospheres. In the case of spherical pseudoline-arrangements in $\mathbb{R}^3$, i.e., rank 3 oriented matroids, its vertices are the intersection points of the lines and two points share an edge if they are adjacent on a line. See Figure 3.1 for an example.

Compared to the set of signed sets of a cell complex, the cocircuit graph is a compact and simple structure. It is a natural question, whether the cocircuit graph of an OM determines the cell complex.

Cordovil, Fukuda and Guedes de Oliveira [28] show that this is not true for general OMs. Nevertheless they show that a *uniform* oriented matroid is determined by its cocircuit graph together with an *antipodal labeling*. Babson, Finschi and Fukuda [7] provide a polynomial time recognition algorithm for cocircuit graphs of uniform oriented matroids, which reconstructs a uniform oriented matroid from its cocircuit graph up to isomorphism.

In [85], Montellano-Ballesteros and Strausz provide a characterization of cocircuit graphs of uniform oriented matroids in terms of a certain connectivity of sign-labeled graphs by signed paths.

After introducing basic notions of oriented matroids, we prove a stronger version of the characterization of [85], i.e., a new characterization of cocircuit graphs of uniform oriented matroids (Theorem 3.1.4). Afterwards, we describe an algorithm which, given a graph G, decides in cubic time if G is the cocircuit We obtain an essentially better runtime than Babson, Finschi and Fukuda [7]. This in particular answers a question of Babson et al. However, some parts of our algorithm are identical to parts of the one in [7].

At the end we will look at another question posed in [7] concerning *antipodality* in cocircuit graphs, which is essential for reducing the runtime of the algorithm furthermore. We support the feeling that the antipodality problem is deep and hard by showing that a quite natural-seeming assumption about cocircuit graphs of uniform oriented matroids implies the Hirsch conjecture. Since the latter was recently disproved by Santos [98] the assumption is false and without it it is not clear how to progress on some of the problems related to antipodality.

The present work is also related to *Perles's conjecture* which says that the 1-skeleton of a simple d-dimensional polytope determines its face lattice; this conjecture was first proved by Blind and Mani-Levitska [23] and then constructively by Kalai [64]. The present work extends the discussion of Perles's conjecture to a class of non-simple polytopes. Joswig [62] conjectured that every cubical polytope can be reconstructed from its dual graph; our result proves this conjecture for the special case of cubical zonotopes up to graph isomorphism. In other words, the face lattice of every cubical zonotope is uniquely determined by its dual graph up to isomorphism.

3.1 Properties of Cocircuit Graphs

In this section we will define the basic concepts needed for the discussion of uniform oriented matroids. The notions introduced here are specialized to uniform oriented matroids, for a more general introduction see [19]. After proving some basic lemmas we will reprove and generalize a characterization of cocircuit graphs of uniform oriented matroids due to [85].

We recall the definition of signed set from the last chapter. A *signed set* $X = (X^+, X^-)$ signed set
is a pair of disjoint subsets $X^+, X^- \subseteq E$ of a ground set E. The *support of X*   support
is $\underline{X} := X^+ \cup X^-$. For $e \in \underline{X}$ we write $X_e = \pm 1$ if $e \in X^{\pm}$, respectively. If
$e \notin \underline{X}$, then we write $X_e = 0$. By X^0 we denote the *zero-support $E \backslash \underline{X}$*. The   zero-support
separator of two signed sets X, Y is defined as $S(X, Y) := \{e \in E \mid \{X_e, Y_e\} = \{+, -\}\}$.   separator
For a signed set X the signed set $-X$ is the one where all signs are reversed, i.e.,
$-X = (X^+, X^-)$.

