Distributive Lattices from Graphs
(Extended Abstract)

Stefan Felsner & Kolja B. Knauer
Institut für Mathematik,
Technische Universität Berlin.
{felsner,knauer}@math.tu-berlin.de

Abstract

Several instances of distributive lattices on graph structures are known. This includes c-orientations (Propp), α-orientations of planar graphs (Felsner/de Mendez) planar flows (Khuller, Naor and Klein) as well as some more special instances, e.g., spanning trees of a planar graph, matchings of planar bipartite graphs and Schnyder woods.

We provide a characterization of upper locally distributive lattices (ULD-lattices) in terms of edge colorings of their Hasse diagrams. In many instances where a set of combinatorial objects carries the order structure of a lattice this characterization yields a slick proof of distributivity or UL-distributivity. This is exemplified by proving a distributive lattice structure on the Δ-bonds of a graph. All the previously known instances of distributive lattices from graphs turn out to be special Δ-bond lattices.

Let a D-polytope be a polytope that is closed under componentwise max and min, i.e., the points of the polytope are an infinite distributive lattice. A characterization of D-polytopes reveals that each D-polytope has an underlying graph model. The associated graph models have two descriptions either edge based or vertex based.

1 Introduction

In the next section we describe the instances of distributive lattices on graph structures that have been known.

In Section 3 a characterization of upper locally distributive lattices (ULD-lattices) and distributive lattices in terms of edge colorings of their Hasse diagrams is described.

This theory is applied in Section 4 to the study of Δ-bonds. It is shown that the Δ-bonds with invariant circular flow-difference Δ of a graph form a distributive lattice. All the previously known instances of distributive lattices from graphs are shown to be special Δ-bond lattices.

In Section 5 we turn the attention to potentials as duals structures to bonds. This leads to the study of D-polytopes, i.e., polytopes that are closed under componentwise max and min. We provide a characterization of the linear equations and inequalities that define D-polytopes. This yields a distributive lattice structure on generalized bonds and a proof that they are the most general structure on graphs that form a distributive lattice.

2 Distributive Lattices from Graphs – Previous Results

2.1 The lattice of c-orientations – Propp [22]

Given an orientation O of a graph $G = (V, E)$ and a vertex $v \in V$ with the property that v is a sink in O a new orientation O' is obtained by reverting the orientation of all edges incident to v while keeping the orientation of all other edges. The operation $O \to O'$ will be called a **push** at vertex v. Pretzel [21] attributes the idea for this operation to Mosesian. If C is a directed cycle in G and O' is obtained from O by pushing a vertex, then $|C_O^+| - |C_O^-| = |C_{O'}^+| - |C_{O'}^-|$, where C_O^+ is the set of edges whose orientation in O and C coincide and C_O^- consists of the the other edges of C. The above formula can be cast in the statement that *pushing a vertex leaves the flow-difference around a cycle invariant*. *push*

Define the **flow-difference** of O as the function $c = c_O$ that associates to each directed cycle C the flow-difference $c(C)$ of O around C. An orientation O with $|C_O^+| - |C_O^-| = c(C)$ is called a c-orientation. An edge is called **rigid** if it has the same orientation in all c-orientations. With any G and c we can associate a G' and c' such that there are no rigid edges in G' and c-orientations of G and c'-orientations of G' are in bijection, such a pair G', c' is called **reduced**. *flow-difference* *rigid* *reduced*

The main result in Propp's article [22] is:

Theorem 1 *Let G, c be reduced and let $v_0 \in V$. If we say that one c-orientation O covers another c-orientation O' exactly when O is obtained from O' by pushing a vertex $v \neq v_0$, then the covering relation makes the set of c-orientations of G into a distributive lattice.*

From duality for planar graphs and the above theorem Propp derives the following two corollaries:

- The set of d-factors of a plane bipartite graph can be enhanced with a distributive lattice structure.
- The set of spanning trees of a plane graph can be enhanced with a distributive lattice structure.

