A correction of a characterization of planar partial cubes

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Abstract

In this note we determine the set of expansions such that a partial cube is planar if and only if it arises by a sequence of such expansions from a single vertex. This corrects a result of Peterin.

1 Introduction

A graph is a *partial cube* if it is isomorphic to an isometric subgraph G of a hypercube graph Q_d, i.e., $\mathrm{dist}_G(v, w) = \mathrm{dist}_{Q_n}(v, w)$ for all $v, w \in G$. Any isometric embedding of a partial cube into a hypercube leads to the same partition of edges into so-called Θ-classes, where two edges are equivalent, if they correspond to a change in the same coordinate of the hypercube. This can be shown using the Djoković-Winkler-relation Θ which is defined in the graph without reference to an embedding, see [5, 6].

Let G^1 and G^2 be two isometric subgraphs of a graph G that (edge-)cover G and such that their intersection $G' := G^1 \cap G^2$ is non-empty. The *expansion* H of G with respect to G^1 and G^2 is obtained by considering G^1 and G^2 as two disjoint graphs and connecting them by a matching between corresponding vertices in the two resulting copies of G'. A result of Chepoi [3] says that a graph is a partial cube if and only if it can be obtained from a single vertex by a sequence of expansions. An equivalence class of edges with resepct to Θ in a partial cube is an inclusion minimal edge cut. The inverse operation of an expansion in partial cubes is called *contraction* and consists in taking a Θ-class of edges E_f and contracting it. The two disjoint copies of the corresponding G^1 and G^2 are just the two components of the graph where E_f is deleted.

2 The flaw and the result

Let H be an expansion of a planar graph G with respect to G^1 and G^2. Then H is a *2-face expansion* of G if G^1 and G^2 have plane embeddings such that $G' := G^1 \cap G^2$ lies on a face in both the respective embeddings. Peterin [4] proposes a theorem stating that a graph is a planar partial cube if and only

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if it can be obtained from a single vertex by a sequence of 2-face expansions. However, his argument has a flaw, since G' lying on a face of G^1 and G^2 does not guarantee that the expansion H be planar. Indeed, Figure 1 shows an example of such a 2-face expansion H of a planar graph G that is non-planar.

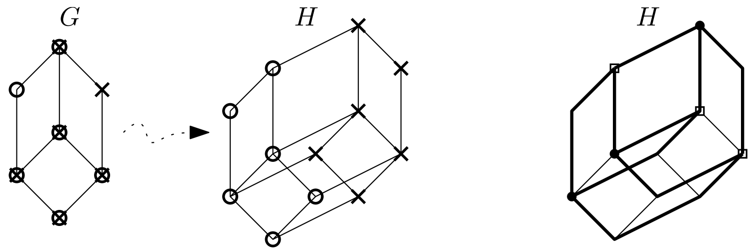


Figure 1: Left: A 2-face expansion H of a planar partial cube G, where G^1 and G^2 are drawn as crosses and circles, respectively. Right: A subdivision of $K_{3,3}$ (bold) in H, certifying that H is not planar.

The correct concept are non-crossing 2-face expansions: We call an expansion H of a planar graph G with respect to subgraphs G^1 and G^2 a *non-crossing 2-face expansion* if G^1 and G^2 have plane embeddings such that $G' := G^1 \cap G^2$ lies on the outer face of both the respective embeddings, such that the orderings on G' obtained from traversing the outer faces of G^1 and G^2 in the clockwise order, respectively, are opposite.

Lemma 1. *For a partial cube $H \ncong K_1$ the following are equivalent:*

(i) H is planar,

(ii) H is a non-crossing 2-face expansion of a planar partial cube G,

(iii) if H is an expansion of G, then G is planar and H is a non-crossing 2-face expansion of G.

Proof.
(ii)$\Longrightarrow$(i): Let G be a planar partial cube and G^1 and G^2 two subgraphs satisfying the preconditions for doing a non-crossing 2-face expansion. We can thus embed G^1 and G^2 disjointly into the plane such that the two copies of $G' := G^1 \cap G^2$ appear in opposite order around their outer face, respectively. Connecting corresponding vertices of the two copies of G' by a matching E_f does not create crossings, because the 2-face expansion is non-crossing, see Figure 2. Thus, if H is a non-crossing 2-face expansion of G, then H is planar.

(i)$\Longrightarrow$(iii): Let H be a planar partial cube, that is an expansion of G. Thus, there is a Θ-class E_f of H such that $G = H/E_f$. In particular, since contraction preserves planarity, G is planar.