Definition 3.1.1. We define a *uniform oriented matroid* of *rank r* as a pair $\mathcal{M} = (E, \mathcal{C}^*)$ uniform oriented
where $\mathcal{C}^*$ is a system of signed sets with ground set E. The elements of $\mathcal{C}^*$ are the *cocircuits*   matroid (UOM)
of $\mathcal{M}$. Denote by n the size of E. Then $\mathcal{C}^*$ must satisfy the following axioms: rank
     cocircuit

(C1) Every $X \in \mathcal{C}^*$ has support of size $n - r + 1$.
(C2) For every $I \subseteq E$ of size $n - r + 1$ there are exactly two cocircuits X, Y with support
     I. Moreover $-X = Y$.
(C3) For every $X, Y \in \mathcal{C}^*$ and $e \in S(X, Y)$ there is a $Z \in \mathcal{C}^*$ with $Z_e = 0$ and
     $Z_f \in \{X_f, Y_f, 0\}$, for every $f \in E \backslash \{e\}$.

In the rest of this chapter we will abbreviate a uniform oriented matroid by UOM. On the
set $\mathcal{C}^*$ of cocircuits of a UOM $\mathcal{M}$ one defines the *cocircuit graph $G_{\mathcal{M}}$* by making X and Y cocircuit graph
adjacent if they differ only a "little bit", i.e., $|X^0 \Delta Y^0| = 2$ and $S(X, Y) = \emptyset$.

A more general notion is the following. Given a graph $G = (V, \mathcal{E})$ with vertices V and
edges $\mathcal{E}$ let $\ell : V \to \mathcal{S}$ be a bijection to a set of signed sets $\mathcal{S}$ on a ground set E, which
satisfies axioms (C1) and (C2). We call ℓ a *sign labeling* of G if we have $\{v, w\} \in \mathcal{E}$ if and   sign labeling
only if $|\ell(v)^0 \Delta \ell(w)^0| = 2$ and $S(\ell(v), \ell(w)) = \emptyset$. Every sign labeling ℓ of G comes with
the two parameters r and n.

If G has a sign labeling with $\mathcal{S}$ satisfying also axiom (C3), i.e., $\mathcal{C}^* := \mathcal{S}$ is the set of
cocircuits of a UOM $\mathcal{M}$, then G is a cocircuit graph $G_{\mathcal{M}}$. We then call ℓ a *UOM-labeling*.   UOM-labeling
In [7] it is shown that $G_{\mathcal{M}} \cong G_{\mathcal{M}'}$ if and only if $\mathcal{M} \cong \mathcal{M}'$.

By (C1) and (C2) $G_{\mathcal{M}}$ clearly has exactly $2\binom{n}{n-r+1}$ vertices. In this section, such as the
following, all the lemmas are well-known.

Lemma 3.1.2. *Let $G_{\mathcal{M}} = (V, \mathcal{E})$ be a cocircuit graph with UOM-labeling ℓ and $v \in V$.
Then for every $f \in \ell(v)^0$ there are exactly two neighbors u, w of v with $\ell(u)_f = -$ and
$\ell(w)_f = +$. In particular $G_{\mathcal{M}}$ is $2(r-1)$-regular.*

Proof. Let ℓ be a UOM-labeling v a vertex and $f \in \ell(v)^0$. Let w be a vertex with $\ell(w)_f = +$ and $\overline{\ell(w)\backslash\{f\}} \subseteq \overline{\ell(v)}$, such that $S(\ell(v), \ell(w))$ is minimal. If there is an $e \in S(\ell(v), \ell(w))$ then we apply (C3) to $\ell(v), \ell(w)$ with respect to e. We obtain $\ell(u)$ with $\overline{\ell(u)\backslash\{f\}} \subseteq \overline{\ell(v)}\backslash\{e\}$ and $\ell(u)_f = +$. Since $\ell(u)_e = 0$ and $\ell(u)_g \in \{\ell(v)_g, \ell(w)_g, 0\}$ for $g \neq e$ we have $S(\ell(v), \ell(u)) \subset S(\ell(v), \ell(w))$, a contradiction. Thus, w is adjacent to v. If there was another neighbor w' of v with $\ell(w')_f = +$ then (C3) applied to $\ell(w')$ and $-\ell(w)$ with respect to f would yield a cocircuit $\ell(u)$ with $\overline{\ell(u)} = \overline{\ell(v)}$. It is easy to see that $\ell(u) \neq \pm\ell(v)$, a contradiction to (C2). $\qquad\square$