2.2 The lattice of flow in planar graphs – Khuller, Naor and Klein [16]

Consider a directed graph $D = (V, A)$, with each arc a having an integer **lower** and **upper** bound on its capacity, denoted $\ell(a)$ and $u(a)$. A **circulation** is a function $f : A \to \mathbb{Z}$ such that $\ell(a) \leq f(a) \leq u(a)$ for each edge and $\sum_{a\in \mathrm{in}(v)} f(a) = \sum_{a\in \mathrm{out}(v)} f(a)$. *circulation*

Define $f < f'$ if f' is obtained from f by pushing a unit of flow around a clockwise cycle in the residual graph of f. Let $P_{\mathcal{F}}$ be the order on flows obtained as the transitive closure of the $<$ relation. An edge is called **rigid** if it carries the same flow in all feasible circulations. Assuming that there are no rigid edges and G is connected the cover relation $f \prec f'$ of $P_{\mathcal{F}}$ is given by the operation of pushing flow in clockwise direction around a bounded facial cycle. *rigid*

The main result in [16] is:

Theorem 2 *The order $P_{\mathcal{F}}$ is a distributive lattice.*

2.3 The lattice of α-orientations in planar graphs – Felsner [11]

Consider a plane graph $G = (V, E)$ with outer face F^*. Given a mapping $\alpha : V \to \mathbb{N}$ an orientation X of the edges of G is called an α**-orientation** if α records the out-degrees of all *α-orientation* vertices, i.e.,$\mathsf{outdeg}_X(v) = \alpha(v)$ for all $v \in V$.

Define $X < X'$ if X' is obtained by reorienting all edges of a clockwise directed cycle in X. Let P_α be the order on α-orientations obtained as the transitive closure of the $<$ relation. An edge of G is called **rigid** if it has the same orientation in all α-orientations. Assuming that there are no rigid edges and G is connected the cover relation $X \prec X'$ of P_α is given by clockwise to counterclockwise reorientations of bounded facial cycle. *rigid*

The main result in [11] is:

Theorem 3 *The order P_α is a distributive lattice.*

In [6] it is noted that such a result was also obtained in the thesis of de Mendez [7]. Special instances of α-orientations on plane graphs yield lattice structures on

- Eulerian orientations of a plane graph.
- Spanning trees and d-factors of a plane graph.
- Schnyder woods of a 3-connected plane graph.

3 Upper Locally Distributive Lattices

Upper locally distributive lattices (ULD) and their duals (lower locally distributive lattices (LLD)) where defined by Dilworth [9]. In the following section we deal with distributive lattices arising from graphs. To prove distributivity we use the following well known characterization: *Distributive lattices are exactly those lattices that are both ULD and LLD.*

ULDs have appeared under several different names, e.g. locally distributive lattices (Dilworth [9]), meet-distributive lattices (Jamison [13, 14], Edelman [10], Björner and Ziegler [4]), locally free lattices (Nakamura [19]). Following Avann [2], Monjardet [18], Stern [23] and others we call them ULDs. The reason for the frequent reappearance of the concept is that there are many instances of ULDs, i.e sets of combinatorial objects that can be *naturally* ordered to form an ULD.

ULDs have first been investigated by Dilworth [8], many different lattice theoretical characterizations of ULDs are known. For a survey on the work until the nineties we refer to Monjardet [18]. We use the original definition of Dilworth:

Definition 1 *Let $(P, \leq)$ be a poset. P is called an* **upper locally distributive lattice (ULD)** *if P is a lattice and each element has a unique minimal representation as meet of meet-irreducibles, i.e., there is a mapping $M : P \to \mathcal{P}(\{m \in P : m \text{ is meet-irreducible}\})$ with the properties* *upper locally distributive lattice (ULD)*

- $x = \bigwedge M_x$ *(representation)*
- $x = \bigwedge A$ *implies* $M_x \subseteq A$ *(minimal).*

Let $D = (V, A)$ be a directed graph, an arc coloring c of D is an U-**coloring** if for every $u, v, w \in V$ with $u \neq w$ and $(v, u), (v, w) \in A$ it holds: *U-coloring*

(U_1) $c(v, u) \neq c(v, w)$.

