Covering of ordinals

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1. Covering graphs
   - Ordinals
   - MSO logic
   - Fundamental sequence
   - MSO-theory of covering graphs

2. Pushdown hierarchy
   - Definition
   - Iteration of exponentiation

3. Higher-order stacks presentation
   - Definition
   - Ordinal presentation
Ordinals

An ordinal is a well-ordering, i.e. an order where
- each set has a smallest element
- each strictly decreasing sequence is finite

During this talk, we confuse ordinal with graph of the order.
Ordinals

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Ordinals

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\[ \omega + 1 \]
An ordinal is a well-ordering, i.e. an order where
- each set has a smallest element
- each strictly decreasing sequence is finite
During this talk, we confuse ordinal with graph of the order.
Theorem (Cantor normal form, 1897)

For $\alpha < \varepsilon_0$, there is a unique decreasing sequence $(\gamma_i)$ such that

$$\alpha = \omega^{\gamma_0} + \cdots + \omega^{\gamma_n}.$$ 

It is enough to define ordinals with

- addition
- operation $\alpha \mapsto \omega^\alpha$. 

Addition

\[ \alpha \quad \beta \]
Addition

\[ \alpha + \beta \]
Addition

\[ \omega \quad \cdots \quad 2 \]
Addition

$$\omega + 2$$
Addition

\[ \omega + 2 \]

\[ 2 + \omega = \omega \]
Addition

\[ \omega + 2 \]

\[ 2 + \omega = \omega \]
Exponentiation

\[ \omega^\alpha \sim \left( \{ \text{decreasing finite sequences of ordinals } < \alpha \}, <_{\text{lex}} \right) \]
Exponentiation

$$\omega^\alpha \simeq \{\text{decreasing finite sequences of ordinals } < \alpha\}, <_{\text{lex}}$$

For instance, $\omega^2 = \omega + \omega + \omega + \omega \ldots$

\[
2 = 0 \rightarrow 1
\]

decreasing sequences $= (1, \ldots, 1, 0, \ldots, 0)$
Exponentiation

\[ \omega^\alpha \simeq (\{ \text{decreasing finite sequences of ordinals } < \alpha \}, <_{\text{lex}}) \]

For instance, \( \omega^2 = \omega + \omega + \omega + \omega \ldots \)

\[
\begin{align*}
2 &= 0 \rightarrow 1 \\
\text{decreasing sequences} &= (1, \ldots, 1, 0, \ldots, 0)
\end{align*}
\]

We restrict to ordinals \( \lessdot \varepsilon_0 = \omega^\varepsilon_0 \).

Notation: \( \omega \uparrow n = \omega^{\omega^\ldots^\omega} \{ n \} \).
Exponentiation

\[ \omega^\alpha \simeq (\{\text{decreasing finite sequences of ordinals} < \alpha\}, <_{\text{lex}}) \]

For instance, \( \omega^2 = \omega + \omega + \omega + \omega \ldots \)

\[
\begin{align*}
2 &= 0 \rightarrow 1 \\
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\end{align*}
\]

![Diagram showing ordinal sequences and their relationships.](diagram.png)

- We restrict to ordinals \( < \varepsilon_0 = \omega^{\varepsilon_0}. \)
- Notation: \( \omega \uparrow n = \omega^{\omega^{\ldots \omega}} \{ n \}. \)
Monadic second-order logic

- first-order variables $x, y \ldots$
- the structure: $<$
- set variables $X, Y \ldots$ and formulas $x \in Y$
- $\land, \lor, \neg, \forall, \exists$
Monadic second-order logic

- first-order variables $x, y \ldots$
- the structure: $<$
- set variables $X, Y \ldots$ and formulas $x \in Y$
- $\land, \lor, \neg, \forall, \exists$

strict order
\[
\{ \begin{align*}
\text{antisymmetry} & : \forall p, q (\neg(p < q \land q < p)) \\
\text{transitivity} & : \forall p, q, r ((p < q \land (q < r) \Rightarrow p < r)
\end{align*} \}
\]

total order
\[
\forall p, q (p < q \lor q < p \lor p = q)
\]

well order
\[
\forall X, \exists z \in X \Rightarrow \\
\exists x (x \in X \land \forall y (y \in X \Rightarrow (x < y \lor x = y)))
\]
MSO-logics and ordinals [Büchi, Shelah]

\[ \text{MTh}(S) = \{ \varphi \mid S \models \varphi \} . \]