A *contraction minor* of a UOM $\mathcal{M} = (E, \mathcal{C}^*)$ is a UOM of the form $\mathcal{M}/E' = (E\backslash E', \mathcal{C}^*/E')$ where $E' \subseteq E$ and $\mathcal{C}^*/E' := \{X_{E\backslash E'} \mid X \in \mathcal{C}^*, E' \subseteq X^0\}$. Here $X_{E\backslash E'}$ denotes the restriction of X to the ground set $E\backslash E'$. Generally $G_{\mathcal{M}/E'}$ is an induced subgraph of $G_{\mathcal{M}}$. The rank of $\mathcal{M}/E'$ is $r - |E'|$. If $\mathcal{M}/E'$ has rank 2 we call it a

coline of $\mathcal{M}$.

Lemma 3.1.3. *Let $G_{\mathcal{M}}$ be a cocircuit graph with UOM-labeling ℓ and $v, w \in V$ with $\ell(v) \neq -\ell(w)$. If $\ell(v)$ and $\ell(w)$ lie in a coline $\mathcal{M}'$ then $d(v, w) = |S(\ell(v), \ell(w))| + \frac{1}{2}|\ell(v)^0 \Delta \ell(w)^0|$ and the unique (v, w)-path of this length lies in $G_{\mathcal{M}'}$.*

Proof. First note that $|S(\ell(v), \ell(w))| + \frac{1}{2}|\ell(v)^0 \Delta \ell(w)^0|$ is a lower bound for the distance in any sign labeled graph. These are just the necessary changes to transform one signed set into the other. To see that in a coline there exists a path of this length we use induction on $|S(\ell(v), \ell(w))| + \frac{1}{2}|\ell(v)^0 \Delta \ell(w)^0|$. The induction base is clear per definition of sign labeling. So we proceed with the induction step.

Since v, w lie in a coline we have $\ell(v)^0 \backslash \ell(w)^0 = \{e\}$ for some $e \in E$. By Lemma 3.1.2 vertex v has a unique neighbor u with $\ell(u)_e = \ell(w)_e$. If $S(\ell(u), \ell(w)) = \emptyset$ we have $u = w$, because otherwise v would have two neighbors with $\ell(u)_e = \ell(w)_e$, a contradiction to Lemma 3.1.2. If $\ell(u)_f = 0$ but $\ell(v)_f = \ell(w)_f \neq 0$ for some $f \in E$ then we can apply (C3) to $\ell(w)$ and $-\ell(u)$ with respect to e. Any resulting signed set has support $\overline{\ell(v)}$. Since $S(\ell(u), \ell(w)) \neq \emptyset$ it cannot have sign labeling $-\ell(v)$. On the other hand its f-entry equals $-\ell(v)_f$, i.e., it cannot have sign labeling $\ell(v)$ either, a contradiction to (C2). This yields $|S(\ell(u), \ell(w))| = |S(\ell(v), \ell(w))| - 1$. Moreover $|\ell(v)^0 \Delta \ell(w)^0|$ is not decreased, since v, w are not adjacent. Applying the induction hypothesis gives the result.