(U_2) There is a $z \in V$ and arcs $(u, z), (w, z)$ such that $c(v, u) = c(w, z)$ and $c(v, w) = c(u, z)$. (see Figure 1)

Definition 2 *A finite poset $(P, \leq)$ is called U-***poset** *if the arcs of the cover graph D_P of P admit a U-coloring.* *U-poset*

Our characterization of ULDs has two parts.

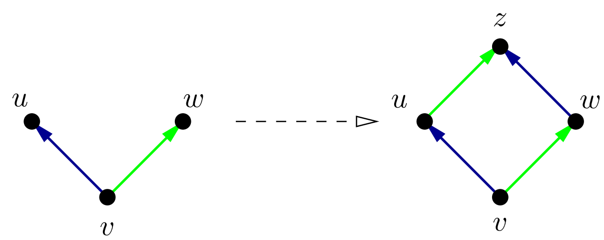

Figure 1: The completion property of U-colorings.

Theorem 4 (a) *If D is a finite, acyclic digraph admitting a U-coloring, then D is a cover graph, hence, the transitive closure of D is a U-poset.*

(b) *Upper locally distributive lattices are exactly the U-posets with a global minimum.*

Dual in the sense of order reversal to U-coloring, U-poset and ULD are L-coloring, L-poset and LLD. The characterization of LLDs dual to Theorem 4 allows easy proofs that the inclusion orders on the following combinatorial structures are lower locally distributive lattices:

- Subtrees of a tree (Boulaye [5]).
- Convex subsets of posets (Birkhoff and Bennett [3]).
- Convex subgraphs of acyclic digraphs, here a set C is convex if $x, y \in C$ implies that all directed (x, y)-paths are in C (Pfaltz [20]).

These combinatorial structures can also be seen as *convex sets of an abstract convex geometry*. This is no coincidence as in fact every LLD is isomorphic to the inclusion order on the convex sets of an abstract convex geometry and vice versa (Edelman [10]).

4 The Lattice of Δ-Bonds

Let $D = (V, A)$ be a connected directed graph with upper and lower integral edge capacities $c_u, c_l : A \to \mathbb{Z}$. We are interested in maps $x : A \to \mathbb{Z}$ such that $c_l(a) \le x(a) \le c_u(a)$ for all $a \in A$. The **circular flow-difference** of x on a cycle C with a prescribed direction is *circular flow-difference*

$$\delta(C, x) := \sum_{a \in C^+} x(a) - \sum_{a \in C^-} x(a).$$

Note that the circular flow-differences $\delta(C, x)$ on the cycles of a basis of the cycle space uniquely determines the flow-difference of x on all cycles of the graph.

For a given $\Delta \in \mathbb{Z}^{\mathcal{C}}$ we consider the set $\mathcal{B}_\Delta(D, c_l, c_u) := \{c_l \le x \le c_u \mid \delta(C, x) = \Delta_C\}$, this is the set of **$\Delta$-bonds** on (D, c_l, c_u). We introduce an order on Δ-bonds with prescribed circular flow-difference, i.e., on the elements of $\mathcal{B}_\Delta(D, c_l, c_u)$ such that: *Δ-bonds*

Theorem 5 *The elements of $\mathcal{B}_\Delta(D, c_l, c_u)$ form a distributive lattice $\mathcal{P}_\Delta$. Moreover, the diagram of $\mathcal{P}_\Delta$ comes with a natural cover preserving lattice embedding into $\mathbb{N}^V$*

In the following we give a sketch of the proof.

