

On tope graphs of complexes of oriented matroids

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Abstract

We give two graph theoretical characterizations of tope graphs of (complexes of) oriented matroids. The first is in terms of excluded partial cube minors, the second is that all antipodal subgraphs are gated. A direct consequence is a third characterization in terms of zone graphs of tope graphs.

Further corollaries include a characterization of topes of oriented matroids due to da Silva, another one of Handa, a characterization of lopsided systems due to Lawrence, and an intrinsic characterization of tope graphs of affine oriented matroids. Furthermore, we obtain polynomial time recognition algorithms for tope graphs of the above and a finite list of excluded partial cube minors for the bounded rank case. In particular, this answers a relatively long-standing open question in oriented matroids.

1 Introduction

A graph $G = (V, E)$ is a *partial cube* if it is (isomorphic to) an isometric subgraph of a hypercube graph Q_n , i.e., $d_G(u, v) = d_{Q_n}(u, v)$ for all $u, v \in V$, where d denotes the distance function of the respective graphs. Partial cubes were introduced by Graham and Pollak [22] in the study of interconnection networks. They form an important graph class in media theory [19], frequently appear in chemical graph theory [18], and quoting [25] present one of the central and most studied classes in metric graph theory.

Important subclasses of partial cubes include median graphs, bipartite cellular graphs, hypercellular graphs, Pasch graphs, and netlike partial cubes. Partial cubes also capture several important graph classes not directly coming from metric graph theory, such as region graphs of hyperplane arrangements, diagrams of distributive lattices, linear extension graphs of posets, tope graphs of oriented matroids (OMs), tope graphs of affine oriented matroids (AOMs), and lopsided systems (LOPs). A recently introduced unifying generalization of these classes are complexes of oriented matroids (COMs), whose tope graphs are partial cubes as well [3].

Partial cubes admit a natural minor-relation (pc-minors for short) and several of the above classes including tope graphs of COMs are pc-minor closed. Complete (finite) lists of excluded

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pc-minors are known for median graphs, bipartite cellular graphs, hypercellular graphs and Pasch graphs, see [8, 9, 10]. Another well-known construction of a smaller graph from a partial cube is the zone graph [25].

In this paper we focus on COMs and their tope graphs. We present two characterizations of the tope graphs and thus two graph theoretical characterizations of COMs. The first characterization is in terms of its complete (infinite) list of excluded pc-minors. As corollaries we obtain excluded pc-minor characterizations for tope graphs of OMs, AOMs, and LOPs. Moreover, in the case of bounded rank the list of excluded pc-minors is finite. We devise a polynomial time algorithm for checking if a given partial cube has another one as pc-minor, leading to polynomial time recognition algorithms for the classes with a finite list of excluded pc-minors. Another consequence is a characterization of tope graphs of COMs in terms of iterated zone graphs, which generalizes a result of Handa [23] about tope sets of OMs.

The second characterization of tope graphs of COMs is in terms of the metric behavior of certain subgraphs. More precisely, we prove that a partial cube is the tope graph of a COM if and only if all of its antipodal (also known as symmetric-even [5]) subgraphs are gated. As corollaries, this theorem specializes to tope graphs of OMs, AOMs, and LOPs. In particular, we obtain a new unified proof for characterization theorems of tope sets of LOPs and OMs due to Lawrence [26] and da Silva [13], respectively. Moreover, this characterization allows to prove that Pasch graphs are COMs, confirming a conjecture of Chepoi, Knauer, and Marc [10]. Finally, our characterization is verifiable in polynomial time, hence gives polynomial time recognition algorithms for tope graphs of COMs, OMs, AOMs, and LOPs, even without bounding the rank.

In particular, we answer two related open questions on OMs, i.e., the question for a graph theoretical characterization and for polynomial time recognition, see e.g. [7, 20].

Structure of the paper:

In Section 2 we introduce partial cubes with some more care, as well as metric subgraphs such as convex, gated, antipodal, and affine subgraphs and we discuss their behavior with respect to pc-minors and expansions. Moreover, we discuss zone graphs of partial cubes, which plays a role in part of our proof. Also, we devise a polynomial time algorithm for checking if a given partial cube has another one as a pc-minor (Proposition 2.4). We give an expansion procedure of how to construct all antipodal partial cubes from a single vertex (Lemma 2.14) and provide an intrinsic characterization of affine partial cubes (Proposition 2.16).

Section 3 is dedicated to the introduction of the systems of sign-vectors relevant to this paper, i.e. COMs, OMs, AOMs, and LOPs, and their behavior under the usual minor-relations.

In Section 4 we bring the content of the first two sections together and explain how systems of sign-vectors lead to partial cubes and vice versa. We show how metric properties of subgraphs and pc-minors correspond to axiomatic properties and minor relations of systems of sign-vectors. In particular we prove that in tope graphs of COMs all antipodal subgraphs are gated (Theorem 4.8), and characterize tope graphs of OMs, AOMs, and LOPs as special tope graphs of COMs. Theorem 4.8 gives the first implication for our characterization theorem.

In Section 5 we introduce the (infinite) class of excluded pc-minors of tope graphs of COMs and provide some of its crucial properties, that will be used throughout the proofs in the following sections. In particular, we show that every member of the class has an antipodal subgraph that is not gated (Lemma 5.1). We conclude Section 5 with the the proof that partial

cubes excluding all pc-minors from the class are tope graphs of COMs (Theorem 5.7). In particular, Theorem 5.7 gives the second implication of our characterization.

Finally, in Section 6 we show that the class of partial cubes in which all antipodal subgraphs are gated is closed under pc-minors (Theorem 6.1). Since the members of our class of excluded pc-minors have non-gated antipodal subgraphs, this yields the third and last implication of our characterization.

Section 7 is dedicated to the corollaries of our theorem, that are announced above. In particular we prove the generalization of Handa's Theorem (Corollary 7.2). We conclude the paper with several further questions in Section 8.

2 Pc-minors, expansions, zone graphs, and metric subgraphs

Let us give an alternative way of characterizing partial cubes. Any isometric embedding of a partial cube into a hypercube leads to the same partition of edges into so-called Θ -classes, where two edges are equivalent, if they correspond to a change in the same coordinate of the hypercube. This can be shown using the Djoković-Winkler-relation Θ which is defined in the graph without reference to an embedding, see [16, 30]. We will describe next, how the relation Θ can be defined independently of an embedding.

A subgraph G' of G is *convex* if for all pairs of vertices in G' all their shortest paths in G indeed stay in G' . For an edge $a = uv$ of G , define the sets $W(u, v) = \{x \in V : d(x, u) < d(x, v)\}$ and $W(v, u) = \{x \in V : d(x, v) < d(x, u)\}$. By a theorem of Djoković [16], a graph G is a partial cube if and only if G is bipartite and for any edge $a = uv$ the sets $W(u, v)$ and $W(v, u)$ are convex. In this case, setting $a\Theta a'$ for $a = uv$ and $a' = u'v'$ if $u' \in W(u, v)$ and $v' \in W(v, u)$ yields Θ .

Let \mathcal{E} be the set of equivalence classes of Θ . For an equivalence class $E_f \in \mathcal{E}$ and an arbitrary edge $uv \in E_f$, let $E_f^- := W(u, v)$ and $E_f^+ := W(v, u)$ the pair of complementary convex halfspaces of G . Now, identifying any vertex v of G with $v \in Q_{\mathcal{E}} = \{\pm\}^{\mathcal{E}}$ which for any class of Θ associates the sign of the halfspace containing v gives an isometric embedding of G into $Q_{\mathcal{E}}$.

2.1 Pc-minors, expansions, and zone graphs

Given $f \in \mathcal{E}$, an (*elementary*) *restriction* consists in taking one of the subgraphs $G(E_f^-)$ or $G(E_f^+)$ induced by the complementary halfspaces E_f^- and E_f^+ , which we will denote by $\rho_{f^-}(G)$ and $\rho_{f^+}(G)$, respectively. These graphs are isometric subgraphs of the hypercube $Q_{\mathcal{E} \setminus \{f\}}$. Now applying two elementary restriction with respect to different coordinates f, g , independently of the order of f and g , we will obtain one of the four (possibly empty) subgraphs induced by $E_f^- \cap E_g^-, E_f^- \cap E_g^+, E_f^+ \cap E_g^-,$ and $E_f^+ \cap E_g^+$. Since the intersection of convex subsets is convex, each of these four sets is convex in G and consequently induces an isometric subgraph of the hypercube $Q_{\mathcal{E} \setminus \{f, g\}}$. More generally, a *restriction* is a subgraph of G induced by the intersection of a set of (non-complementary) halfspaces of G . We denote a restriction by $\rho_X(G)$, where $X \in \{\pm\}^{\mathcal{E}}$ is a signed set of halfspaces of G . For subset S of the vertices of G and $f \in \mathcal{E}$, we denote $\rho_f(S) := \rho_f(G) \cap S$. The smallest convex subgraph of G containing V' is called the *convex hull* of V' and denoted by $\text{conv}(V')$. The following is well-known:

Lemma 2.1 ([1, 2, 8]). *The set of restrictions of a partial cube G coincides with its set of convex subgraphs. Indeed, for any subset of vertices V' we have that $\text{conv}(V')$ is the intersection of all halfspaces containing V' . In particular, the class of partial cubes is closed under taking restrictions.*

For $f \in \mathcal{E}$, we say that the graph G/E_f obtained from G by contracting the edges of the equivalence class E_f is an (*elementary*) *contraction* of G . For a vertex v of G , we will denote by $\pi_f(v)$ the image of v under the contraction in G/E_f , i.e. if uv is an edge of E_f , then $\pi_f(u) = \pi_f(v)$, otherwise $\pi_f(u) \neq \pi_f(v)$. We will apply π_f to subsets $S \subseteq V$, by setting $\pi_f(S) := \{\pi_f(v) : v \in S\}$. In particular we denote the *contraction* of G by $\pi_f(G)$.

It is well-known and in particular follows from the proof of the first part of [11, Theorem 3] that $\pi_f(G)$ is an isometric subgraph of $Q_{\mathcal{E} \setminus \{f\}}$. Since edge contractions in graphs commute, i.e. the resulting graph does not depend on the order in which a set of edges is contracted, we have:

Lemma 2.2. *The class of partial cubes is closed under contractions. Moreover, contractions commute in partial cubes, i.e. if $f, g \in \mathcal{E}$ and $f \neq g$, then $\pi_g(\pi_f(G)) = \pi_f(\pi_g(G))$.*

Consequently, for a set $A \subseteq \mathcal{E}$, we denote by $\pi_A(G)$ the isometric subgraph of $Q(\mathcal{E} \setminus A)$ obtained from G by contracting the classes $A \subseteq \mathcal{E}$ in G . The following can easily be derived from the definitions, see e.g. [10]:

Lemma 2.3. *Contractions and restrictions commute in partial cubes, i.e. if $f, g \in \mathcal{E}$ and $f \neq g$, then $\rho_{g^+}(\pi_f(G)) = \pi_f(\rho_{g^+}(G))$.*

The previous lemmas show that any set of restrictions and any set of contractions of a partial cube G provide the same result, independently of the order in which we perform the restrictions and contractions. The resulting graph G' is also a partial cube, and G' is called a *pc-minor* of G . In this paper we will study classes of partial cubes that are closed under taking pc-minors. Clearly, any such class has a (possibly infinite) set X of minimal excluded pc-minors. We denote by $\mathcal{F}(X)$ the pc-minor closed class of partial cubes excluding X .

Proposition 2.4. *Let X be a finite set of partial cubes. It is decidable in polynomial time if a partial cube G is in $\mathcal{F}(X)$.*

Proof. Let G', G be partial cubes. Denote by n' and n the number of vertices of G' and G , respectively, and with k' and k the number of Θ -classes in G' and G . We will show that testing if G' is a pc-minor of G can be done in polynomial time with respect to n . This clearly implies the result.

For every subset V' of at most n' vertices of G do the following: First compute $\text{conv}(V')$ and count the number of Θ -classes of G crossing it, say it equals to k'' . Then $k'' \leq k$, and if $k'' < k'$ discard the subgraph. On the other hand, if $k'' \geq k'$, then for every subset S of size $k'' - k'$ of the Θ -classes crossing $\text{conv}(V')$, contract in $\text{conv}(V')$ all the Θ -classes of S . Finally, check if the resulting graph is isomorphic to G' .

Using Lemma 2.3, we know that G' is a pc-minor if and only if it can be obtained by first restricting and then contracting, and by Lemma 2.1, taking restrictions coincides with taking convex hulls. This gives the correctness of the algorithm.

For the running time, we have a loop of length $\mathcal{O}(n^{n'})$. In each execution we compute $\text{conv}(V')$ which via Lemma 2.1 can be easily done by intersecting all the halfspaces containing

V' . Then we have $\mathcal{O}\left(\binom{k''}{k''-k'}\right) = \mathcal{O}\left(\binom{k''}{k'}\right) \leq \mathcal{O}\left(\binom{n}{k'}\right) = \mathcal{O}(n^{k'})$ choices for the contractions of the Θ -classes, each of which can clearly be done in polynomial time, too. Note that $k' < n'$. Finally, we check if the obtained graph is isomorphic to G' , which only depends on n' . \square

Later on we will also consider the inverse operation of contraction: a partial cube G is an *expansion* of a partial cube G' if $G' = \pi_f(G)$ for some Θ -class f of G . Indeed expansions can be detected within the smaller graph. Let G' be a partial cube containing two isometric subgraphs G'_1 and G'_2 such that $G' = G'_1 \cup G'_2$, there are no edges from $G'_1 \setminus G'_2$ to $G'_2 \setminus G'_1$, and denote $G'_0 := G'_1 \cap G'_2$. A graph G is an expansion of G' with respect to G_0 if G is obtained from G' by replacing each vertex v of G'_1 by a vertex v_1 and each vertex v of G'_2 by a vertex v_2 such that u_i and v_i , $i = 1, 2$ are adjacent in G if and only if u and v are adjacent vertices of G'_i , and $v_1 v_2$ is an edge of G if and only if v is a vertex of G'_0 . The following is well-known:

Lemma 2.5 ([8, 11]). *A graph G is a partial cube if and only if G can be obtained by a sequence of expansions from a single vertex.*

We will make use of the following lemma about the interplay of contractions and expansions:

Lemma 2.6. *Assume that we have the following commutative diagram of contractions:*

$$\begin{array}{ccc} G & \xrightarrow{\pi_{f_1}} & \pi_{f_1}(G) \\ \downarrow \pi_{f_2} & & \downarrow \pi_{f_2} \\ \pi_{f_2}(G) & \xrightarrow{\pi_{f_1}} & \pi_{f_1}(\pi_{f_2}(G)) \end{array}$$

Assume that G is expanded from $\pi_{f_1}(G)$ along sets $G_1, G_2 \subseteq \pi_{f_1}(G)$. Then $\pi_{f_2}(G)$ is expanded from $\pi_{f_1}(\pi_{f_2}(G))$ along sets $\pi_{f_2}(G_1)$ and $\pi_{f_2}(G_2)$.