For a neighbor u' of v not in a coline with v, w it is easy to check that $|S(\ell(u'), \ell(w))| + \frac{1}{2}|\ell(u')^0 \Delta \ell(w)^0| \geq |S(\ell(v), \ell(w))| + \frac{1}{2}|\ell(v)^0 \Delta \ell(w)^0|$. Hence u' cannot lie on a shortest (v, w)-path. $\qquad\square$

Let G be a graph with sign labeling $\ell : V \to \mathcal{S}$. Let $X, Y \in \mathcal{S}$. We say that a path P in G is (X, Y)-*crabbed* if for every vertex $w \in P$ we have $\ell(w)^+ \subseteq X^+ \cup Y^+$ and

$\ell(w)^- \subseteq X^- \cup Y^-$. We call a (u, v)-path just *crabbed* if it is $(\ell(u), \ell(v))$-crabbed. The following theorem is a strengthening of the main result of [85]:

Theorem 3.1.4. *Let ℓ be a sign labeling of G. Then the following are equivalent:*

(i) ℓ *is a UOM-labeling.*

(ii) *For all* $v, w \in V$ *there are exactly* $|\ell(v)^0 \backslash \ell(w)^0|$ *vertex-disjoint crabbed* v, w*-paths.*

(iii) *For all* $v, w \in V$ *with* $\ell(v)^0 \neq \ell(w)^0$ *there exists a crabbed* v, w*-path.*

Proof. (i)$\Rightarrow$(ii): Let $G = G_\mathcal{M}$ a cocircuit graph. First by Lemma 3.1.2 it is clear that between any two vertices v, w there can be at most $|\ell(v)^0 \backslash \ell(w)^0|$ vertex-disjoint crabbed v, w-paths. For the other inequality we proceed by induction on the size of the ground set E of $\mathcal{M}$. The induction base is skipped. For the inductive step we have to distinguish three cases.

If there is some $e \in \ell(v)^0 \cap \ell(w)^0$ then consider the contraction minor $\mathcal{M}/\{e\}$. By induction hypothesis there are at least $|(\ell(v)^0 \backslash \{e\}) \backslash (\ell(w)^0 \backslash \{e\})| = |\ell(v)^0 \backslash \ell(w)^0|$ vertex-disjoint crabbed v, w-paths in $G_{\mathcal{M}/\{e\}}$. Since the latter is an induced subgraph of $G_\mathcal{M}$ we are done.

Otherwise, if $S(\ell(v), \ell(w)) = \emptyset$ then $\ell(v), \ell(w)$ lie in a *tope* of $\mathcal{M}$. This is the set of signed sets X with $X^+ \subseteq \ell(v)^+ \cup \ell(w)^+$ and $X^- \subseteq \ell(v)^- \cup \ell(w)^-$. Topes of a rank r UOM are $(r-1)$-dimensional PL-spheres and hence their graph is $(r-1)$-connected [19]. A less topological argument for the same fact can be found in [27]. *(tope)*

Otherwise, if there is some $e \in S(\ell(v), \ell(w))$ we consider the *deletion minor* $\mathcal{M}\backslash\{e\}$. It is the oriented matroid on the ground set $E\backslash\{e\}$ with cocircuit set $\mathcal{C}^*\backslash\{e\} := \{X_{E\backslash\{e\}} \mid X \in \mathcal{C}^*, X_e \neq 0\}$. By induction hypothesis there are $|\ell(v)^0 \backslash \ell(w)^0|$ vertex-disjoint crabbed v, w-paths in $G_{\mathcal{M}\backslash\{e\}}$. If on such a path P in $G_{\mathcal{M}\backslash\{e\}}$ two consecutive vertices x, y have $e = S(\ell(x), \ell(y))$ then we apply (C3) with respect to e. We obtain a unique vertex z with $\ell(z)_e = 0$ and $\ell(z)_f = \ell(x)_f$ if $\ell(x)_f \neq 0$ and $\ell(z)_f = \ell(y)_f$ otherwise. The new vertex z is adjacent to x, y in $G_\mathcal{M}$. In this way we can extend P to a crabbed (v, w)-path P' in $G_\mathcal{M}$. *(deletion minor)*

