Chip-Firing, Antimatroids, and Polyhedra

Kolja Knauer [1,2]

Institut für Mathematik
Technische Universtität Berlin
Berlin, Germany

Abstract

Starting from the chip-firing game of Björner and Lovász we consider a generalization to vector addition systems that still admit algebraic structures as sandpile group or sandpile monoid. Every such vector addition language yields an antimatroid. We show that conversely every antimatroid can be represented this way. The inclusion order on the feasible sets of an antimatroid is an upper locally distributive lattice. We characterize polyhedra, which carry an upper locally distributive structure and show that they can be modelled by chip-firing games with gains and losses. At the end we point out a connection to a membership problem discussed by Korte and Lovász.

Keywords: antimatroid, chip-firing game, vector addition language, upper locally distributive lattice, membership problem

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[2] Email: `knauer@math.tu-berlin.de`

1 Introduction

Chip-firing games (CFG) introduced by Björner and Lovász [1] have gained a big amount of attention, because of their relations to many areas of mathematics such as algebra, physics, combinatorics, dynamical systems, statistics, algorithms, and computational complexity, see [6] for a survey.

Here we deal with the fundamental role of CFGs as examples of *antimatroids* or equivalently *upper locally distributive lattices* or *left-hereditary, permutable, locally free languages*. We introduce the notion of *generalized* chip-firing and show that it still carries the nice algebraic properties of CFGs on the one hand but is wide enough to represent the whole class of antimatroids on the other hand.

In the last section we define *upper locally distributive* polyhedra as those polyhedra which as a suborder of the componentwise ordering of $\mathbb{R}^d$ form an upper locally distributive lattice. We give a characterization via the $\mathcal{H}$-description, which implies a central result of [2]. In the spirit of [5], we show that such polyhedra arise as the intersection of CFGs with gains and losses. We feel that in the case that a *polytope of feasible sets* of an antimatroid is an ULD-polytope our approach gives hints on the membership problem discussed by Korte and Lovász [4]

2 Chip-Firing and Antimatroids

Given a directed graph $D = (V, A)$ and a *chip-configuration* $\sigma \in \mathbb{N}^V$, one can play the *chip-firing game (CFG)*. *Firing* v consists of sending one chip along each out-going arc of v to the respective neighbor. This operation is allowed if it yields a new chip configuration, i.e., v can be fired if $\sigma(v) \geq \mathsf{outdeg}(v) > 0$.

The CFGs we are interested in are finite and hence can be assumed to be *simple* (see [3]), i.e., every vertex is fired at most once.

A *vector addition language* is a language $L(X, \sigma)$ given by an *alphabet* $X \subset \mathbb{R}^d$ and a *starting configuration* $\sigma \in \mathbb{R}^d_{\geq 0}$. A word $\ell = (x_1, \ldots, x_k)$ is in $L(X, \sigma)$ if

$$\sigma + x_1 + \ldots + x_i \in \mathbb{R}^d_{\geq 0} \text{ for all } 1 \leq i \leq k.$$

The sequences of vertices which can be fired in a CFG given by (D, σ) can be seen as a *vector addition language* in the following way. The space for the language will be $\mathbb{R}^V$. Now define for every $v \in V$ a vector in

$$x(v)_w := \begin{cases} |\{a = (v,w)\}| & \text{if } v \neq w \\ -\mathsf{outdeg}(v) & \text{otherwise} \end{cases}$$

It is folklore that the alphabet given by the letters $x(v)$ together with the starting configuration σ encodes the CFG. We denote this vector addition language by $L(D, \sigma)$.

The question that we want to address is what vector addition languages share the following important properties of CFGs:
For a word $\ell \in L(X, \sigma)$ denote by $\underline{\ell}$ its *support*, i.e., the multiset of its letters, and by $|\ell| := \sum_{x \in \underline{\ell}} x$ its *norm*. Given a simple CFG the set $\mathcal{A} := \underline{L(D, \sigma)}$ is an *antimatroid* $(V, \mathcal{A})$. That is, $\mathcal{A} \subseteq 2^V$ is closed under union and an *accessible set system*, i.e., for every $\emptyset \neq A \in \mathcal{A}$ there is $v \in A$ such that $A \backslash \{v\} \in \mathcal{A}$. One obvious but in our context fundamental property of an antimatroid is that $\mathcal{A}$ always has a unique inclusion-maximal set. Another property of CFGs is $\underline{L(D, \sigma)} \cong |L(D, \sigma)|$, i.e., different firing sequences yielding the same configuration must fire the same vertices. These properties together allow basic algebraic constructions related to CFGs such as the *sandpile group* and the *sandpile monoid*. Both are operations on norms of maximal elements of the antimatroids (*stable configurations*) arising from different initial configurations on a fixed digraph.