With a partition $(U, \overline{U})$ of the vertices V of D consider the cut $S = S[U] \subset A$. The **forward edges** S^+ of S are those $a \in A$ directed from U to $\overline{U}$, **backward edges** S^- of S are directed from $\overline{U}$ to U. The operation $x \to x'$ described in the next lemma is an **augmentation** at S. *augmentation*

Lemma 1 *If x is in $\mathcal{B}_\Delta(D, c_l, c_u)$ and $S = S[U]$ is a cut such that $x(a) < c_u(a)$ for all $a \in S^+$ and $x(a) > c_l(a)$ for all $a \in S^-$, then the $x' : A \to \mathbb{Z}$ defined as $x'(a) = x(a)$ for all $a \notin S$, $x'(a) = x(a) + 1$ for all $a \in S^+$ and $x'(a) = x(a) - 1$ for all $a \in S^-$ is also in $\mathcal{B}_\Delta(D, c_l, c_u)$.*

Fix an arbitrary vertex v_0 in D as the **forbidden vertex**. For $x, y \in \mathcal{B}_\Delta(D, c_l, c_u)$ define $x < y$ if y can be reached from x via a sequence of augmentations at cuts, such that each of the cuts $S = S[U]$ has $v_0 \in \overline{U}$. A **vertex cut** is a cut $S[v]$ with $v \neq v_0$. *forbidden vertex* *vertex cut*

Lemma 2 *If $x < y$, then y can be obtained from x by a sequence of augmentations at vertex cuts.*

From the lemma it follows that the relation $<$ is acyclic, i.e., an order relation. Otherwise we could linearly combine vertex cuts $S[v]$, with $v \neq v_0$, to zero. But these vertex cuts are a basis of the bond space.

Augmentations at vertex cuts correspond to the cover relations, i.e., edges of the Hasse diagram, of the order defined on $\mathcal{B}_\Delta(D, c_l, c_u)$. This order will be denoted $\mathcal{P}_\Delta$. A coloring of the edges of the Hasse diagram of $\mathcal{P}_\Delta$ is naturally given as a mapping to $V \setminus \{v_0\}$.

Lemma 3 *The above coloring of the edges of the Hasse diagram of the order $\mathcal{P}_\Delta$ is a U-coloring.*

A completely symmetric argument shows that the coloring is also a U-coloring for the reversed order, i.e., a L-coloring. Theorem 4 implies that every connected component of $\mathcal{P}_\Delta$ is an ULD and LLD lattice, hence, a distributive lattice. To complete the proof of Theorem 5 it only remains to show that $\mathcal{P}_\Delta$ is connected.

Lemma 4 *The order $\mathcal{P}_\Delta$ is connected.*

Theorem 6 *The lattice structure $\mathcal{P}_\Delta$ on Δ-bonds generalizes the lattices from Section 2.*

The idea in all the original proofs for the distributive lattice structures from Section 2 was to use a potential function. In the case of c-orientations the potential function is a $p : V \to \mathbb{Z}$ in the other two cases the domain of p is the planar dual set, i.e., the set V^* of faces of the plane graph. In all cases there is a bijection between feasible potentials and the objects of the lattice. The distributive lattice structure on the potentials is easily established by showing that with two feasible potentials their pointwise maximum and minimum are also feasible.

To apply this alternative approach to prove the distributive lattice structure on Δ-bonds we begin with a normalization. Consider a spanning tree T of $D = (V, A)$ and note that the circular flow-differences $\delta(C, x)$ for all cycles are determined by the values taken at cycles belonging to the fundamental cycle basis FCB with respect to T. Given a cycle $C \in FCB(T)$ let a_C be the unique edge in $C \setminus T$. The following simple modification of the data makes $\Delta_C = 0$: if $a_C \in C^+$ redefine the bounds $c_l(a) \leftarrow c_l(a) - \Delta_C$ and $c_u(a) \leftarrow c_u(a) - \Delta_C$, if $a_C \in C^-$ add Δ_C to both bounds. After having done this for all cycles in $FCB(T)$ we have new data (D, c_l', c_u') such that $\mathcal{B}_\Delta(D, c_l, c_u)$ and $\mathcal{B}_{\mathbf{0}}(D, c_l', c_u')$ are in bijection.