Proof. Let $\pi_{f_2}(G)$ be expanded from $\pi_{f_1}(\pi_{f_2}(G))$ along sets H_1, H_2 . Consider $v \in \pi_{f_1}(\pi_{f_2}(G))$. Vertex v is in $H_1 \cap H_2$ if and only if its preimage in $\pi_{f_2}(G)$ is an edge in E_{f_1} , which is equivalent to the property that the preimage of this edge is intersected by $E_{f_1}^+$ and $E_{f_1}^-$ in G . Furthermore, this is equivalent to the image of this subgraph in $\pi_{f_1}(G)$, say I , having at least one vertex in $G_1 \cap G_2$. The image I is contracted to v by π_{f_2} , thus I is an edge or a vertex. Since every edge must have both its endpoints in either G_1 or G_2 , we deduce that I has a vertex in $G_1 \cap G_2$ if and only if v in $\pi_{f_2}(G_1) \cap \pi_{f_2}(G_2)$. This proves that $H_1 \cap H_2 = \pi_{f_2}(G_1) \cap \pi_{f_2}(G_2)$.

Removing $H_1 \cap H_2 = \pi_{f_2}(G_1) \cap \pi_{f_2}(G_2)$ from $\pi_{f_1}(\pi_{f_2}(G))$ cuts it into two connected components, one a subset of H_1 , one a subset of H_2 . On the other hand, removing $G_1 \cap G_2$ from $\pi_{f_1}(G)$ also cuts it into two connected components, one in G_1 and one in G_2 . Since π_{f_2} maps connected subgraphs to connected subgraphs, we see that $H_1 = \pi_{f_2}(G_1)$ and $H_2 = \pi_{f_2}(G_2)$, or the other way around. \square

Let G be a partial cube and $E_f \in \mathcal{E}$ one of its Θ -classes. Assume that a halfspace E_f^+ (or E_f^-) is such that all its vertices are incident with edges from E_f . Then we call E_f^+ (or E_f^-) *peripheral*. In such a case we will also call E_f a peripheral Θ -class, and call G a *peripheral expansion* of $\pi_f(G)$. Note that an expansion along sets G_1, G_2 is peripheral if and only if one of the sets

G_1, G_2 is the whole graph and the other one an isometric subgraph. An expansion is called *full* if $G_1 = G_2$.

For a partial cube G and one of its Θ -classes $E_f \in \mathcal{E}$ the *zone graph* of G with respect to f is the graph $\zeta_f(G)$ whose vertices correspond to the edges of E_f and two vertices are connected by an edge if the corresponding edges of E_f lie in a convex cycle of G , see [25]. In particular, ζ_f can be seen as a mapping from edges of G that are not in E_f but lie on a convex cycle crossed by E_f to the edges of $\zeta_f(G)$. If $\zeta_f(G)$ is a partial cube, then we say that $\zeta_f(G)$ is *well-embedded* if for two edges a, b of $\zeta_f(G)$ we have $a\Theta b$ if and only if the sets of Θ -classes crossing $\zeta_f^{-1}(a)$ and $\zeta_f^{-1}(b)$ coincide and otherwise they are disjoint.

For discussing zone graphs in partial cubes the following will be useful. Let $v_1u_1, v_2u_2 \in E_e$ be edges in a partial cube G with $v_1, v_2 \in E_e^+$. Let $C_1, \dots, C_n, n \geq 1$, be a sequence of convex cycles such that v_1u_1 lies only on C_1 , v_2u_2 lies only on C_n , and each pair C_i and C_{i+1} , for $i \in \{1, \dots, n-1\}$, intersects in exactly one edge and this edge is in E_e , all the other pairs do not intersect. If the shortest path from v_1 to v_2 on the union of C_1, \dots, C_n is a shortest v_1, v_2 -path in G , then we call C_1, \dots, C_n a *convex traverse* from v_1u_1 to v_2u_2 . In [27] it was shown that for every pair of edges v_1u_1, v_2u_2 in relation Θ there exists a convex traverse connecting them.

Lemma 2.7. *Let G be a partial cube and $f \in \mathcal{E}$. Then $\zeta_f(G)$ is a well-embedded partial cube if and only if for any two convex cycles C, C' that are crossed by E_f and some E_g both C and C' are crossed by the same set of Θ -classes.*

Proof. The direction “ \Rightarrow ” follows immediately from the definition of well-embedded.

For “ \Leftarrow ” let G satisfy the property that for any two convex cycles C, C' that are crossed by E_f and some E_g both C, C' are crossed by the same set of Θ -classes.

Define an equivalence relation on the edges of $\zeta_f(G)$ by $a \sim b$ if and only if $\zeta_f^{-1}(a)$ and $\zeta_f^{-1}(b)$ are crossed by the same set of Θ -classes. Let $a', b' \in E_f$ be two edges of G corresponding to vertices of $\zeta_f(G)$. Then, there exists a convex traverse T from a' to b' , i.e., no two cycles in T share Θ -classes apart from f . By the property on convex cycles in G all such paths from a' to b' in $\zeta_f(G)$ are crossed by the same set of equivalence classes and each exactly once. Furthermore, if there was a path in $\zeta_f(G)$ not corresponding to a traverse, its cycles would repeat Θ -classes of G , thus cross several times equivalence classes of $\zeta_f(G)$. Thus, every equivalence class of \sim cuts $\zeta_f(G)$ into two convex subgraphs. We have that $\zeta_f(G)$ is a partial cube and the embedding we defined shows that it is well-embedded. \square

A well-embedded zone graph $\zeta_f(G)$ thus induces an equivalence relation on the Θ -classes of G except f , that are involved in convex cycles crossed by E_f . We denote by $[e]$ the class of Θ -classes containing E_e . Note that $[e]$ corresponds to a Θ -class of $\zeta_f(G)$ and vice versa.

The following will be useful:

Lemma 2.8. *Let $\zeta_f(G)$ be a well-embedded partial cube and g, h two equivalent Θ -classes of G and C a convex cycle crossed by E_f, E_g, E_h . If $a \in E_f$ is an edge such that $a = C \cap E_f \cap E_g^+ = C \cap E_f \cap E_h^+$, then each edge of E_f is either in $E_g^+ \cap E_h^+$ or in $E_g^- \cap E_h^-$.*

Proof. Suppose otherwise, that there is an edge b in $E_f \cap E_g^+ \cap E_h^-$. The interval from a to b is crossed by E_h but not by E_g . Let T be a convex traverse from a to b . Then there exists a convex cycle on T crossed by E_f and E_h but not by E_g contradicting Lemma 2.7. \square

Lemma 2.8 justifies that if $\zeta_f(G)$ is a well-embedded partial cube and $g \in \mathcal{E} \setminus \{f\}$, then we can orient $[g]$ in $\zeta_f(G)$ such that $\rho_{[g]^+}(\zeta_f(G)) = \zeta_f(\rho_{g^+}(G))$.

Lemma 2.9. *Let G be a partial cube, $f \in \mathcal{E}$ such that $\zeta_f(G)$ is a well-embedded partial cube, $A \subseteq \mathcal{E} \setminus \{f\}$ and $X \in \{\pm\}^A$. We have $\pi_{\{[e] \mid e \subseteq A\}}(\zeta_f(G)) = \zeta_f(\pi_A(G))$ and $\rho_{\{[e]^{x_e} \mid e \in A\}}(\zeta_f(G)) = \zeta_f(\rho_X(G))$.*

Proof. For the contractions, clearly any contraction in $\zeta_f(G)$ corresponds to contracting the corresponding equivalence classes in G . Conversely, if some Θ -classes A are contracted in G , this affects only the classes of $\zeta_f(G)$ such that all the corresponding edges in G are contained in A .

Taking a restriction in $\zeta_f(G)$ can be modeled by restricting to the respective sides of all the elements of the corresponding class of Θ -classes of G .

By Lemma 2.8, if a set of restrictions in G leads to a non-empty zone graph, there is an orientation for all the elements of the classes of Θ -classes containing them leading to the same result. \square

2.2 Pc-minors and expansions versus metric subgraphs

In this section we present conditions under which contractions, restrictions, and expansions preserve metric properties of subgraphs. Let $G = (V, E)$ be an isometric subgraph of the hypercube $Q_{\mathcal{E}}$ and let S be a subset of vertices of G . Let f be any coordinate of \mathcal{E} . We will say that E_f is *disjoint* from S if it does not cross S and has no vertices in S .

Lemma 2.10. *If H is a convex subgraph of G and $f \in \mathcal{E}$, then $\rho_{f^+}(H)$ is a convex subgraph of $\rho_{f^+}(G)$. If E_f crosses H or is disjoint from H , then also $\pi_f(H)$ is a convex subgraph of $\pi_f(G)$.*

Lemma 2.11. *If S is a subset of vertices of G and $f \in \mathcal{E}$, then $\pi_f(\text{conv}(S)) \subseteq \text{conv}(\pi_f(S))$. If E_f crosses S , then $\pi_f(\text{conv}(S)) = \text{conv}(\pi_f(S))$.*

Lemma 2.12. *If H' is a convex subgraph of G' and G is obtained from G' by an isometric expansion, then the expansion of H of H' is a convex subgraph of G .*

Let H be a subgraph of G . If for a vertex $x \in H$ there is a vertex $-_H x \in H$ such that $\text{conv}(x, -_H x) = H$ we say that $-_H x$ is the *antipode* of x with respect to H and we omit the subscript H if this causes no confusion. Intervals in a partial cube are convex since intervals in hypercubes equal (convex) subhypercubes, therefore $\text{conv}(x, -_H x)$ consists of all the vertices on the shortest paths connecting x and $-_H x$. Then it is easy to see, that if a vertex has an antipode, it is unique. We call a subgraph H of a partial cube $G = (V, E)$ *antipodal* if every vertex x of H has an antipode with respect to H . Note that antipodal graphs are sometimes defined in a different but equivalent way and then are called symmetric-even, see [5]. By definition, antipodal subgraphs are convex. Their behavior with respect to pc-minors has been described in [10] in the following way:

Lemma 2.13. *Let H be an antipodal subgraph of G and $f \in \mathcal{E}$. If E_f is disjoint from H , then $\rho_{f^+}(H)$ is an antipodal subgraph of $\rho_{f^+}(G)$. If E_f crosses H or is disjoint from H , then $\pi_f(H)$ is an antipodal subgraph of $\pi_f(G)$.*

In particular, Lemma 2.13 implies that the class of antipodal partial cubes is closed under contractions. Next we will deduce a characterization of those expansions that generate all antipodal partial cubes from a single vertex, in the same way as Lemma 2.5 characterizes all partial cubes. Let G be an antipodal partial cube and G_1, G_2 two subgraphs corresponding to an isometric expansion. We say that it is an *antipodal expansion* if and only if $-G_1 = G_2$, where $-G_1$ is defined as the set of antipodes of G_1 .

Lemma 2.14. *Let G be a partial cube and $\pi_e(G)$ antipodal. Then G is an antipodal expansion of $\pi_e(G)$ if and only if G is antipodal. In particular, all antipodal partial cubes arise from a single vertex by a sequence of antipodal expansion.*

Proof. Say $\pi_e(G)$ is expanded to G along sets G_1, G_2 . Let $v \in G_1$ and $v' \in G$ a vertex with $\pi_e(v') = v$.

If G is antipodal, there exists a vertex $-v'$ whose distance to v' is equal to the number of Θ -classes of G . In particular, the shortest path must cross E_e , proving that $\pi_e(-v') \in G_2$. But $\pi_e(-v') = -v$ proving that $-v \in G_2$.

Conversely, if $-G_1 = G_2$ it is easy to see, that the antipode of v' is in $\pi_e^{-1}(-v)$. \square

A further useful property of antipodal subgraphs of partial cubes proved in [10] is the following:

Lemma 2.15. *If H is an antipodal subgraph of G , then H contains an isometric cycle C such that $\text{conv}(C) = H$.*

We call a partial cube *affine* if it is a halfspace of an antipodal partial cube. We can give the following intrinsic characterization of affine partial cubes.

Proposition 2.16. *A partial cube G is affine if and only if for all u, v vertices of G there are $w, -w$ in G such that $\text{conv}(u, w)$ and $\text{conv}(v, -w)$ are crossed by disjoint sets of Θ -classes.*

Proof. Let $G = E_f^+(\tilde{G})$ be a halfspace of an antipodal partial cube \tilde{G} . For $u, v \in G$ consider the antipode $-\tilde{G}v$ of v in $E_f^-(\tilde{G})$. By Lemma 2.15, we can consider an isometric cycle C through $v, u, -\tilde{G}v$ such that $\text{conv}(C) = \tilde{G}$. The two vertices w, z on $C \cap E_f^+(\tilde{G})$ that are incident with edges from $E_f^-(\tilde{G})$ are connected on C by a shortest path crossing all the Θ -classes of G , i.e. $z = -_G w$. By symmetry of $w, -_G w$, we can assume that v appears before u on a shortest path from w to $-_G w$. Thus $w, -_G w \in G$ are such that $\text{conv}(u, w)$ and $\text{conv}(v, -_G w)$ are crossed by disjoint sets of Θ -classes.

Conversely, let G be such that for all $u, v \in G$ there are $w, -w \in G$ such that $\text{conv}(u, w)$ and $\text{conv}(v, -w)$ are crossed by disjoint sets of Θ -classes. We construct \tilde{G} by taking a copy G' of G and join w with an edge to $(-w)'$ for each pair $w, -w \in G$. Associating all these new edges to a new coordinate of the hypercube we get an embedding into a hypercube of dimension one higher. First we show that \tilde{G} is a partial cube. Since G and its copy on their own are partial cubes, suppose now that $u \in G$ and $v' \in G'$. In G we can take $w, -_G w \in G$ such that $\text{conv}_G(u, w)$ and $\text{conv}_G(v, -_G w)$ are crossed by disjoint sets of Θ -classes. Consider a shortest path from u to w , then the edge to $(-w)'$, and finally a shortest path from $(-w)'$ to v' . Since none of the original Θ -classes was crossed twice, this is a shortest path of the hypercube that \tilde{G} is embedded in.

