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Cyclic Orders

Let $\prec$ be a ternary relation over set $\mathcal{X}$. Then $\Gamma = (\mathcal{X}, \prec)$ is a cyclic order (CyO) iff it satisfies, for $a, b, c, d \in \mathcal{X}$:

1. **inversion asymmetry**: $\prec(a, b, c) \Rightarrow \neg \prec(b, a, c)$
2. **rotational symmetry**: $\prec(a, b, c) \Rightarrow \prec(c, a, b)$
3. **ternary transitivity**: $[\prec(a, b, c) \land \prec(a, c, d)] \Rightarrow \prec(a, b, d)$.

A CyO $\Gamma_{\text{tot}} = (\mathcal{X}, \prec_{\text{tot}})$ is called total or a TCO if for all $a, b, c \in \mathcal{X}$,

$$(x \neq y \neq z \neq x) \Rightarrow \prec(a, b, c) \lor \prec(b, a, c).$$

If a total CyO $\Gamma_{\text{tot}}$ on $\mathcal{X}$ extends $\Gamma$, call $\Gamma$ orientable, and $\Gamma_{\text{tot}}$ an orientation of $\Gamma$. 
Cyclic Transitivity
Huntington (1916, 1924, 1938) introduces cyclic (total) order over words of variable length

Partial Cios as ternary relations, orientability problem:

- Galil and Megiddo (1977)
- Genrich (1971): links to synchronization graphs (conflict free Petri nets)
- Petri: Concurrency Theory; Rozenberg et al: Square structures
- Stehr (1998) axiomatizes partial cyclic order over words
- ...
(Some) known results

- Orientability is NP-complete (Megiddo 1976, Galil and Megiddo 1977)
- Compactness: a CyO is orientable iff all its finite sub-CyOs are (Alles, Nešetřil, Poljak 1991)
- Genrich 1971, Stehr 1998: orientable finite CyOs correspond to 1-safe live Synchronization Graphs
- HERE: explore new link with partial orders: WINDING
Non-Orientable CyOs
Preliminaries : Global vs round-orientedness

- \( li \triangleq \{(x, y) \mid \exists z : \triangleleft(x, y, z) \lor \triangleleft(x, z, y)\} \)
- \( co \triangleq X^2 - (id_X \cup li) \)
- **Rounds:** Maximal cliques of \( li \)
- **cuts:** Maximal cliques of \( co \)
- \( \Gamma = (\mathcal{X}, \triangleleft) \) is **round-oriented** (ROCO) iff for any round \( \{a, b, c\} \), either \( \triangleleft(a, b, c) \) or \( \triangleleft(b, a, c) \).

Not a ROCO:
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Shifts and Windings

Definitions

- Fix poset \( \Pi = (\mathcal{X}, <) \). An **order automorphism** is a bijection \( G : \mathcal{X} \to \mathcal{X} \) such that \( \forall x, y \in \mathcal{X} : [x < y] \iff [Gx < Gy] \).
- \( G \) is called a **shift** if \( x < Gx \) for all \( x \in \mathcal{X} \).
- The group \( G \) of \( \Pi \)-automorphisms generated by \( G \) is isomorphic to \( (\mathbb{Z}, +) \).
- Equivalence : \( \overline{x} \sim_G \overline{y} \) iff \( \exists k \in \mathbb{Z} : G^k\overline{x} = \overline{y} \).
- Let \( [\overline{x}] \triangleq [x]_{\sim_G} \).
- Let \( \beta_G : \overline{\mathcal{X}} \to \mathcal{X}, \overline{x} \mapsto [\overline{x}]_{\sim_G} \) be the **winding map**.
Winding a shift-invariant PO to a CyO
Windings and their Properties