Now suppose two different extended paths P_1' and P_2' share a vertex z. Thus there are mutually different x_1, y_1, x_2, y_2 yielding z. This implies that their labels $\ell(x_1)_{E\backslash\{e\}}, \ell(y_1)_{E\backslash\{e\}}, \ell(x_2)_{E\backslash\{e\}}, \ell(y_2)_{E\backslash\{e\}}$ in $\mathcal{M}\backslash\{e\}$ have mutually empty separator. Moreover the zero-supports of these labels have mutually symmetrical difference two. Hence in $G_{\mathcal{M}\backslash\{e\}}$ the vertices x_1, y_1, x_2, y_2 induce a K_4. On the other hand $\ell(x_1)_{E\backslash\{e\}}, \ell(y_1)_{E\backslash\{e\}}, \ell(x_2)_{E\backslash\{e\}}, \ell(y_2)_{E\backslash\{e\}}$ in $\mathcal{M}\backslash\{e\}$ lie together in a rank 2 contraction minor of $\mathcal{M}\backslash\{e\}$. Thus by Lemma 3.1.3, the vertices x_1, y_1, x_2, y_2 should induce a subgraph of a cycle, a contradiction.

(ii)$\Rightarrow$(ii): Obvious.

(iii)$\Rightarrow$(i): We only have to check (C3). So let $\ell(v) \neq \pm\ell(w)$ be two labels and $e \in S(\ell(v), \ell(w))$. On any (v, w)-path P there must be a vertex u with $\ell(u)_e = 0$. If P is crabbed $\ell(u)$ satisfies (C3) for $\ell(v), \ell(w)$ with respect to e. $\qquad\square$

Cocircuit graphs of general oriented matroids are $2(r-1)$-connected [27]. Here we have shown a crabbed analogue for uniform oriented matroids.

Question 3.1.5. An interesting question would be if there is a result similar to Theorem 3.1.4 for non-uniform oriented matroids.

Let us now turn to another basic concept for the recognition of cocircuit graphs. A sign labeling ℓ of $G_{\mathcal{M}}$ induces the map A_ℓ which takes v to the unique vertex w with $\ell(w) = -\ell(v)$. We call A_ℓ the *AP-labeling* induced by ℓ.

AP-labeling

Lemma 3.1.6. *If ℓ is a UOM-labeling then A_ℓ is an involution in $\mathrm{Aut}(G_{\mathcal{M}})$ which satisfies $d(v, A_\ell(v)) = n - r + 2$ for every $v \in V$.*

Proof. Let ℓ be the UOM-labeling inducing the AP-labeling A_ℓ. By the definition of sign labeling it is clear that A_ℓ is an automorphism of order 2. Since any neighbor u of v lies in a coline with $A_\ell(v)$ and $|S(\ell(u), -\ell(v))| + \frac{1}{2}|\ell(u)^0 \Delta \ell(v)^0| = n - r + 1$, by Lemma 3.1.3 we have $d(v, A_\ell(v)) = n - r + 2$. $\qquad\square$

antipodal labeling

More generally, in a context where the parameters r, n are given an *antipodal labeling* of a graph G is an involution $A \in \mathrm{Aut}(G)$ such that the graph distance satisfies $d(v, A(v)) = n - r + 2$ for every $v \in V$.

3.2 The Algorithm

The input is an undirected simple connected graph $G = (V, \mathcal{E})$. The algorithm decides if $G = G_{\mathcal{M}}$ for some UOM $\mathcal{M} = (E, \mathcal{C}^*)$. In the affirmative case it returns $\mathcal{M}$. Otherwise one of the steps of the algorithm fails.