**Theorem**

*For any countable \( \alpha \), \( \text{MTh}(\alpha) \) is decidable.*
MSO-logics and ordinals [Büchi, Shelah]

\[ \text{MTh}(S) = \{ \varphi \mid S \models \varphi \} . \]

**Theorem**

*For any countable \( \alpha \), MTh(\( \alpha \)) is decidable.*

\[ \alpha = \omega^{\gamma_0} + \cdots + \omega^{\gamma_k} + \omega^{\gamma_{k+1}} + \omega^{\gamma_n} \quad \text{where} \quad \begin{cases} \gamma_0, \ldots, \gamma_k \geq \omega \\ \gamma_{k+1}, \ldots, \gamma_n < \omega \end{cases} \]

**Theorem**

*MTh(\( \alpha \)) only depends on \( \delta \) and whether \( \beta > 0 \).*

\[ \text{MTh}(\omega^\omega) = \text{MTh}(\omega^{\omega^\omega}) \ldots \]
Simplifying graphs
Each countable limit ordinal is limit of an \( \omega \)-sequence. How to define this sequence in fixed way?
Fundamental sequence [Cantor]

Let $\alpha = \omega^{\gamma_0} + \cdots + \omega^{\gamma_{k-1}} + \omega^{\gamma_k}$

If $\gamma_n \neq 0$, $\alpha$ a limit ordinal. There is an $\omega$-sequence of limit $\alpha$.

$$\alpha[n] = \begin{cases} 
\delta + \omega^{\gamma'}(n+1) & \text{if } \gamma_k = \gamma' + 1 \\
\delta + \omega^{\gamma_k[n]} & \text{otherwise.}
\end{cases}$$
Fundamental sequence [Cantor]

Let \( \alpha = \omega^\gamma_0 + \cdots + \omega^\gamma_{k-1} + \omega^\gamma_k \)

If \( \gamma_n \neq 0 \), \( \alpha \) a limit ordinal. There is an \( \omega \)-sequence of limit \( \alpha \).

\[
\alpha[n] = \begin{cases} 
\delta + \omega^\gamma'(n+1) & \text{if } \gamma_k = \gamma' + 1 \\
\delta + \omega^\gamma_k[n] & \text{otherwise.}
\end{cases}
\]

Successor ordinals are a degenerate case:

\( \alpha \prec \beta \) if \[
\begin{cases} 
\alpha = \beta[n] \\
\alpha + 1 = \beta.
\end{cases}
\]
Covering graph of $\omega + 2$

$\omega[n] = n + 1$

$\mathcal{G}_{\omega+2}$

0 → 1 → 2 → 3 → ... 

$\omega \rightarrow \omega + 1$
Covering graph of \( \omega^2 + 1 \)

\[ \omega^2[n] = \omega \cdot (n + 1) \]

\[ G_{\omega^2 + 1} \]
Covering graph of $\omega^\omega$
First result

Proposition

\(<\) is the transitive closure of \(<\).

Theorem

For \(\alpha, \beta < \varepsilon_0\), if \(\alpha \neq \beta\), then \(\text{MTh}(G_{\alpha}) \neq \text{MTh}(G_{\beta})\).
Proof sketch

Proposition

For $\alpha \leq \omega \uparrow n$, the out-degree of $G_{\alpha}$ is at most $n$. 
Proof sketch

Proposition

For $\alpha \leq \omega \uparrow n$, the out-degree of $G_\alpha$ is at most $n$. 
Proof sketch

Let \( \sigma \) be the sequence

\begin{itemize}
\item 0 \( \in \sigma \),
\item \( \beta \in \sigma \Rightarrow \) if \( \beta' \) is the largest s.t. \( \beta \prec \beta' \), then \( \beta' \in \sigma \).
\end{itemize}

Degree word : sequence of out-degrees of this sequence.
Proof sketch

Let $\sigma$ be the sequence

- $0 \in \sigma$,
- $\beta \in \sigma \Rightarrow$ if $\beta'$ is the largest s.t. $\beta \prec \beta'$, then $\beta' \in \sigma$.

*Degree word*: sequence of out-degrees of this sequence.