- Set \( \triangleleft (x, y, z) \) iff \( \exists \overline{x} \in x, \overline{y} \in y, \overline{z} \in z : \overline{x} < \overline{y} < \overline{z} < G \overline{x} \).
- \( \Gamma = (\mathcal{X}, \triangleleft) \) is wound from \( \Pi \) via \( G \) (or \( \beta_G \)).
- Note: this leaves more concurrency than the usual construction of CyO from a poset: \( \triangleleft (x, y, z) \) iff
  \( \exists \overline{x} \in x, \overline{y} \in y, \overline{z} \in z : \overline{x}_0 < \overline{y}_0 < \overline{z}_0 \).
- Inheritance: If \( \Gamma_1 \) is obtained by winding, then so are all its Sub-CyOs.
- We say that a winding is loop-free iff for all \( \overline{x} \in \overline{X} \), there exist \( \overline{y}, \overline{z} \in \overline{X} \) that \( \overline{x} < \overline{y} < \overline{z} < G \overline{x} \). Loop-free windings generate ROCOs.
Mind the Gap!

Not all windings preserve successor relations
Mind the Gap!

Let $x \prec y$ iff
1. $x \text{ li } y$ and
2. for all $z \in \mathcal{X} - \{x, y\}$, $x \text{ li } z$ and $z \text{ li } y$ imply $\prec(x, y, z)$.

$\overline{x}$ covers $\overline{y}$ from below, written $x \triangleright y$, iff (a) $\overline{x} < \overline{y}$, and (b) for all $\overline{z} \in \mathcal{X}$, $\overline{z} < \overline{y}$ implies $\overline{z} < \overline{x}$.

$\overline{y}$ covers $\overline{x}$ from above, written $y \sqcap x$, iff (a) $\overline{x} < \overline{y}$, and (b) for all $\overline{z} \in \mathcal{X}$, $\overline{x} < \overline{z}$ implies $\overline{y} < \overline{z}$.

In $(\mathcal{X}, \prec)$, $x$ covers $y$, written $x \triangleright y$, iff (a) $x \text{ li } y$, and (b) for all $z, u \in \mathcal{X}$, $\prec(z, u, y)$ implies $\prec(z, u, x)$.

A gap in $(\overline{\mathcal{X}}, <)$: $x < \cdot y$ holds, but neither $x \sqcap y$ nor $y \triangleright x$.

A gap in $(\mathcal{X}, \prec)$: $x \prec y$ holds, and $x \triangleright y$ does not hold.

Suppose $(\overline{\mathcal{X}}, <)$ is gap-free and $\beta_G$ winds $(\overline{\mathcal{X}}, <)$ to $(\mathcal{X}, \prec)$. If $\beta_G$ is loop-free, then $\beta_G$ maps $\prec$ surjectively to $\simeq$. 

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Separation properties

Definitions

- A cut $c$ is called a separator iff $c \cap O \neq \emptyset$ for all $O \in \mathcal{O}(\Gamma)$, and
- a cycle separator iff $c \cap \gamma \neq \emptyset$ for all $\gamma \in \mathcal{D}(\Gamma)$
- $\Gamma$ is called weakly (cycle) separable iff there exists a (cycle) separator $c \in \mathcal{C}(\Gamma)$, and
- strongly (cycle) separable iff all its cuts are (cycle) cuts.
- If some superstructure cyclic order $\Gamma'$ of $\Gamma$ is (strongly) cycle separable, then $\Gamma$ is called (strongly) saturable.
- Every total CyO is strongly separable.
Separation properties

Cycle separation implies round separation; the converse is not true:

\[ a \xrightarrow{\sim} b, \quad c \xrightarrow{\sim} d, \quad \text{but} \quad u \xrightarrow{\sim} w \text{ is not true.} \]
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THEOREM

Let $\mathcal{X} \neq \emptyset$, and $\Gamma = (\mathcal{X}, \triangleleft)$ a RCO. Then the following are equivalent:

1. $\Gamma$ is weakly saturable;
2. there exists a winding representation for $\Gamma$, i.e., a partial order $\Pi = (\overline{\mathcal{X}}, <)$ with shift $G$ such that $\mathcal{X} = \overline{\mathcal{X}}_G$, and $\phi_G$ winds $\Pi$ to $\Gamma$;
3. $\Gamma$ is orientable.
1 ⇒ 2

Saturability to winding:
Given \((\mathcal{X}, <)\) and \(\mathbf{c}\), set \(\mathcal{X} \triangleq \mathcal{X} \times \mathbb{Z}\), \(x_k \triangleq (x, k)\).

