Cyclic Ordering through Partial Orders

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Cyclic Orders

Let $\lessdot$ be a ternary relation over set $\mathcal{X}$. Then $\Gamma = (\mathcal{X}, \lessdot)$ is a **cyclic order (*CyO*)** iff it satisfies, for $a, b, c, d \in \mathcal{X}$:

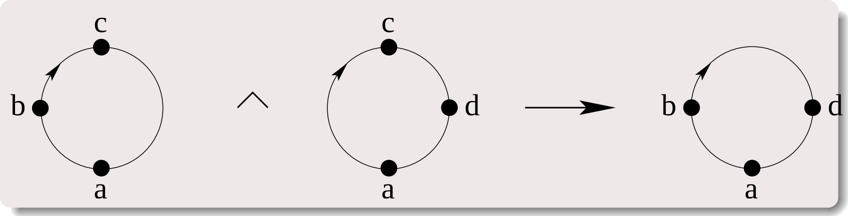
1. **inversion asymmetry**: $\lessdot(a, b, c) \Rightarrow \neg\lessdot(b, a, c)$
2. **rotational symmetry**: $\lessdot(a, b, c) \Rightarrow \lessdot(c, a, b)$;
3. **ternary transitivity**: $[\lessdot(a, b, c) \wedge \lessdot(a, c, d)] \Rightarrow \lessdot(a, b, d)$.

A *CyO* $\Gamma_{tot} = (\mathcal{X}, \lessdot_{tot})$ is called **total** or a *TCO* if for all $a, b, c \in \mathcal{X}$,

$$(x \neq y \neq z \neq x) \Rightarrow \lessdot(a, b, c) \vee \lessdot(b, a, c).$$

If a total *CyO* Γ_{tot} on $\mathcal{X}$ **extends** Γ, call Γ **orientable**, and Γ_{tot} an **orientation** of Γ.

Cyclic Transitivity



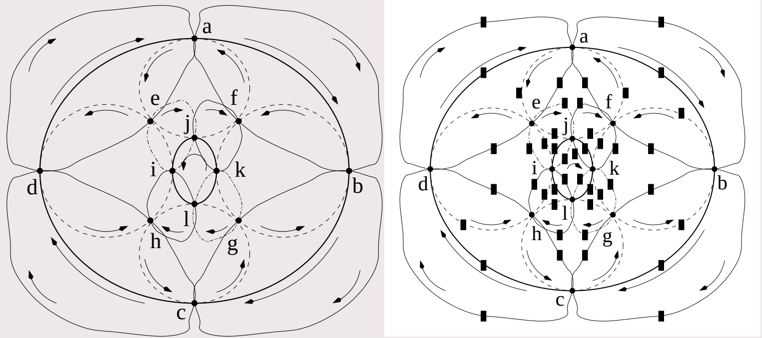
Short History

- Huntington (1916,1924,1938) introduces cyclic (total) order over words of variable length
- Partial Cyos as *ternary relations* , orientability problem:
  - *Power*({Alles, Nešetřil, Novotny, Poljak, Chaida}) (1982,1984,1985,1986,1991,1994,...),
  - Galil and Megiddo (1977)
  - Genrich (1971): links to synchronization graphs (conflict free Petri nets)
  - Petri: Concurrency Theory; Rozenberg et al: Square structures
  - Stehr (1998) axiomatizes partial cyclic order over words
  - . . .

(Some) known results

- Orientability is NP-complete (Megiddo 1976, Galil and Megiddo 1977)
- Compactness: a Cyo is orientable iff all its finite sub-CyOs are (Alles, Nešetřil, Poljak 1991)
- Genrich 1971, Stehr 1998: orientable finite CyOs correspond to 1-safe live Synchronization Graphs
- HERE: explore new link with partial orders : WINDING

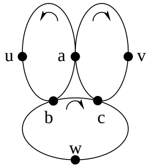
Non-Orientable CyOs



Preliminaries : Global vs round-orientedness

- li $\triangleq \{(x,y) \mid \exists z : \lhd(x,y,z) \vee \lhd(x,z,y)\}$
- $co \triangleq \mathcal{X}^2 - (\mathrm{id}_{\mathcal{X}} \cup \mathrm{li})$
- **Rounds**: Maximal cliques of li
- **cuts**: Maximal cliques of co
- $\Gamma = (\mathcal{X}, \lhd)$ is **round-oriented** ($ROCO$) iff for any round $\{a,b,c\}$, either $\lhd(a,b,c)$ or $\lhd(b,a,c)$.