Let $v_0 \in V$ and let $x \in \mathcal{B}_{\mathbf{0}}(D, c_l', c_u')$, with x associate a potential p such that

$$
\begin{aligned}
p(v_0) &= 0 \quad \text{and} && (1)\\
p(w) - p(v) &= x(a) \text{ for all } a = (v, w) \in A && (2)
\end{aligned}
$$

Starting from v_0 this potential can be computed along a spanning tree. Actually the following holds:

Proposition 1 *Potential functions $p : V \to \mathbb{Z}$ with $p(v_0) = 0$ and $c_l \leq p(w) - p(v) \leq c_u$ for all $a = (v, w) \in A$ are in bijection with $\mathbf{0}$-bonds $\mathcal{B}_{\mathbf{0}}(D, c_l, c_u)$.*

Lemma 5 *If p and q are feasible, then the pointwise maximum and minimum are also feasible.*

Proof. We show one of the calculations needed for the proof. Consider $a = (v, w) \in A$ and suppose that $p(w) \geq q(w)$ and $p(v) \leq q(v)$. Now $c_l \leq q(w) - q(v) \leq q(w) - q(v) + (p(w) - q(w)) = p(w) - q(v) = \max\{p(w), q(w)\} - \max\{p(v), q(v)\} = p(w) - p(v) + (p(v) - q(v)) \leq p(w) - p(v) \leq c_u$. □

The observation that our potentials could as well be real-valued leads to the notion of D-polytopes as studied in the next section.

5 Structure of D-Polytopes

A polyhedron $P \subseteq \mathbb{R}^n$ is called **distributive** if *distributive*

$$x, y \in P \Longrightarrow \min(x, y), \max(x, y) \in P$$

where max and min are taken componentwise.

We will abbreviate distributive polyhedra as **D-polyhedra**. The reason for the name *distributive* and the connection to the earlier sections of this paper are given by the following *D-polyhedra*

Observation 7 *A polyhedron $P \subseteq \mathbb{R}^n$ is a D-polyhedron if and only if it is a distributive lattice with respect to the dominance order, where $x \leq_{\mathsf{dom}} y$ iff $x_i \leq y_i$ $\forall_{1 \leq i \leq n}$*

5.1 Geometric Characterization

We want to find a geometric characterization of D-polyhedra. For this we need the basic

Observation 8 *The property of being a D-polyhedron is invariant under:*

- *translation*
- *scaling*
- *intersection*

In order to give a neat description of D-polyhedra in terms of bounding halfspaces we will pursue the following strategy. We start by characterizing distributive affine subspaces of $\mathbb{R}^n$. Then we provide a characterization of the orthogonal complements of distributive affine spaces. After proving that the induced halfspaces of distributive affine spaces are distributive, we characterize D-polyhedra as intersections of distributive halfspaces.

For a vector $x \in \mathbb{R}^n$ let $\text{supp}(x) := \{i \in [d] \mid x_i \neq 0\}$ be its **support**. Call a set of vectors $B \subseteq \mathbb{R}^n$ **non-negative disjoint (NND)** if the elements of B are non-negative and have pairwise disjoint supports. Note that a NND set of non-zero vectors is automatically linearly independent. With these definitions we can state the first result. *support* *non-negative disjoint (NND)*

Proposition 2 *An affine subspace $A \subseteq \mathbb{R}^n$ is a D-polyhedron if and only if it has a non-negative disjoint basis B.*

The next step will be to provide a class of network matrices N_Λ of certain arc-parameterized digraphs such that an affine space A is distributive if and only if it can be described as $A = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p = c\}$.

