# A correction of a characterization of planar partial cubes

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**Abstract**

In this note we determine the set of expansions such that a partial cube is planar if and only if it arises by a sequence of such expansions from a single vertex. This corrects a result of Peterin.

## 1 Introduction

A graph is a *partial cube* if it is isomorphic to an isometric subgraph $G$ of a hypercube graph $Q_d$, i.e., $\mathrm{dist}_G(v,w) = \mathrm{dist}_{Q_n}(v,w)$ for all $v,w \in G$. Any isometric embedding of a partial cube into a hypercube leads to the same partition of edges into so-called $\Theta$-classes, where two edges are equivalent, if they correspond to a change in the same coordinate of the hypercube. This can be shown using the Djoković-Winkler-relation $\Theta$ which is defined in the graph without reference to an embedding, see [5, 6].

Let $G^1$ and $G^2$ be two isometric subgraphs of a graph $G$ that (edge-)cover $G$ and such that their intersection $G' := G^1 \cap G^2$ is non-empty. The *expansion* $H$ of $G$ with respect to $G^1$ and $G^2$ is obtained by considering $G^1$ and $G^2$ as two disjoint graphs and connecting them by a matching between corresponding vertices in the two resulting copies of $G'$. A result of Chepoi [3] says that a graph is a partial cube if and only if it can be obtained from a single vertex by a sequence of expansions. An equivalence class of edges with resepct to $\Theta$ in a partial cube is an inclusion minimal edge cut. The inverse operation of an expansion in partial cubes is called *contraction* and consists in taking a $\Theta$-class of edges $E_f$ and contracting it. The two disjoint copies of the corresponding $G^1$ and $G^2$ are just the two components of the graph where $E_f$ is deleted.

## 2 The flaw and the result

Let $H$ be an expansion of a planar graph $G$ with respect to $G^1$ and $G^2$. Then $H$ is a *2-face expansion* of $G$ if $G^1$ and $G^2$ have plane embeddings such that $G' := G^1 \cap G^2$ lies on a face in both the respective embeddings. Peterin [4] proposes a theorem stating that a graph is a planar partial cube if and only

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if it can be obtained from a single vertex by a sequence of 2-face expansions. However, his argument has a flaw, since $G'$ lying on a face of $G^1$ and $G^2$ does not guarantee that the expansion $H$ be planar. Indeed, Figure 1 shows an example of such a 2-face expansion $H$ of a planar graph $G$ that is non-planar.

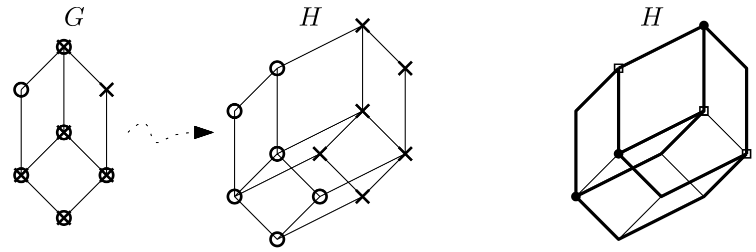


Figure 1: Left: A 2-face expansion $H$ of a planar partial cube $G$, where $G^1$ and $G^2$ are drawn as crosses and circles, respectively. Right: A subdivision of $K_{3,3}$ (bold) in $H$, certifying that $H$ is not planar.

The correct concept are non-crossing 2-face expansions: We call an expansion $H$ of a planar graph $G$ with respect to subgraphs $G^1$ and $G^2$ a *non-crossing 2-face expansion* if $G^1$ and $G^2$ have plane embeddings such that $G' := G^1 \cap G^2$ lies on the outer face of both the respective embeddings, such that the orderings on $G'$ obtained from traversing the outer faces of $G^1$ and $G^2$ in the clockwise order, respectively, are opposite.

**Lemma 1.** *For a partial cube $H \ncong K_1$ the following are equivalent:*

*(i) $H$ is planar,*

*(ii) $H$ is a non-crossing 2-face expansion of a planar partial cube $G$,*

*(iii) if $H$ is an expansion of $G$, then $G$ is planar and $H$ is a non-crossing 2-face expansion of $G$.*

*Proof.*
(ii)$\Longrightarrow$(i): Let $G$ be a planar partial cube and $G^1$ and $G^2$ two subgraphs satisfying the preconditions for doing a non-crossing 2-face expansion. We can thus embed $G^1$ and $G^2$ disjointly into the plane such that the two copies of $G' := G^1 \cap G^2$ appear in opposite order around their outer face, respectively. Connecting corresponding vertices of the two copies of $G'$ by a matching $E_f$ does not create crossings, because the 2-face expansion is non-crossing, see Figure 2. Thus, if $H$ is a non-crossing 2-face expansion of $G$, then $H$ is planar.

(i)$\Longrightarrow$(iii): Let $H$ be a planar partial cube, that is an expansion of $G$. Thus, there is a $\Theta$-class $E_f$ of $H$ such that $G = H/E_f$. In particular, since contraction preserves planarity, $G$ is planar.