**Proposition**

The degree word is

- *ultimately periodic*,
- *MSO-definable*,
- *injective*. 
Pushdown hierarchy

Many definitions:

- higher-order pushdown automata [Müller-Schupp, Carayol-Wöhrle],
- unfolding [Caucal] or treegraph [Carayol-Wöhrle]
  + MSO-interpretations or rational mappings
- prefix-recognizable relations [Caucal-Knapik,Carayol],
- term grammars [Dam, Knapik-Niwiński-Urzyczyn]. . .
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- higher-order pushdown automata [Müller-Schupp, Carayol-Wöhrle],
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- prefix-recognizable relations [Caucal-Knapik,Carayol],
- term grammars [Dam, Knapik-Niwiński-Urzyczyn]...
Pushdown hierarchy

\[ \text{Graph}_0 \ (\text{finite}) \]
Pushdown hierarchy

\[ \text{Graph}_0 \ (\text{finite}) \]
\[ \Rightarrow \]
\[ \mathcal{I} \circ \text{Treegraph} \]
\[ \Downarrow \]
\[ \text{Graph}_1 \ (\text{prefix-recognizable}) \]
Pushdown hierarchy

Graph_0 (finite)

\mathcal{I} \circ \text{Treegraph}

Graph_1 (prefix-recognizable)

\mathcal{I} \circ \text{Treegraph}

Graph_2
Pushdown hierarchy

Each graph in $\text{Graph}_n$ has a decidable MSO-theory.
MSO-interpretations

MSO-interpretation: $\mathcal{I} = \{ \varphi_a(x, y) \}_{a \in \Gamma}$ where $\varphi_a$ is a formula over $G$.

$\mathcal{I}(G) = \{ x \xrightarrow{a} y \mid G \models \varphi_a(x, y) \}$
MSO-interpretations

MSO-interpretation: \( \mathcal{I} = \{ \varphi_a(x, y) \}_{a \in \Gamma} \) where \( \varphi_a \) is a formula over \( G \).

\[ \mathcal{I}(G) = \{ x \xrightarrow{a} y \mid G \models \varphi_a(x, y) \} \]

Exemple: transitive closure.

\[ \varphi_{\prec}(x, y) := \forall X (x \in X \land \text{closed}_{\prec}(X) \Rightarrow y \in X) \land x \neq y \]
\[ \text{closed}_{\prec}(X) := \forall z \in X, \forall z' (z \prec z' \Rightarrow z' \in X) \]

Proposition

For \( \alpha < \varepsilon_0 \), there is an interpretation \( \mathcal{I}(G_\alpha) = \alpha \).
MSO-interpretations

MSO-interpretation: \( \mathcal{I} = \{ \varphi_a(x, y) \}_{a \in \Gamma} \) where \( \varphi_a \) is a formula over \( G \).

\[ \mathcal{I}(G) = \{ x \xrightarrow{a} y \mid G \models \varphi_a(x, y) \} \]

Exemple: transitive closure.

\[ \varphi_<(x, y) := \forall X (x \in X \land \text{closed}_<(X) \Rightarrow y \in X) \land x \neq y \]
\[ \text{closed}_<(X) := \forall z \in X, \forall z' (z < z' \Rightarrow z' \in X) \]

**Proposition**

*For* \( \alpha < \varepsilon_0 \), *there is an interpretation* \( \mathcal{I}(G_\alpha) = \alpha \).

**Proposition**

*There is an interpretation from* \( G_\alpha \) *to* \( G_\beta \) *with* \( \beta \leq \alpha < \varepsilon_0 \).*
Treegraph

\[ G \]
Treegraph
Main result

Theorem (Bloom, Ésik)

If $\alpha < \omega \uparrow 3 = \omega^{\omega \omega}$ then $\alpha \in \text{Graph}_2$. 
Main result

Theorem (Bloom, Ésik)

If $\alpha < \omega \uparrow 3 = \omega^\omega\omega$ then $\alpha \in \text{Graph}_2$.

Proposition

For $\alpha < \omega \uparrow n$, $G_\alpha \in \text{Graph}_{n-1}$.

Theorem

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Main result

Theorem (Bloom, Ésik)

If \( \alpha < \omega \uparrow 3 = \omega^{\omega \omega} \) then \( \alpha \in \text{Graph}_2 \).

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For \( \alpha < \omega \uparrow n \), \( G_\alpha \in \text{Graph}_{n-1} \).

