Existence varieties of regular rings and complemented modular lattices

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Luminy, November 2010
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if \( L \in \text{Mod} \Sigma_l \) has 1, then it is a \textit{complemented modular lattice} (a \textit{CML} for short);
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If \(A \in \text{Mod} \Sigma_\Lambda\), then \(A\) is a (von Neumann) regular algebra.
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$L(R)$ is the lattice of principal right ideals of $R$.
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If $R$ is Artinian, then $L(R)$ is a finite height CML.
(X, \varphi) is a combinatorial geometry, if it has the exchange property:

\[ a \in \varphi(Y \cup \{b\}) \rightarrow b \in \varphi(Y \cup \{a\}) \]

for any \( a, b \in X \) and any \( Y \subseteq X \).
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Let \(V_D\) be a vector space over a division ring \(D\). \(\text{Sub}(V_D)\) is the subspace lattice.
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\(\text{Sub}(V_D) \cong \mathbb{L}(\text{End}(V_D)), \quad \text{End}(V_D)\) is a regular ring.
$\text{Sub}(V_D)$ is a subdirectly irreducible Arguesian SCL:
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\[ \forall x_0 x_1 x_2 y_0 y_1 y_2 \ \bigwedge_{i<3} (x_i \vee y_i) \leq (x_0 \wedge (x_1 \vee c)) \vee (y_0 \wedge (y_1 \vee c)), \]

where

\[ c_i = (x_j \vee x_k) \wedge (y_j \vee y_k), \quad \{i, j, k\} = \{0, 1, 2\}, \]

\[ c = (c_0 \vee c_1) \wedge c_2. \]
Sub($V_\mathcal{D}$) is a subdirectly irreducible Arguesian SCL:

$$\forall x_0 x_1 x_2 y_0 y_1 y_2 \quad \bigwedge_{i<3} (x_i \lor y_i) \leq (x_0 \land (x_1 \lor c)) \lor (y_0 \land (y_1 \lor c)),$$

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If dim $V_\mathcal{D} < \omega$, then Sub($V_\mathcal{D}$) is simple finite height.
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**Theorem (von Neumann, 1939; Jónsson, 1960)**

Let $L$ be a simple Arguesian CL of finite height $n \geq 3$. 
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**Theorem (von Neumann, 1939; Jónsson, 1960)**

Let $L$ be a simple Arguesian CL of finite height $n \geq 3$. Then there is a division ring $\mathbb{D}$ such that $L \cong \text{Sub}(\mathbb{D}^n)$. 
Problem (Dilworth)

Is the class of lattices embeddable into CML-s (SCML-s) a variety?
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Theorem (Jónsson, 1960)

The following are equivalent for a SCML:

1. $L \in S\left(Cl(X, \varphi)\right)$ for a projective geometry $(X, \varphi)$;
2. $L \in S\left(Sub(A)\right)$ for an Abelian group $A$;
3. $L \in S\left(\prod_{i \in I} Sub(V_i)\right)$, $V_i$ is a vector space for all $i \in I$;
4. $L$ is Arguesian.
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Let $\mathcal{K} \subseteq \text{Mod } \Sigma$.

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1. $V_\exists(\mathcal{K}) = H S_\exists P(\mathcal{K})$ is the smallest $\exists$-variety containing $\mathcal{K}$; moreover, $T V_\exists(\mathcal{K}) = V T(\mathcal{K})$.  

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3. Any SI algebra from $V_\exists(\mathcal{K})$ belongs to $HS_\exists P_u(\mathcal{K})$. 
Let $\mathcal{K} \subseteq \text{Mod} \Sigma$. 
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**Definition**
$\mathcal{K} \subseteq \text{Mod} \Sigma$ is an $\exists$-variety, if it is closed under $H$, $S_{\exists}$, and $P$.

**Theorem**

1. $V_{\exists}(\mathcal{K}) = H S_{\exists} P(\mathcal{K})$ is the smallest $\exists$-variety containing $\mathcal{K}$; moreover, $T V_{\exists}(\mathcal{K}) = V T(\mathcal{K})$.
2. The reduct of any free algebra from $V T(\mathcal{K})$ belongs to $P_{s\exists}(\mathcal{K})$.
3. Any SI algebra from $V_{\exists}(\mathcal{K})$ belongs to $H S_{\exists} P_u(\mathcal{K})$.
4. Any $\exists$-variety is generated by its finitely generated SI-s.
Free algebras exist in $\exists$-varieties.
Free algebras exist in $∃$-varieties. Any $∃$-variety can be defined by positive sentences as well as by Horn sentences.
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Any $\exists$-variety can be defined by positive sentences as well as by Horn sentences.