Let \(G : \mathcal{X} \rightarrow \mathcal{X}\) be given by \(Gx_k = x_{k+1}\) for \(k \in \mathbb{Z}\).

\[
\mathcal{R}_0 \triangleq \left\{ (a_0, b_0) \mid a \in \mathbf{c} \wedge a \text{ li } b \right\} \\
\cup \left\{ (a_k, a_{k+l}) \mid a \in \mathcal{X}, k \in \mathbb{Z}, l \in \mathbb{N} \right\}
\]

\(< \triangleq \left\{ (a_{l+k}, b_{m+k}) \mid a_l \mathcal{R}_0 b_m, k \in \mathbb{Z} \right\}
\]

\(< \triangleq \text{ transitive closure of } <\)
Sketch of Proof

2 ⇒ 3

Winding to orientability:
Take a weakly separable SuperPO $\Pi_\#$ of $\Pi$, with separator and $c_0.c_k \triangleq G^k c_0$, and define $U_k \triangleq U_k^\# \cap \overline{X}$, where

$$U_k^\# \triangleq \bigcup \left[ \overline{y}_k, \overline{y}_{k+1} \right],$$

where $\overline{y}_k, \overline{y}_{k+1} \in c_k, c_{k+1}$

- The $U_k$ are pairwise disjoint and cover $\overline{X}$
- Set $\Pi_k \triangleq (U_k, <_k)$; $G$ induces order isomorphisms $G_{n,m} : U_n \rightarrow U_m$ from $\Pi_n$ to $\Pi_m$.
- Take total ordering $\Pi_0^{tot}$ on $U_0$ that embeds $<_0$ (Szpilrajn !)
- Export $\Pi_0^{tot}$ to $U_k$ via $G_{0,k} \Rightarrow$ Done
Sketch of Proof

3 ⇒ 1

Orientability to saturability:
Let $\Gamma_{tot} = (\mathcal{X}, \prec_{tot})$ be an orientation of $\Gamma$; fix $x \in \mathcal{X}$.

- Let $\mathcal{Y}$ be a set disjoint from $\mathcal{X}$, and $\psi : \mathcal{O} \rightarrow \mathcal{Y}$ injective
- Set $A_x \triangleq \{ O \in \mathcal{O}(\Gamma) \mid x \notin O \}$ and $\mathcal{X}_x \triangleq \mathcal{X} \cup \psi(A_x)$
- Let $\iota_x : \mathcal{X} \rightarrow \mathcal{X}_x$ be the insertion of $\mathcal{X}$ into $\mathcal{X}_x$.
- For every cycle $O \in A_x$, set $x_O \triangleq \psi(O)$
- For every edge $[s_i, e_i]$ of $O$, let
  - $\prec_x(s_i, x_O, e_i)$ if $\prec_{tot}(s_i, x, e_i)$, and
  - $\prec_x(s_i, e_i, x_O)$ otherwise.
- Check that $\Gamma_x = (\mathcal{X}_x, \prec_x)$ is a superstructure of $\Gamma$ that has $c_x \triangleq \psi(\mathcal{O}(\Gamma))$ as a separator.
Winding a shift-invariant PO to a CyO

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Cyclic Ordering through Partial Orders
Final Remarks

Outlook

- Characterization provides link to
  - concurrent process dynamics
  - Partial order properties
- The difficulty of cyclic orientation is lifted once a separator is known
- Future Work:
  - formalize abstraction mappings
  - Develop morphisms, study category of CyOs vs Posets
  - Investigate functorial properties (adjunctions etc ?)
  - Correlate (?) with topological coverings etc.
  - Things to explore : e.g. cyclic analogs of lattices ...