It remains to show that \tilde{G} is antipodal. For every vertex $v \in G$ there exists $w, -w \in G$ such that $\text{conv}(v, w)$ and $\text{conv}(v, -w)$ are crossed by disjoint sets of Θ -classes. In fact, in this case $\text{conv}(v, w)$ and $\text{conv}(v, -w)$ together cross all Θ -classes of G . Hence taking a shortest path from v to w , then the edge to $(-w)'$ and from there a shortest path to v' yields a path from v to v' crossing each Θ -class of \tilde{G} exactly once. This implies that v' is an antipode of v . \square

By Lemma 2.3 a contraction of a halfspace is a halfspace and by Lemma 2.13 antipodal partial cubes are closed under contraction, therefore we immediately get:

Lemma 2.17. *The class of affine partial cubes is closed under contraction.*

A subgraph H of G , or just a set of vertices of H , is called *gated* (in G) if for every vertex x outside H there exists a vertex x' in H , the *gate* of x , such that each vertex y of H is connected with x by a shortest path passing through the gate x' . It is easy to see that if x has a gate in H , then it is unique and that gated subgraphs are convex.

In [10] it was shown that gated subgraphs behave well with respect to pc-minors:

Lemma 2.18. *If H is a gated subgraph of G , then $\rho_{f^+}(H)$ and $\pi_f(H)$ are gated subgraphs of $\rho_{f^+}(G)$ and $\pi_f(G)$, respectively.*

In the next lemma we will see that expansions can turn gated graphs into non-gated graphs.

Lemma 2.19. *Let G be an expansion of $\pi_e(G)$ along sets G_1, G_2 . Let H be a subgraph of $\pi_e(G)$, v a vertex of $\pi_e(G)$ and v' the gate of v in H . If $v \in G_1 \cap G_2$, $v' \notin G_1 \cap G_2$ and there exist $v'' \in H$, $v'' \in G_1 \cap G_2$, then the expansion of H in G is not gated.*

Proof. Let v, v', v'', H be as in the lemma and without loss of generality assume that $v' \in G_1 \setminus G_2$. Let E_e^+ correspond to G_1 and E_e^- to G_2 in G . Since vertex $v \in G_1 \cap G_2$, it is expanded to an edge in G . Let u be the vertex on this edge in E_e^+ . Then every shortest path from u to the expansion of H must cross at least the same Θ -classes as a shortest path from v to v' . On the other hand, $v, v' \in G_1$, thus in G_1 there exists a shortest path from v to v' . Then there exists a shortest path from u to the expansion of H , first crossing E_e^- and then all the Θ -classes in the shortest path from v to v' . Note that the expansion of v' is not the gate of u since there is no shortest path from u to the expansion of v'' in E_e^+ passing this vertex. Thus if u has a gate to the expansion of H , it must be at distance $d(v, v')$ to u and the shortest path connecting them must be crossed by exactly those Θ -classes that cross shortest paths from v to v' . Then this gate must be adjacent to the expansion of v' and be in E_e^+ . This is impossible, since $v' \in G_1 \setminus G_2$. \square

3 Systems of sign-vectors

We follow the standard OM notation from [7] and concerning COMs we stick to [3]. Let \mathcal{E} be a non-empty finite (ground) set and let $\emptyset \neq \mathcal{L} \subseteq \{\pm, 0\}^{\mathcal{E}}$. The elements of \mathcal{L} are referred to as *covectors*.

For $X \in \mathcal{L}$, and $e \in \mathcal{E}$ let X_e be the value of X at the coordinate e . The subset $\underline{X} = \{e \in \mathcal{E} : X_e \neq 0\}$ is called the *support* of X and its complement $X^0 = \mathcal{E} \setminus \underline{X} = \{e \in \mathcal{E} : X_e = 0\}$ the *zero set* of X . For $X, Y \in \mathcal{L}$, we call $S(X, Y) = \{f \in \mathcal{E} : X_f Y_f = -\}$ the *separator* of X and Y . The *composition* of X and Y is the sign-vector $X \circ Y$, where $(X \circ Y)_e = X_e$ if $X_e \neq 0$ and

$(X \circ Y)_e = Y_e$ if $X_e = 0$. For a subset $A \subseteq \mathcal{E}$ and $X \in \mathcal{L}$ the *reorientation* of X with respect to A is the sign-vector defined by

$$({}_A X)_e := \begin{cases} -X_e & \text{if } e \in A \\ X_e & \text{otherwise.} \end{cases}$$

In particular $-X :=_{\mathcal{E}} X$. The *reorientation* of \mathcal{L} with respect to A is defined as ${}_A \mathcal{L} := \{{}_A X \mid X \in \mathcal{L}\}$. In particular, $-\mathcal{L} :=_{\mathcal{E}} \mathcal{L}$.

We continue with the formal definition of the main axioms relevant for COMs, AOMs, OMs, and LOPs. All of them are closed under reorientation.

Composition:

(C) $X \circ Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

Since \circ is associative, arbitrary finite compositions can be written without bracketing $X_1 \circ \dots \circ X_k$ so that (C) entails that they all belong to \mathcal{L} . Note that contrary to a convention sometimes made in OMs we do not consider compositions over an empty index set, since this would imply that the zero sign-vector belonged to \mathcal{L} . The same consideration applies for the following two strengthenings of (C).

Face symmetry:

(FS) $X \circ -Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

By (FS) we first get $X \circ -Y \in \mathcal{L}$ and then $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y) \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$. Thus, (FS) implies (C).

Ideal composition:

(IC) $X \circ Y \in \mathcal{L}$ for all $X \in \mathcal{L}$ and $Y \in \{\pm, 0\}^{\mathcal{E}}$.

Note that (IC) implies (C) and (FS). The following axiom is part of all the systems of sign-vectors discussed in the paper:

Strong elimination:

(SE) for each pair $X, Y \in \mathcal{L}$ and for each $e \in S(X, Y)$ there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in \mathcal{E} \setminus S(X, Y)$.

An axiom particular to OMs is:

Zero vector:

(Z) the zero sign-vector $\mathbf{0}$ belongs to \mathcal{L} .

Another operation on sign-vectors is needed to introduce AOMs:

$$(X \oplus Y)_e := \begin{cases} 0 & \text{if } e \in S(X, Y) \\ (X \circ Y)_e & \text{otherwise.} \end{cases}$$

Affinity:

(A) $(X \oplus -Y) \circ Z \in \mathcal{L}$ for all $X, Y, Z \in \mathcal{L}$ such that for each $e \in S(X, -Y)$ and $W \in \mathcal{L}$ with $W_e = 0$ there are $f, g \in \mathcal{E} \setminus S(X, -Y)$ such that $W_f \neq (X \circ -Y)_f$ and $W_g \neq (-X \circ Y)_g$.

We are now ready to define the central systems of sign-vectors of the present paper:

Definition 3.1. A system of sign-vectors $(\mathcal{E}, \mathcal{L})$ is called a:

- complex of oriented matroids (COM) if \mathcal{L} satisfies (FS) and (SE),
- affine matroid (AOM) if \mathcal{L} satisfies (A), (FS), and (SE),
- oriented matroid (OM) if \mathcal{L} satisfies (Z), (FS), and (SE),
- lopsided system (LOP) if \mathcal{L} satisfies (IC) and (SE).

Note that for OMs one can replace (Z) and (FS) by (C) and:

Symmetry:

(Sym) $-X \in \mathcal{L}$ for all $X \in \mathcal{L}$.

Let $\mathcal{L} \subseteq \{0, -1, 1\}^{\mathcal{E}}$ be a system of sign-vectors and $e \in \mathcal{E}$. For $X \in \mathcal{L}$ let $X \setminus e$ be the element of $\{0, -1, 1\}^{\mathcal{E} \setminus \{e\}}$ obtained by deleting the coordinate e from X . Define operations $\mathcal{L}/e = \{X \setminus e \mid X \in \mathcal{L}, X_e = 0\}$ as taking the *hyperplane* of e (usually referred to as contraction) and $\mathcal{L} \setminus e = \{X \setminus e \mid X \in \mathcal{L}\}$ as the *deletion* of e . A sign-system that arises by deletion and taking hyperplanes from another one is called a *minor*. Furthermore denote by $\mathcal{L}_e^+ := \{X \setminus e \mid X \in \mathcal{L}, X_e = +\}$ and $\mathcal{L}_e^- := \{X \setminus e \mid X \in \mathcal{L}, X_e = -\}$ the positive and negative (open) *halfspaces* with respect to e .

A theorem due to Karlander [24] characterizes AOMs as exactly the halfspaces of OMs. However, his proof contains a flaw that has only been observed and fixed recently in [4].

The following is easy to see, see e.g. [3, Lemma 2].

Lemma 3.2. For any system of sign-vectors the operations of taking halfspaces, hyperplanes and deletion commute.

Our systems of sign-vectors behave well with respect to the above operations:

Lemma 3.3. The classes of COMs, AOMs, OMs, and LOPs are minor closed. Moreover, COMs and LOPs are closed under taking halfspaces.

Proof. The result for OMs is folklore and can be found for instance in [6]. For COMs this was shown in [3, Lemma 1] and [3, Lemma 4]. For LOPs it is easy to see, that (IC) is preserved under minors and taking halfspaces. The minor-closedness of AOMs is a little more involved, when only using the axioms given above, see e.g. [14], but using that they are halfspaces of OMs this follows directly from Lemma 3.2. \square

The *rank* of a system of sign-vectors $(\mathcal{E}, \mathcal{L})$ is the largest integer r such that there is subset $A \subseteq \mathcal{E}$ of size $|A| = r$ such that $\mathcal{L} \setminus A = \{\pm, 0\}^r$. In other words, the rank of $(\mathcal{E}, \mathcal{L})$ is just the VC-dimension of \mathcal{L} . Note that this definition of rank coincides with the usual rank definition for OMs, see [13].

A system of sign-vectors $(\mathcal{E}, \mathcal{L})$ is *simple* if it satisfies the following two conditions:

(N1*) for each $e \in \mathcal{E}$, $\{\pm, 0\} = \{X_e : X \in \mathcal{L}\}$;

(N2*) for each pair $e \neq f$ in \mathcal{E} , there exist $X, Y \in \mathcal{L}$ with $\{X_e X_f, Y_e Y_f\} = \{\pm\}$.

An element $e \in \mathcal{E}$ not satisfying (N1*) is called *redundant*. Two elements $e, f \in \mathcal{E}$ are called *parallel* if they do not satisfy (N2*). Note that parallelism is an equivalence relation on \mathcal{E} . We denote by $[e]$ the class of elements parallel to e , for $e \in \mathcal{E}$.

For every COM $(\mathcal{E}, \mathcal{L})$ there exists up to reorientation and relabeling of coordinates a unique simple COM, obtained by successively applying operation $\mathcal{L} \setminus e$ to the redundant coordinates $e \in \mathcal{E}$ and to elements of parallel classes with more than one element. See [3, Proposition 3] for the details. Note that by Lemma 3.2 the order in which these operations are taken is irrelevant and by Lemma 3.3 all the classes of systems of sign-vectors at consideration here, are closed under this operation. We will denote by $\mathcal{S}(\mathcal{E}, \mathcal{L})$ the unique *simplification* of $(\mathcal{E}, \mathcal{L})$.

4 Systems of sign-vectors and partial cubes

The *topes* of a system of sign-vectors $(\mathcal{E}, \mathcal{L})$ are the elements of $\mathcal{T} := \mathcal{L} \cap \{\pm\}^{\mathcal{E}}$. If $(\mathcal{E}, \mathcal{L})$ is simple, we define the *tope graph* $G(\mathcal{L})$ of $(\mathcal{E}, \mathcal{L})$ as the (unlabeled) subgraph of $Q_{\mathcal{E}}$ induced by \mathcal{T} . If $(\mathcal{E}, \mathcal{L})$ is non-simple, we consider $G(\mathcal{L})$ as the tope graph of its simplification $\mathcal{S}(\mathcal{E}, \mathcal{L})$.

In general $G(\mathcal{L})$ is an unlabeled graph and even though it is defined as a subgraph of a hypercube $Q_{\mathcal{E}}$ it could possibly have multiple non-equivalent embeddings in $Q_{\mathcal{E}}$. We call a system $(\mathcal{E}, \mathcal{L})$ a *partial cube system* if its tope graph $G(\mathcal{L})$ is an isometric subgraph of $Q_{\mathcal{E}}$ in which the edges correspond to sign-vectors of \mathcal{L} with a single 0. It is well-known that partial cubes have a unique embedding in $Q_{\mathcal{E}}$ up to automorphisms of $Q_{\mathcal{E}}$, see e.g. [28, Chapter 5]. In other words, the tope graph of a simple partial cube system is invariant under reorientation. For this reason we will, possibly without an explicit note, identify vertices of a partial cube $G(\mathcal{L})$ with subsets of $\{\pm\}^{\mathcal{E}}$. The following was proved in [3, Proposition 2]:

Lemma 4.1. *Simple COMs are partial cube systems.*

Before presenting basic results regarding partial cube systems, we discuss how the minor operations and taking halfspaces as defined in Section 3 affect tope graphs. So let $(\mathcal{E}, \mathcal{L})$ be a simple partial cube system.

First note that deletion does not affect the simplicity of $(\mathcal{E}, \mathcal{L})$. Furthermore, since $(\mathcal{E}, \mathcal{L})$ is a partial cube system, the tope graph $G(\mathcal{L} \setminus e)$ corresponds to $\pi_e(G(\mathcal{L}))$ obtained from $G(\mathcal{L})$ by contracting all the edges in the Θ -class corresponding to coordinate e , as defined in Section 2.

Also, the halfspace \mathcal{L}_e^+ is easily seen to be simple and its tope graph corresponds to the restriction $\rho_{e^+}(G(\mathcal{L}))$ to the positive halfspace of $E_e \in \mathcal{E}$, as defined in Section 2.

The hyperplane \mathcal{L}/e does not need to be a simple system of sign-vectors nor a partial cube system. However, we can establish the following:

Lemma 4.2. *Let $(\mathcal{E}, \mathcal{L})$ be a partial cube system and $e \in \mathcal{E}$. If $\zeta_e(G(\mathcal{L}))$ is a well-embedded partial cube, then $\zeta_e(G(\mathcal{L})) \cong G(\mathcal{L}/e)$.*

Proof. Clearly, both sets of vertices correspond to the set of edges of $G(\mathcal{L})$. If there is an edge in $G(\mathcal{L}/e)$, it corresponds to two edges of $G(\mathcal{L})$ in E_e such that the Θ -classes different than E_e crossing the interval between the two edges form parallel elements of $G(\mathcal{L}/e)$. This means that the interval does not cross E_e in other elements besides the two edges. The interval must include a convex traverse between the two edges, but since there is no other edge in E_e in it, the traverse is a convex cycle. This implies that there is a corresponding edge in $\zeta_e(G(\mathcal{L}))$.