Generally we call a tuple $D_\Lambda = (V, A, \Lambda)$ an **arc-parameterized digraph** if $D = (V, A)$ is a digraph – the **underlying digraph** – with $V = [n]$, $|A| = m$, and $\Lambda \in \mathbb{R}^m_{\geq \mathbf{0}}$ such that $\lambda_a = 0$ iff a is a loop. *arc-parameterized digraph* *underlying digraph*

Given an arc parameterized digraph D_Λ we define its **Λ-network-matrix** to be the matrix $N_\Lambda \in \mathbb{R}^{n \times m}$ with a column $e_j - \lambda_a e_i$ for every arc $a = (i, j)$ with parameter λ_a. *Λ-network-matrix*

If B is a NND basis of an affine space A, we construct an arc-parameterized digraph, such that the resulting Λ-network-matrix N_Λ satisfies $A = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p = c\}$. For every $b \in B$ choose some arbitrary directed spanning tree on $\operatorname{supp}(b)$. To an arc $a = (i, j)$ with $i, j \in \operatorname{supp}(b)$ we associate the arc parameter $\lambda_a := b_j/b_i > 0$. For every $i \notin \bigcup_{b \in B} \operatorname{supp}(b)$ insert a loop $a = (i, i)$ with $\lambda_a := 0$. Collect the λ_a of all the arcs in a vector $\Lambda \in \mathbb{R}^m_{\geq \mathbf{0}}$. The resulting arc-parameterized digraph is a disjoint union of loops and directed trees.

We can state the following:

Proposition 3 *Let $A \subseteq \mathbb{R}^n$ be an affine subspace. Then A has a non-negative disjoint basis if and only if $A = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p = c\}$, where N_Λ is the Λ-network-matrix of an arc parameterized disjoint union of directed trees and loops.*

Now by an easy geometric argument we obtain

Proposition 4 *A halfspace $A^+ = \{p \in \mathbb{R}^n \mid Mp \leq c\}$ is distributive if and only if $A = \{p \in \mathbb{R}^n \mid Mp = c\}$ is distributive.*

Putting the above together we can prove

Theorem 9 *A polyhedron $P \subseteq \mathbb{R}^n$ is a D-polyhedron if and only if*

$$P = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \leq c\}$$

for some Λ-network-matrix N_Λ and $c \in \mathbb{R}^m$.

As an immediate application we obtain that *order polytopes* ($\Lambda = \mathbf{1}$ and $c \in \{0, 1\}^m$) are D-polytopes. More generally ($\Lambda = \mathbf{1}$ and $c \in \mathbb{Z}^m$) one obtains distributivity for *polytropes* [15] or equivalently *alcoved polytopes* [17].

6 Combinatorial Characterization

Since a D-polyhedron P is completely described by an arc-parameterized digraph D_Λ and an arc-capacity vector $c \in \mathbb{R}^m$, the geometric characterization of D-polyhedra suggests a combinatorial viewpoint. Denote by $D(P)$ the tuple (D_Λ, c) of the arc parameterized digraph with arc capacities c given by P. A feasible vertex potential for $D(P)$ is a vector $p \in \mathbb{R}^n$, which assigns a real number p_i to each vertex i of $D(P)$, such that the inequality $p_j - \lambda_a p_i \leq c_a$ holds for every arc $a = (i, j)$ of $D(P)$. Note that the points of the D-polyhedron $P = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \leq c\}$ are exactly the feasible vertex potentials of $D(P)$.

Given a feasible vertex potential p in an arc parameterized digraph with arc capacities (D_Λ, c) and a $\mu \in \mathbb{R}$ denote by $\operatorname{push}_{p,i}(\mu) := p + \mu e_i$ the **μ-push** at vertex i, if this still yields a feasible vertex potential. *μ-push*

Observation 10 *Let P be a D-polytope. Given any feasible vertex potential of $D(P)$, we can obtain any other feasible vertex potential $D(P)$ by a sequence of pushes.*

The next model for a D-polyhedron P will live in the arc space of $D(P)$ but inherit the distributive lattice structure from the vertex potential model.