Not a $ROCO$:



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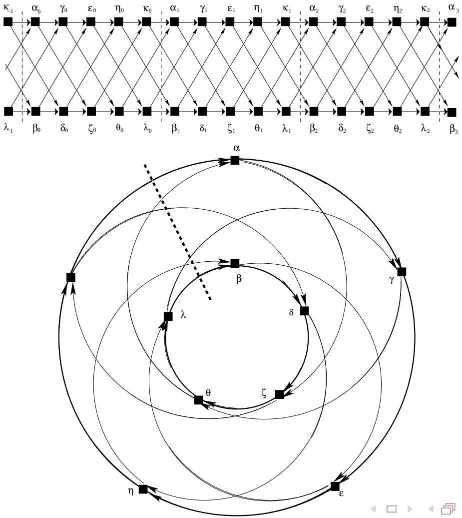
Shifts and Windings

Definitions

- Fix poset $\Pi = (\mathcal{X}, <)$. An **order automorphism** is a bijection $\mathbf{G} : \mathcal{X} \to \mathcal{X}$ such that $\forall\, x, y \in \mathcal{X} : \ [x < y] \Leftrightarrow [\mathbf{G}x < \mathbf{G}y]$.
- $\mathbf{G}$ is called a **shift** if $x < \mathbf{G}x$ for all $x \in \mathcal{X}$
- The group $\mathcal{G}$ of Π-automorphisms generated by $\mathbf{G}$ is isomorphic to $(\mathbb{Z}, +)$
- Equivalence : $\overline{x} \sim_{\mathbf{G}} \overline{y}$ iff $\exists\ k \in \mathbb{Z} \ : \ \mathbf{G}^k\overline{x} = \overline{y}$
- Let $[\overline{x}] \triangleq [x]_{\sim_{\mathbf{G}}}$
- LEt $\beta_{\mathbf{G}} : \overline{\mathcal{X}} \to \mathcal{X}, \overline{x} \mapsto [\overline{x}]_{\sim_{\mathbf{G}}}$ be the **winding map**.

Winding a shift-invariant PO to a CyO

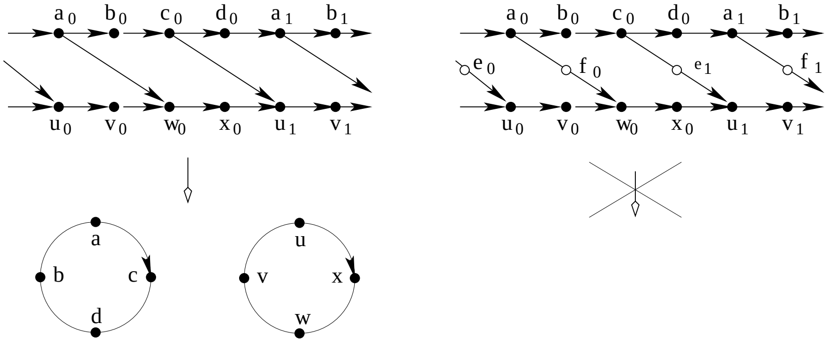


Windings and their Properties

- Set $\lhd(x, y, z)$ iff $\exists\ \overline{x} \in x,\ \overline{y} \in y,\ \overline{z} \in z\ :\ \overline{x} < \overline{y} < \overline{z} < \mathbf{G}\overline{x}$.
- $\Gamma = (\mathcal{X}, \lhd)$ is **wound** from Π via **G** (or $\beta_{\mathbf{G}}$)
- Note : this leaves more concurrency than the **usual** construction of CyO from a poset: $\lhd(x, y, z)$ iff $\exists\ \overline{x} \in x,\ \overline{y} \in y,\ \overline{z} \in z\ :\ \overline{x}_0 < \overline{y}_0 < \overline{z}_0$.
- Inheritance: If Γ_1 is obtained by winding, then so are all its Sub-CyOs.
- We say that a winding is **loop-free** iff for all $\overline{x} \in \overline{\mathcal{X}}$, there exist $\overline{y}, \overline{z} \in \overline{\mathcal{X}}$ that $\overline{x} < \overline{y} < \overline{z} < \mathbf{G}\overline{x}$. Loop-free windings generate *ROCO*s.

Mind the Gap !

Not all windings preserve successor relations



Mind the Gap !