Now, let $\zeta_e(G(\mathcal{L}))$ be a well-embedded partial cube. An edge of it corresponds to a convex cycle C . By Lemma 2.7 all cycles crossing E_e and another Θ -class from C cross all its Θ -classes. By Lemma 2.8, the corresponding elements in \mathcal{L}/e are parallel. Therefore the edge corresponds to an edge of $G(\mathcal{L}/e)$. \square

The correspondences before the lemma in particular give that deletions and halfspaces of partial cube systems coincide with pc-minors, which together with Lemma 3.3 gives:

Proposition 4.3. *Let $(\mathcal{E}, \mathcal{L})$ and $(\mathcal{E}', \mathcal{L}')$ be simple partial cube systems with tope graphs $G(\mathcal{L})$ and $G(\mathcal{L}')$, respectively. If $(\mathcal{E}', \mathcal{L}')$ arises from $(\mathcal{E}, \mathcal{L})$ by deletion and taking halfspaces, then $G(\mathcal{L}')$ is a pc-minor of $G(\mathcal{L})$. Moreover, the families of tope graphs of COMs and LOPs are pc-minor closed.*

In the following, we will describe further how pc-minors and equivalently deletions and halfspaces of partial cube systems translate metric graph properties as introduced in Section 2 into properties of sign-vectors.

For $X \in \mathcal{L}$ we set $\mathcal{T}(X) := \{T \in \mathcal{T} \mid X \circ T = T\}$ and denote by $G(X)$ the subgraph of $G(\mathcal{L})$ induced by $\mathcal{T}(X)$. Furthermore, let $\mathcal{H}(\mathcal{L}) = \{G(X) \mid X \in \mathcal{L}\}$ be the set of subgraphs $G(\mathcal{L})$ obtained by considering $G(X)$ for all $X \in \mathcal{L}$. Conversely, given a convex subgraph G' of a partial cube G with Θ -classes \mathcal{E} denote by $X(G')$ the sign-vector with $X(G')_e = +$ or $X(G')_e = -$ if $G' \subseteq E_e^+$ or $G' \subseteq E_e^-$, respectively, and 0 otherwise for all $E_e \in \mathcal{E}$. Note that for each vertex $v \in G(\mathcal{L})$, $X(v) = v$. Furthermore, let $\mathcal{L}(\mathcal{H}) = \{X(G') \mid G' \in \mathcal{H}\}$ for a set \mathcal{H} of convex subgraphs of G .

Proposition 4.4. *In a simple partial cube system $(\mathcal{E}, \mathcal{L})$ for each $X \in \mathcal{L}$ its tope-graph $G(X)$ is a convex subgraph of $G(\mathcal{L})$. Conversely, if $G = (V, E)$ is a partial cube and \mathcal{H} a set of convex subgraphs of G , such that \mathcal{H} includes all the vertices of G , then there is a simple $(\mathcal{E}, \mathcal{L})$ such that $G = G(\mathcal{L})$ and $\mathcal{H} = \mathcal{H}(\mathcal{L})$.*

Proof. Let $X \in \mathcal{L}$. Since $G(\mathcal{L})$ is an isometric subgraph of $Q_{\mathcal{E}}$, each $e \in \mathcal{E}$ can be identified with a Θ -class $E_e \in \mathcal{E}$ such that the halfspaces E_e^+ and E_e^- of $G(\mathcal{L}) \setminus E_e$ are convex. Now, $\mathcal{T}(X)$ induces the subgraph $\bigcap_{e \in X} E_e^{X_e}$, i.e. is a restriction of $G(\mathcal{L})$ and therefore is convex.

For the converse, by Lemma 2.1 any convex subgraph G' in G may be described as the intersection of convex halfspaces E_e^{\pm} for some $E_e \in \mathcal{E}$. This allows interpreting G' as the sign-vector in $X(G') \in \{\pm 1, 0\}^{\mathcal{E}}$. Since every vertex is contained in \mathcal{H} , we have $G(\mathcal{L}) = G$. \square

The following establishes a connection between the gates of a convex set and the composition operator.

Lemma 4.5. *Let G be a partial cube embedded in a hypercube, G' a convex subgraph of G and v a vertex of G . Then w is the gate for v in G' if and only if $X(w) = X(G') \circ X(v)$. Therefore, a subgraph G' is gated if and only if for all $v \in G$ there is a $w \in G$ such that $X(G') \circ X(v) = X(w)$.*

Proof. First note that $X(w) = X(G') \circ X(v) \Leftrightarrow S(X(v), X(w)) = S(X(v), X(G'))$. If $S(X(v), X(w)) = S(X(v), X(G'))$, then the concatenation of a shortest (v, w) -path and a shortest (w, w') -path for $w' \in G'$ does not cross any Θ -class twice, since by convexity of G' the Θ -classes crossed by the second part all are from $X(G')^0$. Hence it is a shortest path.

Now, let $v \in G$ and w a gate for v in G' . Suppose $e \in S(X(v), X(w)) \setminus S(X(v), X(G'))$. This means a Θ -class $E_e \in \mathcal{E}$ splits G and also G' into two halfspaces, but v and w do not lie on the same side. This is w cannot lie on a shortest path from v to a vertex in the halfspace of G' not containing w . \square

Lemma 4.6. *In an antipodal partial cube G , the antipodal mapping $v \mapsto -v$ is a graph automorphism and for every convex subgraph $-X(G') = X(-G')$.*

Proof. Since every vertex has a unique antipode, $v \mapsto -v$ is indeed a mapping. Since it is an involution it is bijective. To show that it is indeed a homomorphism let vw be an edge of G . Since G is a partial cube and since $\text{conv}(v, -v) = G$ there is a shortest $(v, -v)$ -path starting with the edge vw . Analogously, there is a shortest $(w, -w)$ -path starting with the edge vw . Say this edge is in Θ -class E_e . Now, $S(X(v), X(-v)) \setminus S(X(v), X(-w)) = e$ and hence $S(X(-v), X(-w)) = e$, i.e. $-v - w$ is an edge.

Now, let G' be a convex subgraph of G . Since $X(G')$ just records the signs of the halfspaces of the classes of \mathcal{E} which contain G' and the antipodal mapping sends each vertex v to the vertex sitting on all the other sides, we obtain the result. \square

For a partial cube G isometrically embedded in a hypercube $Q_{\mathcal{E}}$ define $\mathcal{L}(G) = \{X \in \{0, \pm\}^{\mathcal{E}} \mid X \circ (-Y) \in \mathcal{T}(X(G)), \text{ for all } Y \in \mathcal{T}(X(G))\}$.

Lemma 4.7. *Let G be a partial cube. Then $\mathcal{L}(G)$ is a partial cube system that satisfies (FS) (and therefore (C)) and the set $\mathcal{H}(\mathcal{L}(G))$ of corresponding subgraphs coincides with the antipodal gated subgraphs of G .*

Proof. Since (FS) implies (C) (also when restricted to topes) we have that $X \circ Y \in \mathcal{T}(X)$ for $Y \in \mathcal{T}(X(G))$. To show (FS) for $\mathcal{L}(G)$ let $X, Y \in \mathcal{L}(G)$ and $Z \in \mathcal{T}(X(G))$ and note that $(X \circ (-Y)) \circ (-Z) = X \circ (-Y \circ Z)$, which by (C) is in $\mathcal{T}(X(G))$.

We prove now the second part of the statement. For each vertex $v \in G(X)$ we have that $X \circ -X(v) \in \mathcal{T}(X)$ is the antipode of T in $G(X)$. This is, $G(X)$ is antipodal. Furthermore, since (FS) implies (C) we also have that $X \circ X(v) \in \mathcal{T}(X)$, for all T . By Lemma 4.5 we have that $G(X)$ is gated.

Conversely, if A is an antipodal gated subgraph of G , then by Lemma 4.5, we have that $X(A) \circ X(v) \in \mathcal{T}(X(G))$ for all $v \in G$. However, the antipode of the gate of v in A has to correspond to $X(A) \circ -X(v)$. Thus, $X(A) \in \mathcal{L}(G)$. \square

Proposition 4.4 states that in a simple system of sign-vectors there is a correspondence between its vectors and a subset of the set of convex subgraphs of its tope graph. The following proposition determines which convex subgraphs are in the subset if the system is a COM.

Theorem 4.8. *For a simple COM $(\mathcal{E}, \mathcal{L})$ with tope set \mathcal{T} we have*

$$\begin{aligned} \mathcal{L} &= \{X(G') \mid G' \text{ antipodal subgraph of } G(\mathcal{L})\} \\ &= \{X(G') \mid G' \text{ antipodal gated subgraph of } G(\mathcal{L})\} \\ &= \mathcal{L}(G). \end{aligned}$$

In particular, in a tope graph of a COM all antipodal subgraphs are gated.

Proof. Trivially, we have

$$\{X(G') \mid G' \text{ antipodal gated subgraph of } G(\mathcal{L})\} \subseteq \{X(G') \mid G' \text{ antipodal subgraph of } G(\mathcal{L})\}.$$

The equality

$$\mathcal{L}(G) = \{X(G') \mid G' \text{ antipodal gated subgraph of } G(\mathcal{L})\}$$

is precisely Lemma 4.7, while the inclusion $\mathcal{L} \subseteq \mathcal{L}(G)$ follows immediately from (FS).

Thus, we end by proving $\{X(G') \mid G' \text{ antipodal subgraph of } G(\mathcal{L})\} \subseteq \mathcal{L}$. Let G' be an antipodal subgraph of $G(\mathcal{L})$. We show $X(G') \in \mathcal{L}$ by induction on $|X(G')^0|$. If $|X(G')^0| = 0$ then $X(G') \in \mathcal{T}$ and we are done. If $|X(G')^0| > 0$ then take a maximal proper antipodal subgraph G'' of G' . Since vertices are antipodal subgraphs, G'' exists. By induction hypothesis $X(G'') \in \mathcal{L}$. Define G''' to be the subgraph of G' induced by all antipodes of vertices of G'' with respect to G' . By Lemma 4.6, we have $G' \cong G'''$. Moreover, $X(G'') = X(G') \circ -X(G''')$. Since G'' is a proper antipodal subgraph of G' , there exists an $e \in X(G'') \cap X(G')^0$. Then $e \in S(X(G''), X(G'''))$ and we now can apply (SE) to $X(G'')$ and $X(G''')$ with respect to e . We obtain $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X(G'') \circ X(G'''))_f$ for all $f \in \mathcal{E} \setminus S(X(G''), X(G'''))$. By the inclusions that we have shown in the first two parts of the proof, Z corresponds to an antipodal subgraph $G(Z)$. By (SE), $G(Z)$ strictly contains G'' and is contained in G' . By the maximality of G'' , $G(Z) = G'$. \square

As a consequence every simple COM is uniquely determined by its tope set, or up to re-orientation by its tope graph, which had only been proved in a non-constructive way, see [3, Propositions 1 & 3]. The constructive statement here is in fact a generalization of a theorem known for OMs, usually attributed to Mandel, see [12]. We will from now on denote the class of tope graphs of COMs by \mathcal{G}_{COM} and the class of partial cubes in which antipodal subgraphs are gated by AG, i.e. the above theorem implies $\mathcal{G}_{\text{COM}} \subseteq \text{AG}$.

Lemma 4.9. *A partial cube G is in \mathcal{G}_{COM} if and only if $\mathcal{L}(G)$ is a simple COM.*

Proof. If $\mathcal{L}(G)$ is a simple COM, then G is by definition its tope graph, thus G is in \mathcal{G}_{COM} . On the other hand, let G be a graph in \mathcal{G}_{COM} , and \mathcal{L} its up to reorientation unique simple COM. By Theorem 4.8, $\mathcal{L} = \{X \in \{0, -1, 1\}^E \mid X \circ -T \in \mathcal{T} \text{ for all } T \in \mathcal{T}\} = \mathcal{L}(G)$. \square

Lemma 4.10. *In a COM $(\mathcal{E}, \mathcal{L})$ all zone graphs $\zeta_f(G(\mathcal{L}))$ are well-embedded partial cubes. In particular, \mathcal{G}_{COM} is closed under taking zone graphs.*

Proof. Suppose there are two convex cycles C, C' in $G(\mathcal{L})$ that contradict Lemma 2.7, i.e., both are crossed by E_f and E_g and C is crossed by E_h but C' is not. By the second equivalence in Theorem 4.8, the cycles are gated. Without loss of generality assume that C' is completely in E_h^+ . On the other hand, we can reorient E_f and E_g in a way that the vertices on $C \cap E_h^+$ are in $E_f^+ \cap E_g^+, E_f^+ \cap E_g^-$ and $E_f^- \cap E_g^-$. But then any vertex in C' that is in $E_f^- \cap E_g^+$ has no gate to C , a contradiction.

Now, since COM is closed under taking hyperplanes by Lemma 3.3, Lemma 4.2 gives that \mathcal{G}_{COM} is closed under taking zone graphs. \square

Let us finally describe how tope graphs of the other systems of sign-vectors from Section 3 specialize tope graphs of COMs. We will denote the classes of tope graphs of OMs, AOMs, and LOPs by \mathcal{G}_{OM} , \mathcal{G}_{AOM} , and \mathcal{G}_{LOP} . A consequence of Lemma 4.6 is:

Proposition 4.11. *A graph is in \mathcal{G}_{OM} if and only if it is antipodal and in \mathcal{G}_{COM} .*

A not yet intrinsic description of tope graphs of AOMs follows:

Proposition 4.12. *A graph is in \mathcal{G}_{AOM} if and only if it is a halfspace of a graph in \mathcal{G}_{OM} .*

Interpreting axiom (IC) in the partial cube model we also get:

Proposition 4.13. *A graph is in \mathcal{G}_{LOP} if and only if all its antipodal subgraphs are hypercubes and it is in \mathcal{G}_{COM} .*

Noticing that $\mathcal{L}(Q_r)$, for a hypercube Q_r , equals $\{\pm, 0\}^r$ we immediately get the following lemma from the definition of the rank of a system of sign-vectors.