**Problem**

*Can an $\exists$-variety be defined by positive Horn sentences?*
Regular rings
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**Theorem**

\[ \mathbf{V}_\exists (\mathbb{F}^{n \times n} \mid n_0 < n < \omega, \ \mathbb{F} \text{ is a quotient field of } \Lambda) \text{ is the } \exists\text{-variety of regular } \Lambda\text{-algebras.} \]
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Corollary

1. \[ \mathbf{V}_\exists(\mathbb{F}_{p}^{n \times n} \mid n_0 < n < \omega, p \text{ is prime}) \text{ is the } \exists \text{-variety of regular rings.} \]
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2. Free regular rings are residually finite.
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\[ \mathcal{V}(F^{n \times n} \mid n_0 < n < \omega, \ F \text{ is a quotient field of } \Lambda) \text{ is the } \exists\text{-variety of regular } \Lambda\text{-algebras.} \]

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2. Free regular rings are residually finite.
3. The equational theory of regular rings with quasi-inversion as a fundamental operation is decidable.
Theorem

Let $R$ be a SI non-Artinian regular $\Lambda$-algebra.
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Corollary

Any $\exists$-variety of regular $\Lambda$-algebras is generated by its simple Artinian members.

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Free regular $\Lambda$-algebras are residually Artinian.

Goodearl, Menal, and Moncasi (1993) proved the latter statement for algebras with unit.
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By the Wedderburn-Artin theorem, $C(\mathcal{V})$ consists of matrix rings over division rings.
For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

$$D \in D_n(\mathcal{C}) \text{ if and only if } D^{n \times n} \in \mathcal{C}.$$
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$$D \in D_n(\mathcal{C}) \iff D^{n \times n} \in \mathcal{C}.$$ 

**Definition**

$\mathcal{C}$ is **closed**, if the following holds:

1. $D \in D_n(\mathcal{C})$ for all $n > 0$.
2. $D \in D_m(\mathcal{C})$ for all $n \geq m > 0$.
3. If $n = mk > 0$, $F \in D_n(\mathcal{C})$ and $D \in S(F^{k \times k})$ is a division ring, then $D \in D_m(\mathcal{C})$.
4. If $p$ is a prime and there is $D \in D_n(\mathcal{C})$ with $\text{char}(D) = p$, then $F \in \bigcap_{n > 0} D_n(\mathcal{C})$ for any $F$ with $\text{char}(F) = p$.
5. $D_1(\mathcal{C})$ is the class of all division rings.
For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

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4. $p$ is a prime;
   if for any $n > 0$, there is $D \in D_n(\mathcal{C})$ with $\text{char } D = p$, then $F \in \bigcap_{n > 0} D_n(\mathcal{C})$ for any $F$ with $\text{char } F = p$;
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5. $D_1(\mathcal{C})$ is the class of all division rings.
Theorem

Let $\mathcal{C}$ be a class of simple Artinian regular rings. 
$\mathcal{C}$ is closed if and only if $\mathcal{C} = C(\forall)$ for an $\exists$-variety of regular rings.
Sectionally complemented modular lattices
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**Theorem**

*Let $L$ be a SI modular SCL of infinite height.*
Sectionally complemented modular lattices

**Theorem**

Let $L$ be a SI modular SCL of infinite height. There is a unique prime field $\mathbb{F}$ such that $V_\exists(L) = V_\exists(L(\mathbb{F}^{n \times n}) \mid n_0 < n < \omega)$.
Sectionally complemented modular lattices

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Let $L$ be a SI modular SCL of infinite height. There is a unique prime field $\mathbb{F}$ such that $V_\exists(L) = V_\exists(L(\mathbb{F}^{n \times n}) \mid n_0 < n < \omega)$.

**Corollary**

Any $\exists$-variety of SCML is generated by its simple finite height members.
Corollary 1

\[ \forall \exists (\mathbb{L}((\mathbb{F}_p^{n \times n}) \mid n_0 < n < \omega, \ p \text{ is prime}) \text{ is the variety of Arguesian SCL.} \]
Corollary

1. $V_{∃}(L(F_p^{n \times n}) \mid n_0 < n < ω, \ p \ is \ prime)$ is the variety of Arguesian SCL.