Theorem

For \( \alpha < \omega \uparrow n \), \( \alpha \in \text{Graph}_{n-1} \).

Proposition

For \( n > 0 \) and \( \alpha \geq \omega \uparrow (3n + 1) \), \( G_\alpha \notin \text{Graph}_n \).

Corollary

\( G_{\varepsilon_0} \) does not belong to the hierarchy.
Proposition

There is a interpretation $I$ such that $I \circ \text{Treegraph}(G_\alpha) = G_{\omega\alpha}$ for each $\alpha$. 

$G_\omega$
Treegraph on covering graphs

Proposition

There is a interpretation $\mathcal{I}$ such that $\mathcal{I} \circ \text{Treegraph}(G_\alpha) = G_{\omega^\alpha}$ for each $\alpha$. 

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Higher-order stacks [Carayol]

\[ \begin{array}{c}
\begin{array}{c}
  \begin{array}{c}
    b \\
    a
  \end{array} \\
  \rightarrow \\
  \begin{array}{c}
    a \\
    b \\
    a
  \end{array}
\end{array}
\end{array} \]

push\_a

\[ \begin{array}{c}
\begin{array}{c}
  \begin{array}{c}
    a \\
    b \\
    a
  \end{array} \\
  \rightarrow \\
  \begin{array}{c}
    b \\
    a
  \end{array}
\end{array}
\end{array} \]

pop\_1
Higher-order stacks [Carayol]

- **push\(_a\)**:
  
  \[
  \begin{array}{ccc}
  b & a \\
  a & a \\
  \end{array}
  \rightarrow
  \begin{array}{ccc}
  a & b \\
  a & a \\
  \end{array}
  \]

- **copy\(_2\)**:
  
  \[
  \begin{array}{ccc}
  c & b & a \\
  a & b & a \\
  \end{array}
  \rightarrow
  \begin{array}{ccc}
  c & b & a \\
  b & a & a \\
  \end{array}
  \]

- **pop\(_1\)**:
  
  \[
  \begin{array}{ccc}
  a & b & a \\
  \end{array}
  \rightarrow
  \begin{array}{ccc}
  b & a \\
  \end{array}
  \]

- **pop\(_2\)**:
  
  \[
  \begin{array}{ccc}
  c & b & a \\
  a & b & a \\
  \end{array}
  \rightarrow
  \begin{array}{ccc}
  c & b & a \\
  b & a \\
  \end{array}
  \]

Higher-order stacks [Carayol]

- $Ops_1 = \{\text{push}_a, \text{pop}_1\}$
- for $n > 1$, $Ops_n = \{\text{copy}_n, \text{pop}_n\} \cup Ops_{n-1}$

$Ops_n$ is a monoïd for composition. Let $L \in Ops_n^*$. A graph in $\text{Graph}_n$ can be represented with vertices in $\text{Stacks}_n$.

$$
\begin{array}{c}
| s_0 \ldots s_k | \xrightarrow{L} | s'_0 \ldots s'_{k'} |
\end{array}
$$

with $s_i, s'_i \in \text{Stacks}_{n-1}$. 

Presentation of ordinals

- Integers are represented by stacks over a unary alphabet.

\[ \text{push}(i) = i + 1 \quad \text{pop}_1(i + 1) = i \]

For finite integers,

\[ \alpha < \beta \iff s_\alpha \in \text{pop}_1^+(s_\beta) \iff s_\beta \in \text{push}_1^+(s_\alpha) \]

Let \( \text{dec}_1 = \text{pop}_1^+ \), \( \text{inc}_1 = \text{push}_1^+ \).
Presentation of ordinals

- Integers are represented by stacks over a unary alphabet.

\[ \text{push}(i) = i + 1 \quad \text{pop}_1(i + 1) = i \]

For finite integers,

\[ \alpha < \beta \iff s_\alpha \in \text{pop}_1^+(s_\beta) \iff s_\beta \in \text{push}^+(s_\alpha) \]

Let \( \text{dec}_1 = \text{pop}_1^+ \), \( \text{inc}_1 = \text{push}^+ \).