We define a finite an alphabet $X \subset \mathbb{R}^d$ to be a *generalized chip-firing game* if $\underline{L(X, \sigma)}$ is an antimatroid for every σ. The correspondence of norms and supports has not to be part of the definition as

Proposition 2.1 *In a generalized CFG we have* $\underline{L(X, \sigma)} \cong |L(D, \sigma)|$ *for every* σ.

So generalized CFGs have a sandpile group and a sandpile monoid, which can be constructed analogously to the ordinary CFG case. We can prove an easy characterization of generalized CFGs. The *if*-direction already appeared in [1].

Proposition 2.2 *A finite set of letters* $X \subset \mathbb{R}^d$ *is a generalized CFG if and only if for every* $i \in [d]$ *there is at most one* $x \in X$ *with* $x_i < 0$.

A central result of this paper is that the class of generalized CFGs is big enough to represent all antimatroids, something which is not possible with ordinary CFGs.

Theorem 2.3 *For every antimatroid* $(V, \mathcal{A})$ *there is generalized chip-firing game* X *and a starting configuration* σ *both with entries in* $\{0, \pm 1\}$ *such that* $(V, \mathcal{A}) \cong \underline{L(X, \sigma)}$.

An important part of the proof is based on the representation of antimatroids as inclusion order of a poset modulo an antichain-partition due to Nourine [7]. In [5] it was proven that every *upper locally distributive lattice* can be represented as an *extended* CFG. This means that every antimatroid can be written as the intersection of vector addition languages induced by CFGs. This representation generally is in higher dimension as the one produced by Theorem 2.3 but can be easily recovered from it.

3 Upper Locally Distributive Polyhedra

We define an *upper locally distributive polyhedron* or ULD-polyhedron as a polyhedron P such that for all $x, y \in P$ the componentwise maximum $\max(x, y)$ is in P and there is some element $z \in P$ which is componentwise smaller or equal than x and y. This is equivalent to saying that P forms an upper locally distributive lattice as a subposet of the componentwise ordering of $\mathbb{R}^d$. ULDs and inclusion orders of the feasible sets of antimatroids are in one-to-one correspondence in the discrete case. Here we obtain some differences. While keeping the *union-closedness* of an antimatroid we replace the idea of *accessible set system* by the requirement that every pair of sets has a unique maximal common subset, which is weaker in the continuous setting. Analogously to Proposition 2.2 we can characterize ULD-polyhedra via their $\mathcal{H}$-description.

Theorem 3.1 *A polyhedron* P *is an ULD-polyhedron if and only if it can be written as* $P = \{x \in \mathbb{R}^d \mid Mx \geq b\}$, *where* M *is a matrix with at most one negative entry per row and in a row without negative entries there is at most one positive entry.*

An immediate application of Theorem 3.1 is the following. A subset of $\mathbb{R}^d$ is a distributive sublattice of the componentwise ordering if and only if it is closed with respect to componentwise maximum and minimum. Thus we obtain the following result of [2] as a

Corollary 3.2 *A polyhedron* P *is a distributive lattice if and only if it can be written as* $P = \{x \in \mathbb{R}^d \mid Mx \geq b\}$, *where* M *is a matrix with at most one negative and at most one positive entry per row, i.e.,* M *is a generalized network matrix.*

In [2] it is shown that the points of distributive polyhedra correspond to objects that are dual to flows in a digraph with lossy and gainy arcs. Similarly in a digraph with non-negative arc-parameters λ_a, we can modify the notion of CFG in the sense that sending one chip along $a = (v, w)$ yields λ_a new chips at w. This means, the CFG has gains and losses. In the spirit of [5] we can then prove

Theorem 3.3 *Every ULD-polyhedron is the intersection of polyhedra induced by chip-firing games with gains and losses.*

Korte and Lovász [4] define the *feasible polytope* of an antimatroid $(V, \mathcal{A})$ as the convex hull of the characteristic vectors of the elements of $\mathcal{A}$. They discuss the membership problem for these polytopes. Theorem 2.3 yields an $\mathcal{H}$-description of a polytope containing the feasible polytope of the antimatroid in polynomial time. It would be worth investigating under which conditions these polytopes coincide.

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