Lemma 4.14. *The rank of a COM $(\mathcal{E}, \mathcal{L})$ is the largest r such that $G(\mathcal{L})$ contracts to Q_r .*

5 The excluded pc-minors

Let Q_n be the hypercube, $v \in Q_n$ any of its vertices and $-v$ its antipode. Let $Q_n^- := Q_n \setminus -v$ be the hypercube minus one vertex. Consider the set of partial cubes arising from Q_n^- by deleting any subset of $N(v) \cup \{v\}$. It is easy to see that if $n \geq 4$ a graph obtained this way from Q_n^- is a partial cube unless v is not deleted but at least two of its neighbors are deleted. Denote by Q_n^{-*} the partial cube obtained by deleting exactly one neighbor of v , and by $Q_n^{-}(m)$ the graph obtained by deleting v and m neighbors of v , respectively, where for $Q_n^{-}(0)$ we sometimes simply write Q_n^- . It is easy to see that Q_n^- and Q_n^{-} are tope graphs of (realizable) COMs. For $n \leq 3$ all the partial cubes arising by the above procedure are isomorphic to Q_n^- or Q_n^{-} , thus the interesting graphs appear for $n \geq 4$. Denote their collection by $\mathcal{Q}^- = \{Q_n^{-*}, Q_n^{-}(m) \mid 4 \leq n; 1 \leq m \leq n\}$.

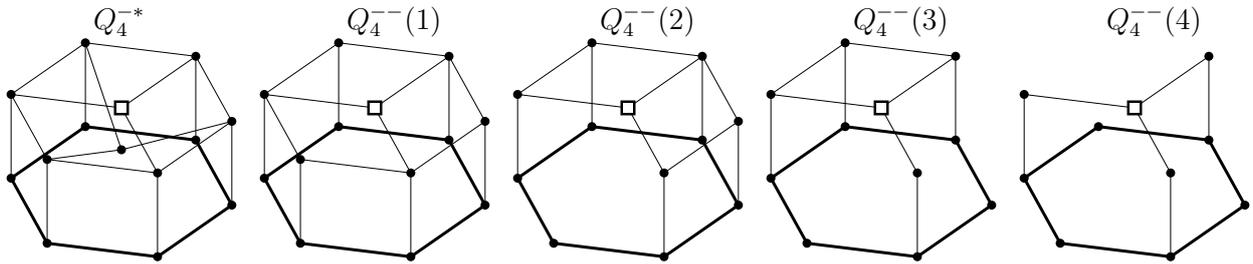


Figure 1: Graphs $Q_4^{-*}, Q_4^{-}(m)$, for $1 \leq m \leq 4$. The square vertex has no gate in the bold C_6 .

Lemma 5.1. *The class \mathcal{Q}^- is pc-minor minimal, i.e. any pc-minor of a graph in \mathcal{Q}^- is not in \mathcal{Q}^- . Furthermore, any graph in \mathcal{Q}^- contains an antipodal subgraph that is not gated, i.e. $\mathcal{Q} \subseteq \overline{\text{AG}}$.*

Proof. For any $G \in \mathcal{Q}^-$ a contraction or a restriction of it is a graph isomorphic to a hypercube, a hypercube minus a vertex, or a hypercube minus two antipodal vertices. All pc-minors of these graphs are isomorphic to a hypercube or a hypercube minus a vertex. Thus, no proper pc-minor of G is in \mathcal{Q}^- .

To see that any graph in \mathcal{Q}^- contains an antipodal subgraph that is not gated, let $G \in \mathcal{Q}^-$ and let w be a neighbor of v that was deleted from Q_n^- to obtain G . The convex subgraph $A \subseteq G$ obtained from restricting Q_n^- to the halfspace containing w and not v is isomorphic to Q_{n-1}^- . In particular, A is antipodal. But A is not gated, since the neighbor u of $-v$ in $G \setminus A$ has no gate in A . In fact, it can be seen by Lemma 4.5, that the gate of u , if existent, must be of the form $X(A) \circ X(u) = X(-v)$ which is not in G . \square

The following lemma will be useful for detecting pc-minors from \mathcal{Q}^- . We define H to be the *full subdivision* of a graph G if every edge of it is replaced by a path of length 2 to obtain H . The vertices of H that correspond to vertices of G are called its *original vertices*.

Lemma 5.2. *Let G be a partial cube and H an isometric subgraph isomorphic to a full subdivision of K_m such that*

- *no vertex of G is adjacent to all the original vertices of H ,*
- *the convex hull of H is neither isomorphic to Q_m^- nor Q_m^{--} ,*
- *H is inclusion minimal with this properties.*

Then the convex hull of H is in \mathcal{Q}^- .

Proof. Let H be as in the lemma. First note that $m \geq 4$ since for $m = 2$, H is isomorphic to $P_3 = Q_2^-$, and for $m = 3$, H is isomorphic to C_6 . Taking into account that there is no vertex of G adjacent to all the original vertices of H , the convex hull in both cases is isomorphic to Q_m^- or to Q_m^{--} .

Now consider H embedded in a hypercube Q_m with $m \geq 4$. Since H is an isometric subgraph of G , it is a partial cube and thus has up to reorientation a unique embedding in Q_m . Note that a possible embedding, and thus the only embedding, is such that there exists a vertex in Q_m adjacent to precisely all the original vertices of H . Let v be such a vertex. By the assumption, v is not in G . Let $\{v_1, \dots, v_{m-1}\}$ be any subset of the original vertices of H of size $m-1$. Since H is inclusion minimal, the convex hull of $\{v_1, \dots, v_{m-1}\}$ is a graph isomorphic to Q_{m-1}^- or Q_{m-1}^{--} in which v is a missing vertex. Note that the antipode of v in Q_m is at distance m from v . We have proved that all the vertices of Q_m at distance at most $m-2$ from v are in G while some of the neighbors of the antipode of v in Q_m are possibly not in G . Thus the convex hull of H is a graph in \mathcal{Q}^- . \square

A second useful lemma, tells us how to find excluded pc-minors in non-antipodal graphs with antipodal contractions.

Lemma 5.3. *Let G be a partial cube with $v \in G$ such that $-v \notin G$, and E_{e_1}, \dots, E_{e_k} be Θ -classes of G such that $\pi_{e_i}(G)$ is antipodal for all $1 \leq i \leq k$. Then either G contains a convex subgraph from \mathcal{Q}^- or G contains a convex Q_k^- or Q_k^{--} crossed by precisely E_{e_1}, \dots, E_{e_k} such that $-v$ is a missing vertex of it. In particular, if the latter holds and E_{e_1}, \dots, E_{e_k} are all the Θ -classes of G , then G is isomorphic to Q_k^- .*

Proof. Let $v \in G$ be a vertex without an antipode in G . Let E_{e_1}, \dots, E_{e_k} be the Θ -classes as above. Every contraction $\pi_{e_i}(G)$ is antipodal, therefore $\pi_{e_i}(v)$ has an antipode in $\pi_{e_i}(G)$. Let $v_i \in G$ be the preimage of the antipode $-\pi_{e_i}(G)\pi_{e_i}(v)$ of $\pi_{e_i}(v)$ in $\pi_{e_i}(G)$ for $1 \leq i \leq k$, respectively. By definition, v_1, \dots, v_k are pairwise at distance 2, thus a part of an isometric subgraph isomorphic to the subdivision of K_k . Moreover, there is no vertex adjacent to all of them, since v has no antipode in G . By Lemma 5.2, the convex hull of v_1, \dots, v_k is isomorphic to one of Q_k^-, Q_k^- , or there is an inclusion minimal subdivided $K_{k'}$ for $k' \leq k$ whose convex closure is not isomorphic to $Q_{k'}^-$ or $Q_{k'}^-$. In the former case we are done while in the latter case the convex closure of the inclusion minimal subdivided $K_{k'}$ is in \mathcal{Q}^- .

Assuming that E_{e_1}, \dots, E_{e_k} are all the classes of G , the convex hull of v_1, \dots, v_k is crossed by all the Θ -classes of G , hence it is G . Then G is isomorphic to Q_k^- or Q_k^- if it has no convex subgraph in \mathcal{Q}^- . Since G is not antipodal, it must be isomorphic to Q_k^- . \square

Lemma 5.4. *Let G be a partial cube. If a zone graph $\zeta_e(G)$ is not a well-embedded partial cube, then G has a pc-minor in $\{Q_4^*, Q_4^-(m) \mid 1 \leq m \leq 4\}$.*

Proof. For the sake of contradiction assume G does not satisfy the conditions of Lemma 2.7, i.e., there are convex cycles C_1, C_2 both crossed by E_e and E_f and C_1 is crossed by E_g but C_2 is not. Without loss of generality assume that C_2 is completely in E_g^+ . On the other hand, we can reorient E_e and E_f in a way that the vertices on $C_1 \cap E_h^+$ are in $E_e^+ \cap E_f^+, E_e^+ \cap E_f^-$ and $E_e^- \cap E_f^-$. But then any vertex v in C_2 that is in $E_e^- \cap E_f^+$ has no gate to C_1 .

We will now see, that this leads to the existence of a pc-minor in $\{Q_4^* \} \cup \{Q_4^-(m) \mid 1 \leq m \leq 4\}$. First contract any Θ -class different from E_e, E_f, E_g but crossing C_1 , obtaining graph G' . Then C_1 is contracted to a convex 6-cycle C'_1 in G' , by Lemma 2.11. It is non-gated since the image of v in G' still is in $E_e^- \cap E_f^+ \cap E_g^+$ while $E_e^- \cap E_f^+ \cap E_g^+ \cap C'_1 = \emptyset$. Now consider a maximal sequence S of contractions of Θ -classes different from E_e, E_f, E_g such that for the image C''_1 of C'_1 we have that $\text{conv}(C''_1)$ is a 6-cycle. Since v has no gate to C'_1 , contracting all the Θ -classes different from E_e, E_f, E_g maps v to a vertex in $\text{conv}(C''_1) \setminus C''_1$ contradicting that $\text{conv}(C''_1)$ is a 6-cycle. Thus S is not equal to all the Θ -classes different from E_e, E_f, E_g .

Pick any Θ -class E_h not in $S \cup \{E_e, E_f, E_g\}$. By maximality $\text{conv}(\pi_h(C''_1))$ is not a 6-cycle thus it must be isomorphic to Q_3^- or Q_3 . Let u be a vertex in Q_3^- or Q_3 adjacent to three vertices $\pi_h(u_1), \pi_h(u_2), \pi_h(u_3) \in \pi_h(C''_1)$ with $u_i \in C''_1$ for $i \in \{1, 2, 3\}$. No preimage u' of u is adjacent to any u_i , for $i \in \{1, 2, 3\}$, since otherwise u' would be in $\text{conv}(C''_1)$. Thus, u', u_1, u_2, u_3 are pairwise at distance two. Since $\text{conv}(C''_1)$ is a 6-cycle, there is no vertex adjacent to all of them. Together with their connecting 2-paths they form an isometric K_4 . Moreover, their convex hull is not isomorphic to Q_4^- or Q_4^- since such graphs have no convex 6-cycles. By Lemma 5.2, G has a pc-minor in $\{Q_4^*, Q_4^-(m) \mid 1 \leq m \leq 4\}$. \square

Lemma 5.5. *If $G \in \mathcal{Q}^-$, then there is a sequence (e_1, \dots, e_k) of Θ -classes such that $\zeta_{e_1, \dots, e_k}(G)$ is not a partial cube.*

Proof. Let $G \in \{Q_n^*, Q_n^-(m) \mid 1 \leq m \leq n\}$ for some $n \geq 4$. Let $v \in Q_n$ the vertex from the definition of \mathcal{Q}^- .

If $n = 4$ one can easily see in Figure 1 that G has a zone graph containing a C_5 if $G \in \{Q_4^*, Q_4^-(1), Q_4^-(2)\}$ or a C_3 if $G \in \{Q_4^-(3), Q_4^-(4)\}$ i.e., the zone graph is not a partial cube.

Let now $n > 4$. If $G \in \{Q_n^{-*}, Q_n^-(m) \mid 1 \leq m \leq n-1\}$, let w be one of the neighbors of v that is in G and let e be the Θ -class of G coming from the edge vw in Q_n . It is easy to see that $\zeta_e(G) \in \{Q_{n-1}^{-*}, Q_{n-1}^-(m) \mid 1 \leq m \leq n-1\}$. If otherwise $G = Q_n^-(n)$, then let w be any of the neighbors of v . Then w is missing in G and let e be the Θ -class of G coming from the edge vw in Q_n . One can check that $\zeta_e(G) = Q_{n-1}^-(n-1)$. The lemma follows by iterating the given zone graphs until arriving at a non partial cube. \square

A useful sort of converse of the proof of Lemma 5.5 is the following:

Lemma 5.6. *If G is partial cube and $\zeta_e(G) \in \mathcal{Q}^-$ for some Θ -class e , then G has a pc-minor in \mathcal{Q}^- .*

Proof. Assume that $\zeta_e(G)$ is well embedded since otherwise G has a pc-minor in \mathcal{Q}^- , by Lemma 5.4. Let G be pc-minor minimal, without affecting $H := \zeta_e(G)$ and let N be the number of Θ -classes of H . Then by minimality G is the convex hull of E_e and has $N+1$ Θ -classes. There exist two isometric copies of H in B with edges of E_e being a matching of them. Let v_1, \dots, v_N be the original vertices of the subdivided K_N in the first copy and v'_1, \dots, v'_N the original vertices of the subdivided K_N in the second copy. Not both subdivisions can have a vertex adjacent to all of the original vertices of the subdivisions since then there would be an edge in E_e between the two vertices, which is not true by the definition of H . Without loss of generality, assume that v_1, \dots, v_N have no common neighbor. Then by Lemma 5.2, either G has a pc-minor in \mathcal{Q}^- and we are done, or the convex hull of v_1, \dots, v_N is isomorphic to Q_N^- or Q_N^{-*} . Analogously, if v'_1, \dots, v'_N have no common neighbor their convex hull is isomorphic to Q_N^- or Q_N^{-*} . But then H is isomorphic to Q_N^- or Q_N^{-*} which is not the case. Thus v'_1, \dots, v'_N have a common neighbor, say u . Then u, v_1, \dots, v_N are pairwise at distance 2, without a common neighbor. Then by Lemma 5.2, either G has a pc-minor in \mathcal{Q}^- or their convex hull H is isomorphic to Q_{N+1}^- or Q_{N+1}^{-*} . The latter cannot be since then H is not in \mathcal{Q}^- . Therefore, G has the pc-minor in \mathcal{Q}^- . \square

We are ready to prove the main theorem of this section.

Theorem 5.7. *We have $\mathcal{F}(\mathcal{Q}^-) \subseteq \mathcal{G}_{\text{COM}}$.*

Proof. For the sake of contradiction, assume that we can pick G , a smallest graph that is not in \mathcal{G}_{COM} but in $\mathcal{F}(\mathcal{Q}^-)$. In particular, since $\mathcal{F}(\mathcal{Q}^-)$ is pc-minor closed, every pc-minor of G is in \mathcal{G}_{COM} .