- For infinite \( \alpha \),

\[ \alpha = \omega^{\gamma_0} + \cdots + \omega^{\gamma_k} \]

Each \( \gamma_i \) is representable by \( s_{\gamma_i} \in \text{Stacks}_{n-1} \)

\[ s_\alpha = \begin{array}{c}
  s_{\gamma_0} \\
  \ldots \\
  s_{\gamma_k}
\end{array} \]
For $n > 1$, if $\alpha < \beta$, 

$$s_\alpha = s_{\gamma_0} \ldots s_{\gamma_i} \ldots s_{\gamma_k}$$
For $n > 1$, if $\alpha < \beta$,

$$s_\alpha = \underbrace{s_{\gamma_0} \ldots s_{\gamma_i} \ldots s_{\gamma_k}}$$

either $$s_\beta = \underbrace{s_{\gamma_0} \ldots s_{\gamma_i} \ldots s_{\gamma_k s_{\gamma_{k+1}}} \ldots s_{\gamma_h}}$$

$$s_\beta \in (copy_n.(id + dec_{n-1}))^+ (s_\alpha)$$
For \( n > 1 \), if \( \alpha < \beta \),

\[
\begin{align*}
\sigma_\alpha &= \begin{bmatrix} s_{\gamma_0} \ldots s_{\gamma_i} \ldots s_{\gamma_k} \end{bmatrix} \\
\text{either } \sigma_\beta &= \begin{bmatrix} s_{\gamma_0} \ldots s_{\gamma_i} \ldots s_{\gamma_k} s_{\gamma_{k+1}} \ldots s_{\gamma_h} \end{bmatrix} \\
\sigma_\beta &\in (\text{copy}_n.(id + \text{dec}_{n-1}))^+ (\sigma_\alpha) = \text{tail}_n^+ (\sigma_\alpha)
\end{align*}
\]
For $n > 1$, if $\alpha < \beta$,

$$s_\alpha = \underbrace{s_{\gamma_0} \ldots s_{\gamma_i} \ldots s_{\gamma_k}}_{\text{either}}$$

either

$$s_\beta = \underbrace{s_{\gamma_0} \ldots s_{\gamma_i} \ldots s_{\gamma_k} s_{\gamma_{k+1}} \ldots s_{\gamma_h}}_{\text{so}}$$

$$s_\beta \in \left(\text{copy}_n.(id + \text{dec}_{n-1})\right)^+ (s_\alpha) = \text{tail}_n^+(s_\alpha)$$

or

$$s_\beta = \underbrace{s_{\gamma_0} \ldots s_{\gamma_i-1} s_{\gamma'_i > \gamma_i} \ldots s_{\gamma'_{k'}}}_{\text{so}}$$

$$s_\beta \in \text{pop}_n^*.\text{inc}_{n-1}.(\text{copy}_n.(id + \text{dec}_{n-1}))^* (s_\alpha)$$
For $n > 1$, if $\alpha < \beta$,

$$s_\alpha = \begin{array}{c}s_{\gamma_0} \ldots s_{\gamma_i} \ldots s_{\gamma_k} \end{array}$$

either

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or

$$s_\beta = \begin{array}{c}s_{\gamma_0} \ldots s_{\gamma_i-1} s_{\gamma_i'} \gamma_i \ldots s_{\gamma_k'} \end{array}$$

$$s_\beta \in (\text{copy}_n.(id + \text{dec}_{n-1}))^+(s_\alpha) = \text{tail}_n^+(s_\alpha)$$

or

$$s_\beta \in \text{pop}_n^*.\text{inc}_{n-1}.(\text{copy}_n.(id + \text{dec}_{n-1}))^*(s_\alpha)$$

$$\text{inc}_n = [\text{pop}_n^*.\text{inc}_{n-1} + \text{tail}_n].\text{tail}_n^*$$

$$\text{dec}_n = \text{pop}_n^*. [\text{pop}_n + \text{dec}_{n-1}].\text{tail}_n^*$$

**Theorem**

*If $(\alpha, <, >)$ is in Graph$_n$, then $(\omega^\alpha, <, >)$ is in Graph$_{n+1}$.***
Future work

- Obtain the stronger result

\[ \alpha < \omega \uparrow (n + 1) \iff \alpha \in \text{Graph}_n. \]

- Is there a definition of fundamental sequence so that the result

\[ \alpha \neq \beta \Rightarrow \text{MTh}(G_\alpha) \neq \text{MTh}(G_\beta) \]

remains true for further ordinals?

- Ordinals greater than \( \omega^\omega \) are not selectable [Rabinovich, Shomrat]. Are covering graphs selectable?

- Extend the method to other linear orderings.