Consider now $H$ with some planar embedding. Since $H$ is a partial cube, $E_f$ is an inclusion-minimal edge cut of $H$. Thus, $H \setminus E_f$ has precisely two components corresponding to $G^1$ and $G^2$, respectively. Since $E_f$ is a minimal cut its planar dual is a simple cycle $C_f$. It is well-known, that any face of a planar embedded graph can be chosen to be the outer face without changing the combinatorics of the embedding. We change the embedding of $H$, such

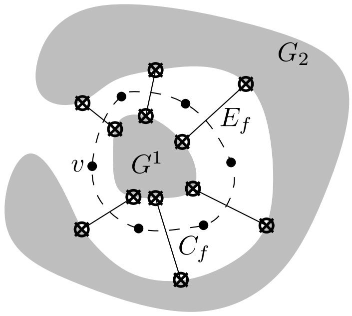


Figure 2: Two disjoint copies of subgraphs $G^1$ and $G^2$ in a planar partial cube $H$.

that some vertex $v$ of $C_f$ corresponds to the outer face of the embedding, see Figure 2.

Now, without loss of generality $C_f$ has $G^1$ and $G^2$ in its interior and exterior, respectively. Since $C_f$ is connected and disjoint from $G^1$ and $G^2$ it lies in a face of both. By the choice of the embedding of $H$ it is their outer face. Moreover, since every vertex from a copy of $G'$ in $G^1$ can be connected by an edge of $E_f$ to its partner in $G^2$ crossing an edge of $C_f$ but without introducing a crossing in $H$, the copies of $G'$ in $G^1$ and $G^2$ lie on this face, respectively.

Furthermore, following $E_f$ in the sense of clockwise traversal of $C_f$ gives the same order on the two copies of $G'$, corresponding to a clockwise traversal on the outer face of $G^1$ and a counter-clockwise traversal on the outer face of $G^2$. Thus, traversing both outer faces in clockwise order the obtained orders on the copies of $G'$ are opposite. Hence $H$ is a non-crossing 2-face expansion of $G$.

(iii)$\Longrightarrow$(ii): Since $H \not\cong K_1$, it is an expansion of some partial cube $G$. The rest is trivial. $\square$

Lemma 1 yields our characterization.

**Theorem 2.** *A graph $H$ is a planar partial cube if and only if $H$ arises from a sequence of non-crossing 2-face expansions from $K_1$.*

*Proof.*
$\Longrightarrow$: Since $H$ is a partial cube by the result of Chepoi [3] it arises from a sequence of expansions from $K_1$. Moreover, all these sequences have the same length correspodning to the number of $\Theta$-classes of $H$. We proceed by induction on the length $\ell$ of such a sequence. If $\ell = 0$ the sequence is empty and there is nothing to show. Otherwise, since $H \not\cong K_1$ is planar we can apply Lemma 1 to get that $H$ arises by a non-crossing 2-face expansions from a planar partial cube $G$. The latter has a sequence of expansions from $K_1$ of length $\ell - 1$ which by induction can be chosen to consist of non-crossing 2-face expansions. Together with the expansion from $G$ to $H$ this gives the claimed sequence from $H$.
$\Longleftarrow$: Again we induct on the length $\ell$ of the sequence. If $\ell = 0$ we are fine since $K_1$ is planar. Otherwise, consider the graph $G$ in the sequence such that $H$ is its non-crossing 2-face expansion. Then $G$ is planar by induction and $H$ is planar by Lemma 1, since it is a non-crossing 2-face expansion of $G$. $\square$

## 3 Remarks

We have characterized planar partial cubes graphs by expansions. Planar partial cubes have also been characterized in a topological way as dual graphs of non-separating pseudodisc arrangements [1]. There is a third interesting way of characterizing them. The class of planar partial cubes is closed under *partial cube minors*, see [2], i.e., contraction of $G$ to $G/E_f$ where $E_f$ is a $\Theta$-class and restriction to a component of $G \setminus E_f$. What is the family of minimal obstructions for a partial cube to being planar, with respect to this notion of minor? The answer will be an infinite list, since a subfamily is given by the set $\{G_n \Box K_2 \mid n \geq 3\}$, where $G_n$ denotes the *gear graph* (also known as *cogwheel*) on $2n+1$ vertices and $\Box$ is the Cartesian product of graphs. See Figure 3 for the first three members of the family.

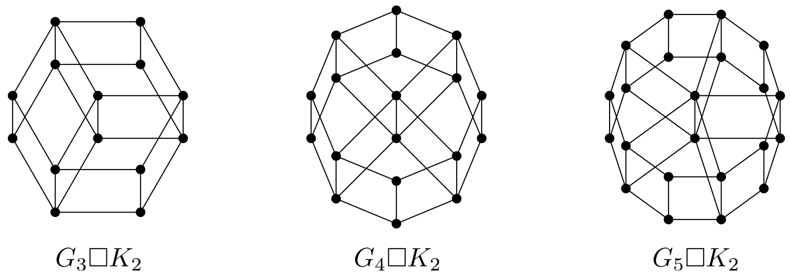


Figure 3: The first three members of an infinite family of minimal obstructions for planar partial cubes.

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