This is, given a D-polyhedron P we look at $\hat{P} := \{x \in \mathbb{R}^m \mid x \leq c \text{ and } x \in \operatorname{Im}(N_\Lambda^\top)\}$ instead of $P = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \leq c\}$. The elements of $\hat{P}$ will be called **generalized bonds**; in the spirit of *generalized flow*, c.f. [1]. *generalized bonds*

How do generalized bonds look like in the digraph? Clearly a generalized bond x consists of real values $x_a \leq c_a$ for every arc of $D(P)$. Since $x \in \operatorname{Im}(N_\Lambda^\top) = \operatorname{Kern}(N_\Lambda)^\perp$ we have $\langle x, f\rangle = 0$ $\forall_{f \in \operatorname{Kern}(N_\Lambda)}$. As $\operatorname{Kern}(N_\Lambda)$ is a subset of the arc space of $D(P)$ too, understanding the elements of $\mathrm{Kern}(N_\Lambda)$ is vital to our analysis. We will then be able to describe the generalized bonds of $D(P)$ as capacity-respecting arc values, which satisfy a certain *flow-difference-condition* around elements $f \in \mathrm{Kern}(N_\Lambda)$.

6.1 Bonds

Consider as an example the case where $\Lambda = \mathbf{1}$ is the all-ones vector. Now $D_{\mathbf{1}} \cong D$ in the sense that $N_{\mathbf{1}}$ is nothing else but the network-matrix of D. It is an elementary fact from algebraic graph theory that $\mathrm{Kern}(N_{\mathbf{1}})$ is the **flow space** $\mathcal{F}(D)$ of D, see for instance [12]. Its elements are the circulations of D, i.e those real arc values $f \in \mathbb{R}^m$ which respect flow-conservation at every vertex of D. Moreover each support-minimal element of $\mathcal{F}(D)$ is a scalar multiple of the signed incidence vector $\mathrm{sign}(C)$ of a cycle C of D, where $\mathrm{sign}(C)_a$ is 1 if a is a forward arc of C, and -1 if a is a backward arc, and 0 otherwise. *flow space*

Now the set of generalized bonds of $D_{\mathbf{1}}$ consists of these $x \in \mathbb{R}^m$ such that $x \leq c$ and $\langle x, f\rangle = 0\ \forall_{f \in \mathcal{F}(D)}$, which after the above comments is equivalent to $\langle x, \mathrm{sign}(C)\rangle = 0\ \forall_{C \in \mathcal{C}(D)}$. In other words $\hat{P}$ can be seen as the set of *real-valued* Δ-bonds of $D_{\mathbf{1}}$ within the arc capacities c and $\Delta = \mathbf{0}$.

The only difficulty when trying to shift the distributive lattice property from P to $\hat{P}$ consist in the fact that for network matrices P is generally unbounded, but $\hat{P}$ is not. By intersecting P with the appropriate hyperplanes one obtains a bounded D-polytope P' with $\hat{P}' \cong \hat{P}$. This transformation corresponds to prohibiting pushes at the forbidden vertices and yields the desired distributive lattice structure on the set of *real-valued* bonds of an arbitrary digraph $D_{\mathbf{1}}$.

To deduce the full strength of the results of the previous section one remaining difference is the integrality. Clearly if $P \subseteq \mathbb{R}^n$ is a D-polyhedron then also $P \cap \mathbb{Z}^n$ is a distributive lattice, which yields a distributive lattice structure on the image $(N_{\mathbf{1}}^\top)\mathbb{Z}^n$.

But to obtain a distributive lattice on the integral bonds we need such a structure on $\hat{P} \cap \mathbb{Z}^m$. Luckily $\Lambda = \mathbf{1}$ hence $N_{\mathbf{1}}^\top$ is a totally unimodular matrix, which yields $\hat{P} \cap \mathbb{Z}^m = (N_{\mathbf{1}}^\top)\mathbb{Z}^n$, i.e. the integral bonds carry a distributive lattice structure.

Remark This approach does not yield the fact, that these bonds are all connected by vertex pushes.