By Lemma 4.9, G is a graph in \mathcal{G}_{COM} if and only if $\mathcal{L}(G)$ is a COM. Note that for every edge $a = uv$ of G , the corresponding sign-vector $X(a)$, with all the coordinates equal to the coordinates of u and v except for the coordinate e in which they differ equal to zero, is included in $\mathcal{L}(G)$ since $X(a) \circ (-Y)$ equals u or v for every $Y \in G$. Since G is not in \mathcal{G}_{COM} , but G is a partial cube and by Lemma 4.7 $\mathcal{L}(G)$ satisfies (C), [3, Theorem 3] gives that $\mathcal{L}(G)$ has a hyperplane $\mathcal{L}(G)/e$ that is not a COM.

Since $G \in \mathcal{F}(\mathcal{Q}^-)$, by Lemma 5.4 we have that $G' := \zeta_e(G)$ is a well-embedded partial cube. By Lemma 4.2 we get $G' \cong G(\mathcal{L}(G)/e)$, i.e., it is the tope graph of the hyperplane.

Claim 5.8. *We have $\mathcal{S}(\mathcal{L}(G)/e) = \mathcal{L}(G')$.*

Proof. By Lemma 4.7, the elements of $\mathcal{L}(G')$ correspond to antipodal subgraphs of G' . Furthermore, it is not hard to see that elements of $\mathcal{S}(\mathcal{L}(G)/e)$ correspond to antipodal subgraphs of G that are crossed by E_e , where redundant coordinates have been deleted. If A is an antipodal subgraph of G crossed by E_e , then by Lemma 4.6, each edge $uv \in E_e$ of A has an antipodal edge $-_A u -_A v \in E_e$. Thus the zone graph of A corresponding to E_e is an antipodal subgraph of G' and we get that $\mathcal{S}(\mathcal{L}(G)/e) \subseteq \mathcal{L}(G')$.

Conversely, assume that there is an antipodal graph A' in G' that does not correspond to a zone graph of an antipodal subgraph of G . By definition of G' we can identify its vertices with edges of G in E_e . Let A be the convex hull of those edges in G that correspond to vertices of A' , and let $E_e, E_{e_1}, \dots, E_{e_k}$ be the Θ -classes crossing A . Since A' does not correspond to a zone graph of an antipodal graph, A is not antipodal.

By minimality of G , for every E_{e_j} the contraction $\pi_{e_j}(G)$ is the tope graph of a COM. This is, $\mathcal{L}(\pi_{e_j}(G)) = \mathcal{L}(G) \setminus e_j$ is a COM. Hence, by Lemma 3.3 its hyperplane $(\mathcal{L}(G) \setminus e_j)/e$ is a COM as well. Now Lemma 3.2 gives $\mathcal{S}((\mathcal{L}(G) \setminus e_j)/e) = \mathcal{S}(\mathcal{L}(G)/e) \setminus e_j$. We have proved that $\mathcal{S}(\mathcal{L}(G)/e) \setminus e_j$ is a COM for every $E_{e_j} \in \{E_{e_1}, \dots, E_{e_k}\}$. Note that $\pi_{e_j}(G')$ is the tope graph of $\mathcal{S}(\mathcal{L}(G)/e) \setminus e_j$ and therefore it is in \mathcal{G}_{COM} .

By Theorem 4.8, the covectors of the COM corresponding to $\pi_{e_j}(G')$ are precisely its antipodal subgraphs. Since $\pi_{e_j}(A')$ is antipodal, it corresponds to a covector. But then this covector is in $\mathcal{S}((\mathcal{L}(G) \setminus e_j)/e)$, i.e. there is an antipodal graph in $\pi_{e_j}(G)$ whose zone graph is $\pi_{e_j}(A')$. By definition this must be $\pi_{e_j}(A)$, proving that $\pi_{e_j}(A)$ is antipodal for every $E_{e_j} \in \{E_{e_1}, \dots, E_{e_k}\}$.

Let $v \in A$ be a vertex without an antipode in A . Without loss of generality, $v \in E_e^+$. By Lemma 5.3, A either has a pc-minor in \mathcal{Q}^- and we are done, or there is a Q_k^- or Q_k^- in A crossed by precisely E_{e_1}, \dots, E_{e_k} and its missing vertex is the missing antipode of v in A . Then this convex subgraph is precisely $E_e^- \cap A$.

First assume that $E_e^- \cap A$ is isomorphic to Q_k^- . Thus, v has a neighbor in E_e^- , say u . If u has no antipode in A we deduce as above that E_e^+ is isomorphic to Q_k^- or Q_k^- . Since v in A it must be isomorphic to Q_k^- and thus $A \cong Q_k^- \square K_2$. But then the zone graph of A corresponding to E_e is not antipodal and A' is not antipodal. A contradiction.

Finally, assume that E_e^- is isomorphic to Q_k^- . Then there are k vertices in E_e^- at distance 2 from v and pairwise also at distance 2 but there is no vertex adjacent to all of them. By Lemma 5.2, A either has a pc-minor in \mathcal{Q}^- and we are done, or A is isomorphic to Q_{k+1}^- or Q_{k+1}^- . In the first case, none of the zone graphs of Q_{k+1}^- is antipodal, while in the second case A is antipodal. A contradiction. This finishes the proof that $\mathcal{L}(G)/e = \mathcal{L}(G')$. □

Now we can assume that G' is a well-embedded partial cube, but is not in \mathcal{G}_{COM} since $\mathcal{L}(G') = \mathcal{S}(\mathcal{L}(G)/e)$, and $\mathcal{L}(G)/e$ is not a COM. By minimality of G , $G' = \zeta_e(G)$ has a pc-minor $H' \in \mathcal{Q}^-$, i.e., $H' = \rho_X(\pi_A(\zeta_e(G)))$ for some Θ -classes A and an oriented set X of Θ -classes of G' . By Lemma 2.9 we have $H' = \zeta_e(\rho_{X'}(\pi_{A'}(G)))$ for Θ -classes A' and an oriented set X' of Θ -classes of G . Let H be the graph $\rho_{X'}(\pi_{A'}(G))$. Lemma 5.6 gives that H has a pc-minor in \mathcal{Q}^- , contradicting that $G \in \mathcal{F}(\mathcal{Q}^-)$. □

6 AG is pc-minor closed

We assert in Lemma 5.1 that none of the graphs in \mathcal{Q}^- is in AG. To show $AG \subseteq \mathcal{F}(\mathcal{Q}^-)$ it is thus sufficient to prove that AG is closed under taking pc-minors. We will prove three statements.

Theorem 6.1. *If G is in AG, then all pc-minors of G are in AG, as well.*

Lemma 6.2. *Let G be an antipodal graph from AG such that all its pc-minors are in AG as well and G' an expansion of G . Then one of the following occurs:*

1. G' is antipodal,
2. G' is a peripheral expansion of G ,
3. G' is not in AG.

Lemma 6.3. *If Lemma 6.2 holds for all pairs G, G' where G' is on less than n vertices, then Theorem 6.1 holds for all the partial cubes on at most n vertices.*

Proof. Suppose that Theorem 6.1 does not hold and let G be a minimal counterexample, while Lemma 6.2 holds for all the expansions of size less than the size of G . First, observe that AG is closed under restrictions, since restrictions cannot create new antipodal subgraphs and gated subgraphs remain gated. So, let $\pi_e(G) \notin AG$ be a contraction of G that is not in AG. Let A be a smallest antipodal subgraph of $\pi_e(G)$, that is not gated in $\pi_e(G)$. In particular A is a proper subgraph. By the minimality in the choice of A , itself is in AG. Now, by the minimality in the choice of G , all pc-minors of A are also in AG. Let A' denote the expansion of A with respect to e , that appears as a proper subgraph of G . If E_e does not cross A' , then $A' \cong A$ is antipodal subgraph and is non-gated, since otherwise $A = \pi_e(A')$ would be gated as well by Lemma 2.18. This contradicts $G \in AG$. If E_e crosses A' , we can apply Lemma 6.2 to A , since A' has less vertices than G . We get that either A' is antipodal, A' is a peripheral expansion of A , or A' is not in AG. The latter cannot be since G in AG. In the former two cases, either A' is antipodal or has A as a subgraph. In both cases, we have an antipodal subgraph that is contracted to A in $\pi_e(G)$. By Lemma 2.18, the antipodal subgraph in G is non-gated contradicting $G \in AG$. \square

Proof of Lemma 6.2. Suppose that the lemma is false. Let G, G' be a minimal counterexample, i.e. G is an antipodal graph from AG such that all its pc-minors are in AG. Furthermore, G' is an expansion of G that is not antipodal, not peripheral, but in AG, with minimal number of vertices possible. Let E_c be the Θ -class such that $\pi_c(G') = G$.

Claim 6.4. *Any pc-minor of G' is in AG.*

Proof. Let n be the number of vertices in G' . Since G' is a minimal counterexample to Lemma 6.2, the lemma holds for all the graphs on less than n vertices. Then by Lemma 6.3, Theorem 6.1 holds for all graphs on at most n vertices. In particular it holds for G' , thus all its pc-minors are in AG. \square

We will call a contraction of a partial cube *antipodal* if the contacted graph is antipodal. The following claim is immediate since a contraction of an antipodal graph is antipodal and contractions commute.

Claim 6.5. *If $\pi_e(H)$ is an antipodal contraction of H , then $\pi_e(\pi_f(H))$ is an antipodal contraction of $\pi_f(H)$ for all $f \in \mathcal{E}$.*

Every contraction $\pi_e(G')$ of G' , $E_e \neq E_c$, makes E_c peripheral or it is antipodal, since otherwise the contraction $\pi_e(G')$ together with the contraction $\pi_c(\pi_e(G'))$ would yield a smaller counterexample to the lemma. We can divide the Θ -classes of G' into two sets: call the set of the Θ -classes of the antipodal contractions \mathcal{A} , and the set of the remaining Θ -classes \mathcal{B} . By the above, a contraction $\pi_e(G')$ of G' , for every $E_e \in \mathcal{B}$, makes E_c peripheral in $\pi_e(G')$. Note that $E_c \in \mathcal{A}$, i.e. in particular \mathcal{A} is non-empty. Also \mathcal{B} is non-empty, because otherwise, every contraction of G' is antipodal, thus by Lemma 5.3, G' is isomorphic to Q_n^- or Q_n^{--} . The latter cannot be since G' is not antipodal. The former is impossible since then $G = \pi_c(G') \cong Q_{n-1}$, and G' is a peripheral expansion.

Furthermore, note that peripherality of a θ -class is preserved under contraction.

Claim 6.6. *For every $E_e \in \mathcal{B}$ and every $E_f \in \mathcal{A}$, the Θ -class E_f is peripheral in $\pi_e(G')$.*

Proof. Assume that this is not the case. Then $\pi_e(G')$ is not antipodal by definition, while $\pi_f(\pi_e(G'))$ is antipodal by Claim 6.5. Moreover, E_f is not peripheral in $\pi_e(G')$ by assumption, while all pc-minors $\pi_e(G')$ are in AG by Claim 6.4. Thus $\pi_e(G')$ and $\pi_f(\pi_e(G'))$ are a smaller counterexample to the lemma, a contradiction. \square

Now, consider the halfspaces E_c^+ and E_c^- in G' . Contracting any Θ -class e from \mathcal{B} makes E_c peripheral in $\pi_e(G')$, which implies that either $\pi_e(E_c^+)$ or $\pi_e(E_c^-)$ is a peripheral halfspace in $\pi_e(G')$. For this reason denote by \mathcal{B}^+ those Θ -classes in \mathcal{B} that make $\pi_e(E_c^+)$ peripheral and with \mathcal{B}^- the remaining elements of \mathcal{B} (which make $\pi_e(E_c^-)$ peripheral).

Claim 6.7. *Let $E_e \in \mathcal{B}^+$ and $E_f \in \mathcal{B}^-$. Then $\pi_f(\pi_e(G'))$ is antipodal. Moreover, E_f is peripheral in $\pi_e(G')$.*

Proof. Since peripherality is closed under contraction, both $\pi_f(\pi_e(E_c^+))$ and $\pi_f(\pi_e(E_c^-))$ are peripheral in $\pi_f(\pi_e(G'))$. Thus $\pi_f(\pi_e(G')) \cong K_2 \square A$ for some graph A . On one hand contracting E_c in $\pi_f(\pi_e(G'))$ gives an antipodal graph by the choice of E_c , on the other hand it is isomorphic to A . Thus A is an antipodal graph. Then also $\pi_f(\pi_e(G')) \cong K_2 \square A$ is antipodal.

Now consider the pair of graphs $\pi_f(\pi_e(G'))$ and $\pi_e(G')$. The first is antipodal by the above, while the second is its expansion. Since both are pc-minors of G' , $\pi_e(G')$ is in AG and $\pi_f(\pi_e(G'))$ and all its pc-minors are in AG as well, by Claim 6.4. Furthermore, $\pi_e(G')$ is not antipodal since $e \in \mathcal{B}$. By the minimality of G' the expansion $\pi_e(G')$ must be peripheral, proving that E_f is peripheral in $\pi_e(G')$. \square

Let G'' be obtained from G' by a maximal chain of contractions of Θ -classes $\mathcal{C} = \{E_{e_1}, \dots, E_{e_p}\}$, such that G'' is non-antipodal. Note that $\mathcal{C} \subseteq \mathcal{B}$ since every contraction from a Θ -class in \mathcal{A} makes the graph antipodal. Moreover, $\mathcal{C} \subseteq \mathcal{B}^+$ or $\mathcal{C} \subseteq \mathcal{B}^-$ by Claim 6.7. Without loss of generality assume that $\mathcal{C} \subseteq \mathcal{B}^+$.

Claim 6.8. *$\pi_{e_1}(G')$ contains a hypercube exactly crossed by the Θ -classes $\mathcal{B}^+ \setminus \{e_1\}$.*

Proof. By assumption E_c^+ is not peripheral in G' , hence there exists a vertex $v \in E_c^+$ not incident with any edge of E_c^+ . For every $E_e \in \mathcal{B}^+$, E_c^+ is peripheral in $\pi_e(G')$, thus there is an edge in E_e connecting v and an edge in E_c . Thus for every $E_e \in \mathcal{B}^+$, there is a path from v to E_c^- first

crossing an edge in E_e and then an edge in E_c . Vertex v together with the end-vertices of these paths form a collection of vertices pairwise at distance 2. They have no common neighbor, since this neighbor would have to be in E_c^- , but v has no neighbor in E_c^- . Since all pc-minors of G' are in AG, Lemma 5.2 implies that the convex hull C of these vertices is isomorphic to Q_n^- or Q_n^{--} for some $n > 0$. This convex subset is crossed exactly by all the Θ -classes of \mathcal{B}^+ and E_c .