1. Check if G is $2(r-1)$-regular for some r.
2. Check if $|V| = 2\binom{n}{n-r+1}$ for some n.
3. Calculate the $V \times V$ distance matrix of G.
4. Fix $v \in V$ and define $D(v) := \{w \in V \mid d(v, w) = n - r + 2\}$.
5. For all $w \in D(v)$ do

    A. Construct an antipodal labeling A with $A(v) = w$.

    B. Construct a sign labeling ℓ of G with $A = A_\ell$.

    C. Check if ℓ is a UOM-labeling. If so, define $\mathcal{C}^* := \ell(V)$, return $(E, \mathcal{C}^*)$ and stop.

6. Return that G is no cocircuit graph.

Parts 1 and 2 run in time $|\mathcal{E}| = (r-1)|V|$ and are necessary to determine the parameters r and n of $\mathcal{M}$. The distance matrix is computed to avoid repeated application of shortest path algorithms during the main part of the algorithm. Since G is unweighted and undirected we can obtain its distance matrix in $O(|V||\mathcal{E}|)$, see for instance Chapter 6.2 of [100]. Hence we can do the first three parts in $O(r|V|^2)$.

For the rest of the algorithm we have to execute steps A to C at most $|D(v)|$ times. These will be explained in the following.

A. Construct an antipodal labeling.

Lemma 3.2.1. *Let $G_{\mathcal{M}}$ be a cocircuit graph with AP-labeling A_ℓ. If $A(v) = w$ and u is a neighbor of v then $A(u)$ is the unique neighbor u' of w with $d(u, u') = n - r + 2$.*

Proof. Suppose that, besides $u' = A(u)$, there is another neighbor u'' of w with $d(u, u'') = n - r + 2$. Since $d(u', u'') \leq 2$, we have $|\ell(u')^0 \Delta \ell(u'')^0| \leq 4$. Since ℓ is a UOM-labeling, $\ell(u), \ell(u'), \ell(u'')$ lie in a rank $r' = 3$ contraction minor on n' elements. Contraction in $\mathcal{M}$ just means deletion of vertices in $G_{\mathcal{M}}$. This implies that we have $d \leq d'$ for the distance functions of the cocircuit graphs of $\mathcal{M}$ and the contraction minor $\mathcal{M}'$, respectively. On the other hand $n' - r' + 2 = n - (r - 3) - 3 + 2 = n - r + 2$. This yields $d'(u, u') = d'(u, u'') = n - r + 2$ in a UOM of rank 3, which contradicts Lemma 3.3.1. $\square$

We obtain a simple breadth-first search algorithm that given $A(v) = w$ determines A in time $O(r^2|V|)$. Just walk from the root v through a breadth-first search tree. For vertex u with father $f(u)$ the vertex $A(f(u))$ is known. Look through the $2(r - 1)$ neighbors of $A(f(u))$. For the unique neighbor u' having $d(u, u') = n - r + 2$ set $A(u) = u'$. If u' is not unique or does not exist, then no AP-labeling A of G with $A(v) = w$ exists.

B. Construct a sign labeling of G.

We use the algorithm presented in [7], which given an antipodal labeling A tries to construct a sign labeling ℓ such that $A = A_\ell$. We will not go into detail here. Just to have a vague idea how to do this: Based on Lemma 3.1.3 and Lemma 3.1.6 the algorithm constructs an edge-partition of G into the subgraphs induced by the colines. Every coline may be defined by an r-set as the set of vertices with zero-support contained in that set. Using this and two intersection-lemmas for colines one assigns zero-supports to the vertices of G. This is done in a way unique up to isomorphism. In a last step one assigns signed-sets to the vertices, which then is unique up to isomorphism again. The algorithm either encounters a problem, i.e., a contradiction to some of the properties of cocircuit graphs, and returns that G is not a cocircuit graph or it returns a sign labeling ℓ with $A = A_\ell$. If A is an AP-labeling for some UOM-labeling ℓ, then the algorithm finds such an ℓ. If A is not the AP-labeling for a UOM-labeling the algorithm might return a non UOM-labeling with $A = A_\ell$. The algorithm runs in time $O(rn|V|)$.

C. Check if a sign labeling is a UOM-labeling.

We check for every $u \in V(G)$ if there is a crabbed path to every vertex $w \in V(G) \backslash \{A_\ell(u)\}$. By Theorem 3.1.4 this is equivalent to ℓ being a UOM-labeling. To improve running time we need the following simple lemma.

