Simple Treewidth

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1 Introduction

A *k-tree* is a graph that can be constructed starting with a $(k+1)$-clique and in every step attaching a new vertex to a k-clique of the already constructed graph. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the minimum k such that G is a *partial k-tree*, i.e., G is a subgraph of some k-tree [7].

We consider a variation of treewidth, called *simple treewidth*. A simple k-tree is a k-tree with the extra requirement that there is a construction sequence in which no two vertices are attached to the same k-clique. (Equivalently, a k-tree is simple if it has a tree representation of width k in which every $(k-1)$-set of subtrees intersects at most 2 tree-vertices.) Now, the *simple treewidth* $\mathrm{stw}(G)$ of G is the minimum k such that G is a partial simple k-tree, i.e., G is a subgraph of some simple k-tree.

We have encountered simple treewidth as a natural parameter in questions concerning geometric representations of graphs, i.e., representing graphs as intersection graphs of geometrical objects where the quality of the representation is measured by the complexity of the objects. E.g., we have shown that the maximal *interval-number*([3]) of the class of treewidth k graphs is $k+1$, whereas for the class of simple treewidth k graphs it is k, see [6]. Another example is the *bend-number*([4]), which for treewidth 3 graphs is 4 and for simple treewidth 3 graphs is 3, see [5] and a corresponding statement for higher values is conjectured in [4].

The aim of this note is to compare these two parameters and to motivate simple treewidth by indicating that it endows treewidth with a topological flavor. We pose several interesting open problems.

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2 Properties of simple treewidth

Let us first observe, that both parameters cannot differ too much.

Observation 2.1. *For every G we have* $\mathrm{tw}(G) \leq \mathrm{stw}(G) \leq \mathrm{tw}(G) + 1$.

Proof. The first inequality is clear. For the second inequality we show that every k-tree G is a subgraph of a simple $(k+1)$-tree H. Whenever in the construction sequence of G several vertices $\{v_1, \dots, v_n\}$ are attached to the same k-clique C we consider $C \cup \{v_1\}$ as a $(k+1)$-clique in the construction sequence for H. Attaching v_i to C can be interpreted as attaching v_i to $C \cup \{v_{i-1}\}$ and omitting the edge $\{v_{i-1}, v_i\}$. This way we avoid that several vertices are attached to the same k-clique by considering $(k+1)$-cliques. $\square$

Simple treewidth endows the notion of treewidth with a more topological flavor, as indicated for small k in the table below:

	≤ 1	≤ 2	≤ 3
stw	paths	outerplanar	planar & tw ≤ 3, [2]
tw	forest	series-parallel	tw ≤ 3

A *linkless embedding* of G is an embedding into $\mathbb{R}^3$ with the property that no two cycles of G form a *link*, see [9].

Observation 2.2. *If* $\mathrm{stw}(G) \leq 4$ *then G has a linkless embedding.*

Proof. It suffices to show that simple 4-trees have linkless embeddings, since edge-deletion does not destroy the linkless embedding. Therefore consider K_5 embedded in $\mathbb{R}^3$ as a tetrahedron plus a vertex in its interior. In each step of the construction sequence every available 4-clique is represented by a tetrahedron with empty interior, where we insert the new vertex. It is easy to see that the resulting embedding is linkless. $\square$

Non-simple 4-trees do not have linkless embeddings, which is easy to see using the forbidden-minor chracterization of linkless embeddable graphs [8].

Problem 2.3. $\mathrm{stw}(G) \leq 4 \Leftrightarrow G$ *is linkless embeddable and* $\mathrm{tw}(G) \leq 4$.

Simple treewidth also has connections to discrete geometry. In [1] a *stacked polytope* is defined to be a polytope admitting a triangulation whose dual graph is a tree. In that paper it is proved that a full-dimensional polytope $P \subset \mathbb{R}^d$ is stacked if and only if $\mathrm{tw}(G(P)) \leq d$, were $G(P)$ denotes the 1-skeleton of P. Indeed, we strongly suspect:

Problem 2.4. *A graph G is the* 1*-skeleton of a stacked d-polytope if and only if* $\operatorname{stw}(G) = d$ *and G is d-connected.*

One can show that the class of simple treewidth at most k graphs is minor-closed. A proof of the following statement would imply that for planar graphs with treewidth at least 3 treewidth and simple treewidth coincide.

Problem 2.5. *If G has no $K_{3,k}$-minor and* $\operatorname{tw}(G) = k$ *then* $\operatorname{stw}(G) = k$.

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