6.2 General Parameters

Lets now look at the case of general bonds in an arc-parameterized digraph D_Λ, i.e. general $\Lambda \in \mathbb{R}^m_{\geq \mathbf{0}}$. We want to obtain an analog to cycle and flow spaces for arc-parameterized graphs. Let D be the underlying digraph of D_Λ. For $C \in \mathcal{C}(D)$ define $\tau(C) := (\Pi_{a \in C} \lambda_a^{\mathrm{sign}(C)_a})$ the **twist** of C. Call C **weak** if it has twist 1 and **strong** otherwise. Moreover call a triple $B = (C, P, C')$ of two cycles $C, C' \in \mathcal{C}(D)$ together with a (possibly trivial) path P connecting them a **barbell**. Call a barbell $B = (C, P, C')$ **strong** if both cycles C and C' are strong. *twist* *barbell*

Define by $\mathcal{F}(D_\Lambda) := \mathrm{Kern}(N_\Lambda)$ the **generalized flow space** of D_Λ and let $\mathcal{C}(D_\Lambda)$ be the set of support minimal vectors of $\mathcal{F}(D)$. We call $\mathcal{C}(D_\Lambda)$ the **generalized cycle space** of D_Λ. *generalized cycle space*

Proposition 5 *Let D_Λ be an arc-parameterized digraph. If $f \in \mathcal{C}(D_\Lambda)$ then either supp(f) = supp(C) for some weak cycle $C \in \mathcal{C}(D)$ or supp(f) = supp(B) for some strong barbell $B = (C, P, C')$.*

On the other hand given a weak cycle or a strong barbell it is easy to give the family of $f \in \mathcal{C}(D)$ with the corresponding support. Given a weak cycle C just assign some non-zero values f_a to its arcs such that

$$(\Sigma_{a=(i,j)} f_a) - (\Sigma_{a=(j,k)} \lambda_a f_a) = 0$$

for consecutive vertices i, j, k of C. Since C is weak this equation has a one-dimensional solution space.

Given a strong barbell one can balance the twists of the strong cycles along the connecting path and obtains a one-dimensional solution space as well.

The preceeding argument justifies to call $\underline{\mathcal{C}}(D_\Lambda)$ the set of weak cycles and strong barbells of D_Λ the **combinatorial support** of the generalized cycle space of D_Λ. Given $H \in \underline{\mathcal{C}}(D_\Lambda)$ denote by $f(H)$ the unique $f \in \mathcal{C}(D)$ with $\mathrm{supp}(f) = \mathrm{supp}(H)$ and $f_i = 1$ for the minimal $i \in \mathrm{supp}(f)$. Let $x \in \mathbb{R}^m$ and $H \in \underline{\mathcal{C}}(D_\Lambda)$. Write $\delta(H, x) := \langle x, f(H) \rangle$ for the **flow-difference** of x around H.

combinatorial support

flow-difference

We thus have obtained:

Theorem 11 *Let P be a D-polyhedron and $D(P)$ its arc-parameterized digraph with capacities $c \in \mathbb{R}^m$, then $x \in \mathbb{R}^m_{\leq c}$ is a generalized bond of $D(P)$ if and only if $\delta(H, x) = 0$ for every $H \in \underline{\mathcal{C}}(D_\Lambda)$.*

Curiously, given an arc-parameterized digraph D_Λ, its strong cycles are to a *strong one-tree*, what usual cycles are to a tree. For simplicity assume the underlying digraph D to be connected. A subgraph of D is called a **strong one-tree** if it consists of a spanning tree T of D together with an additional arc a such that $T \cup \{a\} := \widetilde{T}$ contains a strong cycle. Adding an arc a' to $\widetilde{T}$ yields either a unique strong barbell or a weak cycle. Denote this set of weak cycles and strong barbells by $\underline{\mathcal{C}}(\widetilde{T}, D_\Lambda) \subseteq \underline{\mathcal{C}}(D_\Lambda)$. Then $\{f(H) \mid H \in \underline{\mathcal{C}}(\widetilde{T}, D_\Lambda)\}$ is a basis of $\mathcal{F}(D)$, called the **fundamental basis** induced by $\widetilde{T}$.

strong one-tree

fundamental basis

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