Lemma 3.2.2. *If a (u, v)-path P and a (v, w)-path P' are $(\ell(u), Y)$-crabbed and $(\ell(v), Y)$-crabbed, respectively, then their concatenation (P, P') is $(\ell(u), Y)$-crabbed.*

Proof. The vertices in P – in particular v – satisfy the conditions for being $(\ell(u), Y)$-crabbed. Hence for every $w \in P'$ we have $\ell(w)^+ \subseteq \ell(v)^+ \cup Y^+ \subseteq \ell(u)^+ \cup Y^+$. The analogue statement holds for $\ell(w)^+$. □

We are ready to describe the algorithm:

- For every $u \in V(G)$ do
    1. For every edge $\{v, w\}$ do
        – Delete the undirected edge $\{v, w\}$.
        – If (v, w) is $(\ell(v), \ell(u))$-crabbed insert that directed edge.
        – If (w, v) is $(\ell(w), \ell(u))$-crabbed insert that directed edge.
    2. Start a breadth-first search on the resulting directed graph G' at u such that only backward arcs are traversed.
    3. If not every vertex is reached by the search, return that ℓ is no UOM-labeling and stop.
- Return that ℓ is a UOM-labeling of G

Lemma 3.2.2 tells us that for checking if there is a crabbed (u, v)-path for every $v \in V(G)$ it is enough to check that if the directed graph G' has a directed path from every vertex to u. Step 2 does exactly this. Loop 1 will be executed $(r - 1)|V|$ times and each round costs $O(n - r)$ many comparisons. Step 2 runs in time linear in the edges. Since the whole process has to be repeated $|V|$ times, we need $O(r(n - r)|V|^2)$ many operations.

Overall Runtime

We add the runtimes of the single parts of the algorithm. We see that part C dominates all other parts of the algorithm, thus we obtain an overall runtime of $O(|D(v)|r(n - r)|V|^2)$. So far the best known upper bounds for the size of $D(v)$ are in $O(|V|)$, hence our runtime is $O(r(n - r)|V|^3)$. For comparison, the runtime of the algorithm in [7] is $O(rn^2|V|^4)$. The improvement of runtime comes from approaching step C in a new way. Already in [7] it was asked if in that part some better algorithm was possible.

3.3 Antipodality

The problem of bounding the size of $D(v)$ is hard. In the present section we will point out some open problems concerning this value. The following lemma is well-known.

Lemma 3.3.1. *If the rank of $\mathcal{M}$ is at most 3 then for $v, w \in V(G_{\mathcal{M}})$ we have $d(v, w) = n - r + 2$ if and only if $-\ell(v) = \ell(w)$. Moreover $n - r + 2$ is the diameter of $G_{\mathcal{M}}$.*

Proof. Let ℓ be a UOM-labeling of $G_{\mathcal{M}}$ with induced AP-labeling A_ℓ. Let v, w be vertices with $A_\ell(v) \neq w$. We observe the following:

For any shortest (v, w)-path P in $G_{\mathcal{M}}$ we have $P \cap A_\ell(P) = \emptyset$. On a shortest path there cannot occur anything like $(u', u, \ldots, A_\ell(u))$, because by Lemma 3.1.3, for every neighbor u' of u, we have $d(u', A_\ell(u)) = n - r + 1$ since u' and $A_\ell(u)$ lie in a coline. Hence, u' lies on a shortest $(u, A_\ell(u))$-path.

Every shortest path $P = (v = v_0, \ldots, v_k = w)$ satisfies $\ell(v_i)_e = 0$ and $\ell(v_{i+1})_e \neq 0$, implying $\ell(w)_e = \ell(v_{i+1})_e$. Otherwise there would be v_i, v_j in P lying in a coline $\mathcal{M}/e$, but the part of P connecting v_i, v_j would leave $\mathcal{M}/e$. Since $\ell(v_i) \neq -\ell(v_j)$, this contradicts Lemma 3.1.3.