2. Free Arguesian SCL are residually finite.
Corollary

1. \( V_\exists \left( \mathbb{L}(\mathbb{F}_p^{n\times n}) \mid n_0 < n < \omega, \ p \text{ is prime} \right) \) is the variety of Arguesian SCL.

2. Free Arguesian SCL are residually finite.

3. Equational theory of Arguesian lattices with sectional complementation as a fundamental operation is decidable.
Corollary

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Corollary

Equational theory of modular lattices with sectional complementation is decidable.
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By the von-Neumann-Jónsson coordinatization theorem, any $L \in C(\mathcal{V})$ with $\text{ht } L \geq 3$ is of the form $\mathbb{L}(D^n_D)$. 
For a class $\mathcal{C}$ of simple Arguesian finite height SCL and for $n > 0$:

$$D \in D_n(\mathcal{C}) \text{ if and only if } \mathbb{L}(D^n_D) \in \mathcal{C}.$$
Definition

Let $\mathcal{C}$ be closed, if the following holds:

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2. $D_n(\mathcal{C}) \subseteq D_m(\mathcal{C})$ for all $n \geq m > 0$;
3. If $n = mk > 0$, $F \in D_n(\mathcal{C})$, and $D \in S(F^k \times k)$ is a division ring, then $D \in D_m(\mathcal{C})$;
4. $p$ is a prime; if for any $n > 0$, there is $D \in D_n(\mathcal{C})$ with char $D = p$, then $F \in \bigcap_{n > 0} D_n(\mathcal{C})$ for any $F$ with char $F = p$;
5. If $D \in D_2(\mathcal{C})$ and $|F| \leq |D|$, then $F \in D_2(\mathcal{C})$;
6. $M_k \in \mathcal{C}$ for $k < \omega$, then $M_n \in \mathcal{C}$ for all $2 \leq n \leq k$. 

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**Definition**

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1. $D_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $D_n(\mathcal{C}) \subseteq D_m(\mathcal{C})$ for all $n \geq m > 0$;
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3. if \( n = mk > 0 \), \( F \in D_n(\mathcal{C}) \), and \( D \in S(F^{k \times k}) \) is a division ring, then \( D \in D_m(\mathcal{C}) \);
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4. \( p \) is a prime;
   if for any \( n > 0 \), there is \( D \in \mathbf{D}_n(\mathcal{C}) \) with \( \text{char} \ D = p \), then \( F \in \bigcap_{n>0} \mathbf{D}_n(\mathcal{C}) \) for any \( F \) with \( \text{char} \ F = p \);
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5. if $D \in D_2(\mathcal{C})$ and $|F| \leq |D|$, then $F \in D_2(\mathcal{C})$;
   - if $M_k \in \mathcal{C}$ for $k < \omega$, then $M_n \in \mathcal{C}$ for all $2 < n \leq k$;
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3. if \( n = mk > 0 \), \( F \in D_n(C) \), and \( D \in S(F^{k\times k}) \) is a division ring, then \( D \in D_m(C) \);
4. \( p \) is a prime;
   if for any \( n > 0 \), there is \( D \in D_n(C) \) with \( \text{char} \ D = p \), then \( F \in \bigcap_{n>0} D_n(C) \) for any \( F \) with \( \text{char} \ F = p \);
5. if \( D \in D_2(C) \) and \( |F| \leq |D| \), then \( F \in D_2(C) \);
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Definition

$\mathcal{C}$ is **closed**, if the following holds:

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   if for any $n > 0$, there is $D \in D_n(\mathcal{C})$ with $\text{char} D = p$, then $F \in \bigcap_{n > 0} D_n(\mathcal{C})$ for any $F$ with $\text{char} F = p$;
5. if $D \in D_2(\mathcal{C})$ and $|F| \leq |D|$, then $F \in D_2(\mathcal{C})$;
   if $M_k \in \mathcal{C}$ for $k < \omega$, then $M_n \in \mathcal{C}$ for all $2 < n \leq k$;
6. $D_1(\mathcal{C})$ is the class of all division rings.
Theorem

Let $\mathcal{C}$ be a class of simple Arguesian finite height SCL. $\mathcal{C}$ is closed if and only if $\mathcal{C} = C(\mathcal{V})$ for an $\exists$-variety of Arguesian SCL.
Problem

Is the class of lattices embeddable into SCML-s a variety?
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Corollary

If L embeds into a SCML, then $\text{Id}(L)$ does.