Contracting to $\pi_{e_1}(G')$, C is contracted into a hypercube crossed by all the Θ -classes of $\mathcal{B}^+ \setminus \{e_1\}$ and E_c . Then the lemma holds. \square

The graph G'' and all its pc-minors are in AG by Claim 6.4 and all its contractions are antipodal. By Lemma 5.3, we have $G'' \cong Q_n^-$. Now, we will consider the sequence of expansions of G'' leading back the first contraction $\pi_{e_1}(G')$. Note that G'' is crossed by precisely those Θ -classes that are not in \mathcal{C} . These are the Θ -classes $\mathcal{A} \cup \mathcal{B}^-$, and the Θ -classes $\mathcal{B}^+ \setminus \mathcal{C}$.

By Claims 6.6 and 6.7, every Θ -class in $\mathcal{A} \cup \mathcal{B}^-$ is peripheral in $\pi_{e_1}(G')$ thus also in every contraction of it. For each $E_e \in \mathcal{A} \cup \mathcal{B}^- \setminus \{E_c\}$, without loss of generality, say that E_e^+ is the peripheral halfspace of it. For E_c the halfspace E_c^+ is peripheral, since $E_{e_1} \in \mathcal{B}^+$, so we can assume that E_e^+ is peripheral for each $E_e \in \mathcal{A} \cup \mathcal{B}^-$. Since $G'' \cong Q_n^-$ is crossed by each E_e and E_e^- is non-peripheral in it, E_e^- is also non-peripheral in every expansion in the sequence.

Let $|\mathcal{A} \cup \mathcal{B}^-| = k$ and $|\mathcal{B}^+ \setminus \mathcal{C}| = \ell$. Then $G'' \cong Q_n^-$ with $n = k + \ell$, can be seen as a collection of 2^k disjoint subgraphs spanned on the edges of $\mathcal{B}^+ \setminus \mathcal{C}$ each isomorphic to Q_ℓ , except for one isomorphic to Q_ℓ^- , and all connected in a hypercube manner by edges of $\mathcal{A} \cup \mathcal{B}^-$. The subgraph isomorphic to Q_ℓ^- is precisely the subgraph $\bigcap_{E_e \in \mathcal{A} \cup \mathcal{B}^-} E_e^+$.

In other words $\pi_{\mathcal{B}^+ \setminus \mathcal{C}}(G'') \cong Q_k$ and for $X \in \{\pm\}^{\mathcal{A} \cup \mathcal{B}^-}$ we have $\rho_X(G'') \cong Q_\ell$ unless if $X = (+, \dots, +)$ in which case $\rho_X(G'') \cong Q_\ell^-$. Next we prove that this structure is preserved when expanding back towards $\pi_{e_1}(G')$.

Claim 6.9. *Let $\bar{\ell} = \ell + p - 1$, where $p = |\mathcal{C}|$. We have $\pi_{\mathcal{B}^+ \setminus \mathcal{C}}(\pi_{e_1}(G')) \cong Q_k$ and for $X \in \{\pm\}^{\mathcal{A} \cup \mathcal{B}^-}$ we have $\rho_X(\pi_{e_1}(G')) \cong Q_{\bar{\ell}}$ if and only if $X \neq (+, \dots, +)$.*

Proof. We will prove the claim by induction on the number p , where G_j is the j th expansion, starting from $G'' = G_0$ and ending in $\pi_{e_1}(G') = G_{p-1}$. By the paragraph before the claim, the claim holds for G'' .

By induction assumption let the claim hold for G_j , $j \geq 0$, and let G_{j+1} be its expansion, in the sequence of the expansions leading to $\pi_{e_1}(G')$. Let $E_f \in \mathcal{C}$ be such that $\pi_f(G_{j+1}) = G_j$ and H_1, H_2 be the subgraphs of G_j we expand along.

First, we will prove that each copy of $Q_{\ell+j}$ in G_j not crossed by the Θ -classes $\mathcal{A} \cup \mathcal{B}^-$ is contained in $H_1 \cap H_2$. Denote $\mathcal{A} \cup \mathcal{B}^- = \{E_{f_1}, \dots, E_{f_s}\}$. Consider an edge $uv \in E_{f_i}$ in G_j such that $u \in H_1 \cap H_2$. If $u \in E_{f_i}^+$ and $v \in E_{f_i}^-$, then $E_{f_i}^+$ is peripheral in the expanded graph G_{j+1} . Thus, $v \in H_1 \cap H_2$.

Now assume that $u \in E_{f_i}^-$ and $v \in E_{f_i}^+$. We will prove that if additionally $v \notin \bigcap_{E_e \in \mathcal{A} \cup \mathcal{B}^-} E_e^+$, then we can also conclude $v \in H_1 \cap H_2$. So let $u, v \in E_{f_{i'}}^-$ for some other $E_{f_{i'}} \in \mathcal{A} \cup \mathcal{B}^- \setminus \{E_{f_i}\}$. For the sake of contradiction assume that $v \notin H_1 \cap H_2$. Without loss of generality $v \in H_1$. If v has a neighbor in $E_{f_{i'}}^+$ – say v' , then by the above arguments, if $v' \in H_1 \cap H_2$ then also $v \in H_1 \cap H_2$. Thus, if v' exists, then it is in $H_1 \setminus H_2$.

Consider the contractions $\pi_{f_{i'}}(G_{j+1})$ and $\pi_{f_{i'}}(G_j)$. Since $E_{f_{i'}} \in \mathcal{A} \cup \mathcal{B}^-$ both graphs are antipodal. By Lemma 2.6, the expansion from $\pi_{f_{i'}}(G_j)$ to $\pi_{f_{i'}}(G_{j+1})$ corresponds to sets $\pi_{f_{i'}}(H_1)$

and $\pi_{f_{i'}}(H_2)$. Since $v, v' \in H_1 \setminus H_2$, their image $\pi_{f_{i'}}(v) = \pi_{f_{i'}}(v')$ is in $\pi_{f_{i'}}(H_1) \setminus \pi_{f_{i'}}(H_2)$. Since $u \in H_1 \cap H_2$, its image is in $\pi_{f_{i'}}(H_1) \cap \pi_{f_{i'}}(H_2)$. Let $-\pi_{f_{i'}}(v), -\pi_{f_{i'}}(u)$ be the antipodes of $\pi_{f_{i'}}(v), \pi_{f_{i'}}(u)$ in $\pi_{f_{i'}}(G_j)$, respectively. Since $\pi_{f_{i'}}(G_{j+1})$ is an antipodal expansion of $\pi_{f_{i'}}(G_j)$ by Lemma 2.14 we have, $-\pi_{f_{i'}}(v) \in \pi_{f_{i'}}(H_2) \setminus \pi_{f_{i'}}(H_1)$ and $-\pi_{f_{i'}}(u) \in \pi_{f_{i'}}(H_1) \cap \pi_{f_{i'}}(H_2)$.

Since $v \in E_{f_i}^+$ also $\pi_{f_{i'}}(v) \in E_{f_i}^+$, thus $-\pi_{f_{i'}}(v) \in E_{f_i}^-$. Similarly, since $u \in E_{f_i}^-$ also $\pi_{f_{i'}}(u) \in E_{f_i}^-$, thus $-\pi_{f_{i'}}(u) \in E_{f_i}^+$.

Consider the contraction from G_j to $\pi_{f_{i'}}(G_j)$. By induction hypothesis, the structure of G_j is such that $E_{f_i}^-$ is isomorphic to a hypercube crossed by $E_{f_{i'}}^+$ thus two vertices x, z of G_j are contracted to $-\pi_{f_{i'}}(v)$. Since $-\pi_{f_{i'}}(v) \in \pi_{f_{i'}}(H_2) \setminus \pi_{f_{i'}}(H_1)$, we have $x, z \in H_2 \setminus H_1$.

On the other hand, at least one vertex $w \in H_1 \cap H_2$ of G_j is contracted to $-\pi_{f_{i'}}(u)$. It also holds that the vertices $x, z \in E_{f_i}^-$ and $w \in E_{f_i}^+$. Since uv is an edge, $-\pi_{f_{i'}}(u) - \pi_{f_{i'}}(v)$ is an edge. Hence, xw or zw has to be an edge - say without loss of generality zw is an edge. Then we have a pair of adjacent vertices w, z such that $w \in H_1 \cap H_2 \cap E_{f_i}^+$ and $z \in H_1 \setminus H_2 \cap E_{f_i}^-$. We have already shown that this is impossible. Thus, $u \in H_1 \cap H_2$ implies $v \in H_1 \cap H_2$.

Now consider the copies of $Q_{\ell+j}$ in G_j . By Claim 6.8, $\pi_{e_1}(G')$ contains a hypercube crossed by exactly all the Θ -classes of $\mathcal{B}^+ \setminus \{e_1\}$. The latter implies that G_{j+1} contains a hypercube crossed by precisely the Θ -classes crossing copies $Q_{\ell+j}$ in G_j plus the Θ -class obtained while expanding to G_{j+1} . In particular this implies that one of the copies of $Q_{\ell+j}$ in G_j gets completely expanded, i.e. that copy is completely in $H_1 \cap H_2$. Since none of the copies of $Q_{\ell+j}$ in G_j is in $\bigcap_{E_e \in \mathcal{A} \cup \mathcal{B}^-} E_e^+$, by the above paragraph this property propagates to all the copies of $Q_{\ell+j}$, i.e. all of them are completely in $H_1 \cap H_2$. Since the non-cube $\bigcap_{E_e \in \mathcal{A} \cup \mathcal{B}^-} E_e^+$ cannot expand to a cube, G_{j+1} is as stated in the claim. \square

The following claim will suffice to finish the proof of the lemma.

Claim 6.10. *If the expansion G' of $\pi_{e_1}(G')$ is such that $\pi_c(G')$ is antipodal and E_c is not peripheral in G' , then G' is not in AG.*

Proof. Consider the expansion along sets G_1 and G_2 . Since $e_1 \in \mathcal{B}^+$, the halfspace E_c^+ is peripheral in $\pi_{e_1}(G')$. Moreover, by Claim 6.9 we have $E_c^- \cong Q_{k+\bar{\ell}-1}$ and $E_c^+ \not\cong Q_{k+\bar{\ell}-1}$. On the other hand, since E_c is not peripheral in G' , there is an edge $zz' \in \pi_{e_1}(G') \cap E_c$ such that $z \in (G_1 \setminus G_2) \cap E_c^-$ and $z' \in G_1 \cap G_2 \cap E_c^+$.

Note that $\pi_c(G')$ is an antipodal expansion of $\pi_{e_1}(\pi_c(G')) \cong Q_{k+\bar{\ell}-1}$. By Lemma 2.6, it is an expansion along sets $\pi_c(G_1)$ and $\pi_c(G_2)$. In the following we will use Lemma 2.6 to get properties of G_1, G_2 from the expansion $\pi_c(G_1), \pi_c(G_2)$.

We consider two cases, either $\pi_c(G')$ is a full expansion of $\pi_{e_1}(\pi_c(G')) \cong Q_{k+\bar{\ell}-1}$ or not. In the first case, every vertex of $\pi_{e_1}(\pi_c(G'))$ is in $\pi_c(G_1) \cap \pi_c(G_2)$, thus for any edge E_c in $\pi_{e_1}(G')$ we have that one of its endpoints is in $G_1 \cap G_2$ and any vertex not incident to E_c is in $G_1 \cap G_2$, as well. Since $E_c^+ \not\cong Q_{k+\bar{\ell}-1}$ there is a missing vertex in it.

Let v be a vertex in $(G_1 \setminus G_2) \cap E_c^-$ that is as close as possible to a missing vertex. We have proved above that v exists (one candidate would be z). Since the expansion is full, by the above paragraph, v is not adjacent to the missing vertex. Then by the above paragraph its neighbor in E_c^+ and by the choice of v all its neighbors in E_c^- closer to the missing vertex are in $G_1 \cap G_2$. In the expansion G' , the expansions of these vertices together with v give rise to vertices pairwise at distance 2. Since v is not expanded they have no common neighbor. By Claim 6.4 G' and all its pc-minors are in AG, thus by Lemma 5.2, their convex hull C

is a cube minus a vertex or a cube minus two antipodes. Moreover, by the choice of v and since $E_c^- \cong Q_{k+\bar{\ell}-1}$ in $\pi_{e_1}(G_1)$, the hull C also has the missing vertex. However, now also its expansion is missing, thus two adjacent vertices are missing from C . Thus, C is not isomorphic to a cube minus a vertex or a cube minus two antipodes.

Now consider that $\pi_c(G')$ is not a full expansion of $\pi_{e_1}(\pi_c(G')) \cong Q_{k+\bar{\ell}-1}$. Then there exists a vertex v of $\pi_{e_1}(\pi_c(G'))$ with $v \in \pi_c(G_2) \setminus \pi_c(G_1)$. Then there is a vertex w of $\pi_{e_1}(G')$ in the preimage of v such that $w \in (G_2 \setminus G_1) \cap E_c^-$. Let again $zz' \in \pi_{e_1}(G')$ be an edge as in the first paragraph, i.e. $zz' \in E_c$ such that $z \in (G_1 \setminus G_2) \cap E_c^-$ and $z' \in G_1 \cap G_2 \cap E_c^+$. Then let $y \in E_c^-$ be the closest vertex to z such that $y \in G_2 \setminus G_1$ (one candidate would be w). Now, let x be a vertex in the interval from z to y such that $x \in G_1 \setminus G_2$ and x is as close as possible to y . Then all the vertices in the interval from x to y are in $G_1 \cap G_2$ except for x, y . Then since $E_c^- \cong Q_{k+\bar{\ell}-1}$ the interval from x to y is isomorphic to a hypercube, i.e. antipodal. Then, also its expansion is an antipodal graph, but one preimage of z' has no gate in it by Lemma 2.19. \square

By the above claim the sequence of contraction cannot exist which gives a contradiction. Thus there is no G', G and the lemma holds. \square

Proof of Theorem 6.1. Since Lemma 6.2 holds, by Lemma 6.3 we obtain the theorem. \square

7 Corollaries

Let us begin by resuming our main theorem:

Theorem 7.1. *For a graph G the following conditions are equivalent:*

- (i) $G \in \mathcal{G}_{\text{COM}}$,
- (ii) G is a partial cube and all its antipodal subgraphs are gated,
- (iii) $G \in \mathcal{F}(\mathcal{Q}^-)$.