This yields that, on a shortest (v, w)-path, we will have $\ell(v_i)_e = 0$ and $\ell(v_{i+1})_e \neq 0$ at most once per $e \notin \ell(v)^0 \cap \ell(w)^0$ and never if $e \in \ell(v)^0 \cap \ell(w)^0$. This yields $d(v, w) \leq |\underline{\ell(v)}| = n - r + 1$. $\square$

The proof actually shows that in a UOM of rank 3 every shortest path is crabbed. In [7] it was asked if the statement of Lemma 3.3.1 holds for every rank:

Question 3.3.2. Given a UOM of rank r on n elements, does $d(X, Y) = n - r + 2$ imply $-X = Y$? Is $n - r + 2$ the diameter of $G_{\mathcal{M}}$?

One could hope that the signed sets of two vertices u, v give some crucial information about how to connect them by a path. As in the case of rank 3 matroids one would like to use crabbed paths to prove something about the distance function of $G_{\mathcal{M}}$. More precisely, a *tope* in $\mathcal{M}$ is a maximal set $T \subseteq \mathcal{C}^*$ such that $S(X, Y) = \emptyset$ for every $X, Y \in T$. In particular cocircuits X, Y with $-X = Y$ are not contained in a common tope. So if the answers to the questions posed in Question 3.3.2 are both "yes", then cocircuits X, Y being contained in a common tope T must have distance less or equal to $n - r + 1$. In order to prove such thing one might hope, that even stronger there exists a crabbed (X, Y)-path of length at most $n - r + 1$ for all $X, Y \in T$. But unfortunately this is not generally true:

Proposition 3.3.3. *The assumption that for a fixed tope T of $\mathcal{M}$ every $X, Y \in T$ are connected by a crabbed path of length $n - r + 1$ implies the Hirsch conjecture. Hence the assumption is false [98].*

Proof. The Hirsch conjecture says that the graph of a d-dimensional simple polytope with f facets has diameter at most $f - d$. Take a d-dimensional simple polytope P with f facets in $\mathbb{R}^d$. The f facet-defining hyperplanes of P form an affine hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^d$. Put $\mathcal{H}$ into the $(x_{d+1} = 1)$-hyperplane of $\mathbb{R}^{d+1}$ and extend $\mathcal{H}$ to a central hyperplane arrangement $\mathcal{H}'$ in $\mathbb{R}^{d+1}$. The arrangement $\mathcal{H}'$ encodes a rank $d+1$ uniform oriented matroid

$\mathcal{M}$ on f elements. We obtain cone(P) as a maximal cell of $\mathcal{H}'$. This means that the vertices of P correspond to the cocircuits of a tope T_{P}. The graph of P is the subgraph of $G_{\mathcal{M}}$ induced by T_{P}. Now for two cocircuits in T_{P} a path connecting them is crabbed if and only if it is contained in T_{P}. The parameters of $\mathcal{M}$ are just so that the existence of a crabbed path of the desired length in T_{P} is equivalent to the existence of a path of length $f - d$ in the graph of P. □

The Hirsch conjecture is true in dimension 3 and topes of UOMs of rank 4 are combinatorially equivalent to simple polytopes of dimension 3. Thus, for UOMs of rank 4 the assumption of the above proposition is true for every tope. Still this does not immediately yield a positive answer to Question 3.3.2 for $r = 4$. Actually $r = 4$ is the first rank for which Question 3.3.2 is open.

Another question, which already seems to be hard and is similar to one posed in [7] is the following:

Question 3.3.4. How many different antipodal labelings that pass through steps A and B of the algorithm does a cocircuit graph admit?

Every answer to Question 3.3.4 better than $O(|V|)$ would improve the runtime of our algorithm.

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