Consider now H with some planar embedding. Since H is a partial cube, E_f is an inclusion-minimal edge cut of H. Thus, $H \setminus E_f$ has precisely two components corresponding to G^1 and G^2, respectively. Since E_f is a minimal cut its planar dual is a simple cycle C_f. It is well-known, that any face of a planar embedded graph can be chosen to be the outer face without changing the combinatorics of the embedding. We change the embedding of H, such

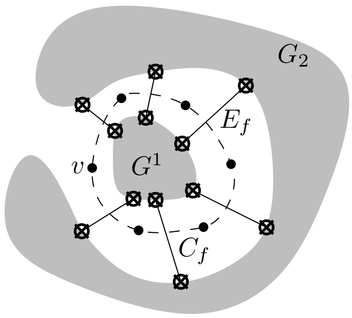


Figure 2: Two disjoint copies of subgraphs G^1 and G^2 in a planar partial cube H.

that some vertex v of C_f corresponds to the outer face of the embedding, see Figure 2.

Now, without loss of generality C_f has G^1 and G^2 in its interior and exterior, respectively. Since C_f is connected and disjoint from G^1 and G^2 it lies in a face of both. By the choice of the embedding of H it is their outer face. Moreover, since every vertex from a copy of G' in G^1 can be connected by an edge of E_f to its partner in G^2 crossing an edge of C_f but without introducing a crossing in H, the copies of G' in G^1 and G^2 lie on this face, respectively.

Furthermore, following E_f in the sense of clockwise traversal of C_f gives the same order on the two copies of G', corresponding to a clockwise traversal on the outer face of G^1 and a counter-clockwise traversal on the outer face of G^2. Thus, traversing both outer faces in clockwise order the obtained orders on the copies of G' are opposite. Hence H is a non-crossing 2-face expansion of G.

(iii)$\Longrightarrow$(ii): Since $H \ncong K_1$, it is an expansion of some partial cube G. The rest is trivial. $\square$

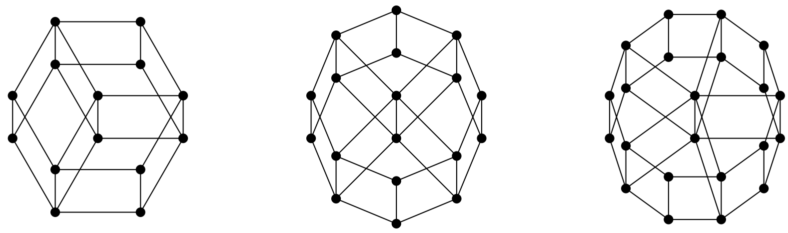
Lemma 1 yields our characterization.

Theorem 2. *A graph H is a planar partial cube if and only if H arises from a sequence of non-crossing 2-face expansions from K_1.*

Proof.
$\Longrightarrow$: Since H is a partial cube by the result of Chepoi [3] it arises from a sequence of expansions from K_1. Moreover, all these sequences have the same length correspodning to the number of Θ-classes of H. We proceed by induction on the length ℓ of such a sequence. If $\ell = 0$ the sequence is empty and there is nothing to show. Otherwise, since $H \ncong K_1$ is planar we can apply Lemma 1 to get that H arises by a non-crossing 2-face expansions from a planar partial cube G. The latter has a sequence of expansions from K_1 of length $\ell - 1$ which by induction can be chosen to consist of non-crossing 2-face expansions. Together with the expansion from G to H this gives the claimed sequence from H.
$\Longleftarrow$: Again we induct on the length ℓ of the sequence. If $\ell = 0$ we are fine since K_1 is planar. Otherwise, consider the graph G in the sequence such that H is its non-crossing 2-face expansion. Then G is planar by induction and H is planar by Lemma 1, since it is a non-crossing 2-face expansion of G. $\square$

3 Remarks

We have characterized planar partial cubes graphs by expansions. Planar partial cubes have also been characterized in a topological way as dual graphs of non-separating pseudodisc arrangements [1]. There is a third interesting way of characterizing them. The class of planar partial cubes is closed under *partial cube minors*, see [2], i.e., contraction of G to G/E_f where E_f is a Θ-class and restriction to a component of $G \setminus E_f$. What is the family of minimal obstructions for a partial cube to being planar, with respect to this notion of minor? The answer will be an infinite list, since a subfamily is given by the set $\{G_n \Box K_2 \mid n \geq 3\}$, where G_n denotes the *gear graph* (also known as *cogwheel*) on $2n+1$ vertices and $\Box$ is the Cartesian product of graphs. See Figure 3 for the first three members of the family.



$G_3 \Box K_2$ $G_4 \Box K_2$ $G_5 \Box K_2$

Figure 3: The first three members of an infinite family of minimal obstructions for planar partial cubes.

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