Proof. The implication (i) \Rightarrow (ii) is Theorem 4.8. The implication (ii) \Rightarrow (iii) follows from the fact that that all the graphs in \mathcal{Q}^- have a non-gated antipodal subgraph asserted in Lemma 5.1 and that the class AG is pc-minor closed, see Theorem 6.1. Finally, (iii) \Rightarrow (i) is Theorem 5.7. \square

Theorem 7.1 can be used to obtain a characterization of \mathcal{G}_{COM} in terms of zone graphs.

Corollary 7.2. *A graph G is in \mathcal{G}_{COM} if and only if it is a partial cube such that all iterated zone graphs are well-embedded partial cubes.*

Proof. If $G \in \mathcal{G}_{\text{COM}}$, then by Lemma 4.10 all its zone graphs are well-embedded partial cubes and in \mathcal{G}_{COM} . Hence, the argument can be iterated to prove that $\zeta_{e_1, \dots, e_k}(G)$ is a partial cube for any sequence of hyperspaces.

If $G \notin \mathcal{G}_{\text{COM}}$, then by Theorem 7.1 there is an $H \in \mathcal{Q}^-$ such that $H = \pi_A(\rho_X(G))$ for some Θ -classes of G . By Lemma 5.5 we can thus find a sequence e_1, \dots, e_k such that $\zeta_{e_1, \dots, e_k}(\pi_A(\rho_X(G)))$ is not a partial cube. If $\zeta_{e_i, \dots, e_k}(G)$ was a well-embedded partial cube for all $1 \leq i \leq k$, then by Lemma 2.9 we could find sets of Θ -classes A' and X' in it such that $\zeta_{e_1, \dots, e_k}(\pi_{A'}(\rho_{X'}(G))) =$

$\pi_{A'}(\rho_{X'}(\zeta_{e_1, \dots, e_k}(G)))$. But then the latter would be a partial cube, contradicting the choice of e_1, \dots, e_k . Therefore there is an iterated zone graph of G that is not a partial cube. \square

Theorem 7.1 and Corollary 7.2 specialize to other systems of sign-vectors. Using Proposition 4.11 we immediately get:

Corollary 7.3. *For a graph G the following conditions are equivalent:*

- (i) $G \in \mathcal{G}_{\text{OM}}$,
- (ii) G is an antipodal partial cube and all its antipodal subgraphs are gated,
- (iii) G is in $\mathcal{F}(\mathcal{Q}^-)$ and antipodal,
- (iv) G is an antipodal partial cube and all its iterated zone graphs are well-embedded partial cubes.

Note that the equivalence (i) \Leftrightarrow (ii) corresponds to a characterization of tope sets of OMs due to da Silva [13] and (i) \Leftrightarrow (vi) corresponds to a characterization of tope sets of Handa [23].

Let us call an affine subgraph G' of an affine partial cube G *conformal* if for all $v \in G'$ we have $-_{G'}v \in G' \Leftrightarrow -_Gv \in G$. We give an intrinsic characterization of \mathcal{G}_{AOM} :

Corollary 7.4. *For a graph G the following conditions are equivalent:*

- (i) $G \in \mathcal{G}_{\text{AOM}}$,
- (ii) G is an affine partial cube and all its antipodal and conformal subgraphs are gated,
- (iii) G is in $\mathcal{F}(\mathcal{Q}^-)$ (or $G \in \mathcal{G}_{\text{COM}}$), affine, and all its conformal subgraphs are gated.

Proof. By Proposition 4.12, $G \in \mathcal{G}_{\text{AOM}}$ if and only if G is an affine partial cube such that the antipodal \tilde{G} containing G as a halfspace is in \mathcal{G}_{OM} . By Corollary 7.3 this is equivalent to all antipodal subgraphs of \tilde{G} being gated. It is easy to see, that the antipodal subgraphs of \tilde{G} correspond to the antipodal and the conformal subgraphs of G and they are gated if and only if the corresponding antipodal and conformal subgraphs of G are. This proves (i) \Leftrightarrow (ii). \square

For lopsided sets the (i) \Leftrightarrow (ii) part of the corollary corresponds to a characterization due to Lawrence [26]. For the complete statement denote $\mathcal{Q}^- := \{Q_n^- \mid n \geq 3\}$.

Corollary 7.5. *For a graph G the following conditions are equivalent:*

- (i) $G \in \mathcal{G}_{\text{LOP}}$,
- (ii) G is a partial cube and all its antipodal subgraphs are hypercubes,
- (iii) G is in $\mathcal{F}(\mathcal{Q}^-)$.

Proof. By Proposition 4.13 and Theorem 7.1 $G \in \mathcal{G}_{\text{LOP}}$ is equivalent to the property that G has all antipodal subgraphs gated and isomorphic to hypercubes. But hypercubes are always gated. This gives (i) \Leftrightarrow (ii).

The implication (ii) \Rightarrow (iii) follows from the fact that \mathcal{G}_{LOP} is pc-minor closed while the graphs from \mathcal{Q}^- are (pc-minor minimal) non-hypercube antipodal graphs.

To prove (iii) \Rightarrow (ii), let G be a pc-minor-minimal partial cube with an antipodal subgraph not isomorphic to a hypercube. Then G is antipodal and not isomorphic to a hypercube. Let G be embedded in a Q_n for some minimal $n \in \mathbb{N}$. Now, G can be seen as Q_n minus a set of antipodal pairs of vertices. Take a pair u, v of vertices in Q_n not in G at a minimal distance in Q_n . If $d(u, v) = 1$, contracting the Θ -class of uv contract G to an antipodal subgraph not isomorphic to a hypercube, contradicting that G is pc-minor-minimal. The interval $\text{conv}(u, v)$ in Q_n intersected with G is a convex subgraph of G isomorphic to a hypercube minus a vertex. Note that $d(u, v) > 2$, since otherwise the subgraph is not connected. Since G is pc-minor-minimal, the subgraph is the whole G and thus G is in \mathcal{Q}^- . \square

Recall that by Lemma 4.14 the rank of a COM is the dimension of a maximal hypercube to which its tope graph can be contracted. Considering COMs of bounded rank, we can reduce the set of excluded pc-minors to a finite list. For any $r \geq 3$ define the following finite sets

$$\mathcal{Q}_r^- := \{Q_n^*, Q_n^-(m), Q_{r+2}^-(r+2), Q_{r+1} \mid 4 \leq n \leq r+1; 1 \leq m \leq n\} \subset \mathcal{Q}^- \cup \{Q_{r+1}\},$$

and

$$\mathcal{Q}_r^- := \{Q_n^-, Q_{r+1} \mid 3 \leq n \leq r+1\} \subset \mathcal{Q}^- \cup \{Q_{r+1}\}.$$

Corollary 7.6. *For a graph G and an integer $r \geq 3$ we have:*

- $G \in \mathcal{G}_{\text{COM}}$ of rank at most $r \Leftrightarrow G \in \mathcal{F}(\mathcal{Q}_r^-)$.
- $G \in \mathcal{G}_{\text{OM}}$ of rank at most $r \Leftrightarrow G \in \mathcal{F}(\mathcal{Q}_r^-)$ and G is antipodal.
- $G \in \mathcal{G}_{\text{AOM}}$ of rank at most $r \Leftrightarrow G \in \mathcal{F}(\mathcal{Q}_r^-)$, G is affine and all its conformal subgraphs are gated.
- $G \in \mathcal{G}_{\text{LOP}}$ of rank at most $r \Leftrightarrow G \in \mathcal{F}(\mathcal{Q}_r^-)$.

Proof. The respective proofs follow immediately from Theorem 7.1 and Corollaries 7.3, 7.4, and 7.5 together with the observation that the largest hypercube that is a contraction minor of Q_n^* and $Q_n^-(m)$ is of dimension $n-1$ for $0 \leq m \leq n-1$ and of $Q_n^-(n) \in \mathcal{Q}^-$ is of dimension $n-2$. Now, the graphical interpretation of the rank as given in Lemma 4.14 gives the result. \square

Note that using Proposition 2.4 can easily be seen to yield a polynomial time recognition algorithm for the recognition of the bounded rank classes above. However, Theorem 7.1 also yields polynomial time recognition algorithms for the unrestricted classes.

Corollary 7.7. *The recognition of graphs from any of the classes $\mathcal{G}_{\text{COM}}, \mathcal{G}_{\text{AOM}}, \mathcal{G}_{\text{OM}}, \mathcal{G}_{\text{LOP}}$ can be done in polynomial time.*

Proof. By [17], partial cubes can be recognized and embedded in a hypercube in quadratic time. For a partial cube embedded in a hypercube checking if it is antipodal can be done in linear time by checking if every vertex has its antipode. Note that the convex hull of any subset can be computed in linear time in the number of edges (for instance by using Lemma 2.1 for a graph embedded in a hypercube) and checking if a convex subgraph is gated is linear by Lemma 4.5.

We proceed by designing recognition algorithms using the (i) \Leftrightarrow (ii) part of the respective characterizations. We start by computing $\text{conv}(u, v)$ for all pairs of vertices u, v and storing them. We check if $\text{conv}(u, v)$ is antipodal, and if so we check if it is gated and if it is isomorphic to a hypercube, i.e. $|\text{conv}(u, v)| = 2^{d(u,v)}$. If all the antipodal graphs obtained in this way are gated, then $G \in \mathcal{G}_{\text{COM}}$, otherwise we do not proceed. If G is among the antipodal subgraphs, then \mathcal{G}_{OM} . Moreover, if all the antipodal subgraphs are isomorphic to hypercubes, then $G \in \mathcal{G}_{\text{LOP}}$.

We continue by checking for each $\text{conv}(u, v)$ if it is an affine subgraphs. For each pair $u', v' \in \text{conv}(u, v)$ such that $|\text{conv}(u', v')| < |\text{conv}(u, v)|$ we store the pair in $NA(u, v)$, and we search for a pair $w, -_{\text{conv}(u,v)}w \in \text{conv}(u, v)$ such that the set of Θ -classes on a shortest (u', w) -path and on a shortest $(v', -_{\text{conv}(u,v)}w)$ -path are disjoint. Note that the convex hulls are already computed. If this is the case for all such u', v' , we store $\text{conv}(u, v)$ as an affine subgraph. The correctness of this part follows from Proposition 2.16.

Now we check if the whole graph is affine, in this case say $G = \text{conv}(u, v)$. Then for every affine subgraph $\text{conv}(u', v')$ and vertex $w \in \text{conv}(u', v')$, we check if the pair $w, -_{\text{conv}(u',v')}w$ is a pair in $NA(u', v')$ if and only if $w, -_G w$ is a pair in $NA(u, v)$. If this is the case, $\text{conv}(u', v')$ is a conformal subgraph and we check if it is gated. Finally, if all conformal subgraphs are gated, then $G \in \mathcal{G}_{\text{AOM}}$. \square

A partial cube is called *Pasch*, their class is denoted by \mathcal{S}_4 , if any two disjoint convex subgraphs lie in two disjoint halfspaces. We confirm a conjecture from [10]:

Corollary 7.8. *The class \mathcal{S}_4 of Pasch graphs is a subclass of \mathcal{G}_{COM} .*

Proof. In [10] the list of excluded pc-minors of Pasch graphs has been given. With this it is easy to verify that $\mathcal{S}_4 \subset \mathcal{F}(\mathcal{Q}^-)$. Another way of seeing the inclusion is using [10, Theorem A], which confirms that the convex closure of any isometric cycle in a Pasch graph is gated. By Lemma 2.15 in particular we have that all antipodal subgraphs are gated and the claim follows from Theorem 7.1. \square

8 Problems

Note that any partial cube is a convex subgraph of an antipodal partial cube. To see this, consider an arbitrary partial cube G embedded in a hypercube Q_n . Construct an induced subgraph A_G of the hypercube Q_{n+3} where the convex subgraph with the last three coordinates equal to 1 is G , the convex subgraph with the last three coordinates equal to -1 is $-G$ and the other convex subgraphs given by fixing the last three coordinates are isomorphic to Q_n . It is easily checked that A_G is an antipodal partial cube with G a convex subgraph of it. Nevertheless even if $G \in \mathcal{G}_{\text{COM}}$, A_G is not necessary in \mathcal{G}_{COM} , e.g., it is easy to check that A_{C_6} has a Q_4^{*-} -minor. Thus, let us restate a conjecture from [3, Conjecture 1] in terms of tope graphs.

Conjecture 1. *Every $G \in \mathcal{G}_{\text{COM}}$ is a convex subgraph of a graph $G' \in \mathcal{G}_{\text{OM}}$.*

Note that an affirmative answer to this question would immediately give a topological representation theorem for COMs.

A question arising from Corollaries 7.3 and 7.4 is based on the observation that \mathcal{G}_{AOM} and \mathcal{G}_{OM} are closed under contraction.

Problem 1. *Find the list of minimal excluded affine and antipodal contraction-minors for the classes \mathcal{G}_{AOM} and \mathcal{G}_{OM} , respectively.*

Using Proposition 2.16 it is easy to see, that the affine partial cubes in \mathcal{Q}^- are exactly the graphs of the form $Q_n^-(m)$. So this gives a family of excluded contraction minors for \mathcal{G}_{AOM} . However, it is not complete since some graphs of this family, e.g. the one arising from $Q_4^-(1)$, have halfspaces that are COMs which therefore have to be excluded as well. Moreover, the antipodal subgraphs obtained as in the proof of Proposition 2.16 from the affine graphs $Q_n^-(m)$ give a family of excluded contraction minors for \mathcal{G}_{OM} . In particular, this implies that the only non-matroidal antipodal graphs with at most five Θ -classes are the ones coming from the four graphs $Q_4^-(1), \dots, Q_4^-(4)$ – a result originally due to Handa [23]. However, also this family is incomplete, since the graph A_{C_6} constructed above has no halfspace in \mathcal{Q}^- .

Several characterizations of planar partial cubes are known [1, 15], where the latter corrects a characterization from [29]. A particular consequence of Corollary 7.6 is that tope graphs of OMs of rank at most three are characterized by the finite list of excluded pc-minors \mathcal{Q}_3^- . By a result of Fukuda and Handa [21] this graph class coincides with the class of antipodal planar partial cubes. Since planar partial cubes are closed under pc-minors, we wonder about an extension of this result to general planar partial cubes:

Problem 2. *Find the list of minimal excluded pc-minors for the class of planar partial cubes.*

It is easy to see, that any answer here will be an infinite list, since a (strict) subfamily is given by the set $\{G_n \square K_2 \mid n \geq 3\}$, where G_n denotes the *gear graph* (also known as *cogwheel*) on $2n + 1$ vertices. This in particular shows, that if a pc-minor closed class is contained in an (ordinary) minor closed graph class it can still have an infinite list of excluded minors. A tempting probably quite hard problem is the following:

Problem 3. *Give a criterion for when a pc-minor closed class has a finite list of excluded pc-minors.*

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