- Let $x \mathrel{\lhd\!\cdot}\; y$ iff
  1. x li y and
  2. for all $z \in \mathcal{X} - \{x, y\}$, x li z and z li y imply $\lhd(x, y, z)$.
- $\overline{x}$ **covers** $\overline{y}$ **from below**, written $x \,\underline{\vee}\, y$, iff (a) $\overline{x} < \overline{y}$, and (b) for all $\overline{z} \in \mathcal{X}$, $\overline{z} < \overline{y}$ implies $\overline{z} < \overline{x}$.
- $\overline{y}$ **covers** $\overline{x}$ **from above**, written $y \,\overline{\wedge}\, x$, iff (a) $\overline{x} < \overline{y}$, and (b) for all $\overline{z} \in \mathcal{X}$, $\overline{x} < \overline{z}$ implies $\overline{y} < \overline{z}$.
- In $(\mathcal{X}, \lhd)$, x **covers** y, written $x \rhd y$, iff (a) x li y, and (b) for all $z, u \in \mathcal{X}$, $\lhd(z, u, y)$ implies $\lhd(z, u, x)$.
- A **gap** in $(\overline{\mathcal{X}}, <)$: $x \lessdot y$ holds, but neither $x \,\overline{\wedge}\, y$ nor $y \,\underline{\vee}\, x$.
- A **gap** in $(\mathcal{X}, \lhd)$: $x \mathrel{\lhd\!\cdot}\, y$ holds, and $x \rhd y$ does not hold.
- Suppose $(\overline{\mathcal{X}}, <)$ is gap-free and $\beta_{\mathbf{G}}$ winds $(\overline{\mathcal{X}}, <)$ to $(\mathcal{X}, \lhd)$. If $\beta_{\mathbf{G}}$ is loop-free, then $\beta_{\mathbf{G}}$ maps $\lessdot$ surjectively to $\mathrel{\lhd\!\cdot}$.

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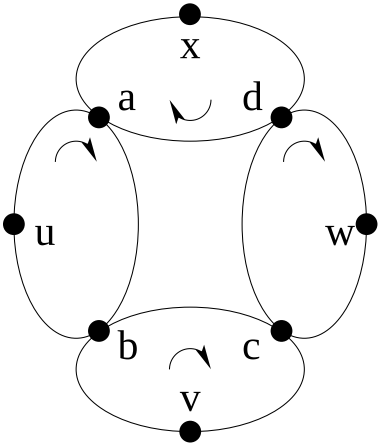
Separation properties

Definitions

- A cut **c** is called a **separator** iff $\mathbf{c} \cap O \neq \emptyset$ for all $O \in \mathcal{O}(\Gamma)$, and
- a **cycle separator** iff $\mathbf{c} \cap \gamma \neq \emptyset$ for all $\gamma \in \mathcal{D}(\Gamma)$
- Γ is called **weakly (cycle) separable** iff there exists a (cycle) separator $\mathbf{c} \in \mathcal{C}(\Gamma)$, and
- **strongly (cycle) separable** iff all its cuts are (cycle) cuts.
- If some superstructure cyclic order Γ' of Γ is (strongly) cycle separable, then Γ is called **(strongly) saturable.**
- Every total *CyO* is strongly separable.

Separation properties

Cycle separation implies round separation; the converse is not true:



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Main Result

THEOREM

Let $\mathcal{X} \neq \emptyset$, and $\Gamma = (\mathcal{X}, \lhd)$ a *ROCO*. Then the following are equivalent:

1. Γ is **weakly** saturable;
2. there exists a winding representation for Γ, i.e. a partial order $\Pi = (\overline{\mathcal{X}}, <)$ with shift $\mathbf{G}$ such that $\mathcal{X} = \overline{\mathcal{X}}_{/\mathbf{G}}$, and $\phi_{\mathbf{G}}$ winds Π to Γ ;
3. Γ is orientable.

Sketch of Proof

$1 \Rightarrow 2$

Saturability to winding:
Given $(\overline{\mathcal{X}}, <)$ and $\mathbf{c}$, set $\overline{\mathcal{X}} \triangleq \mathcal{X} \times \mathbb{Z}$, $x_k \triangleq (x, k)$.
Let $\mathbf{G} : \overline{\mathcal{X}} \to \overline{\mathcal{X}}$ be given by $\mathbf{G} x_k = x_{k+1}$ for $k \in \mathbb{Z}$.

$$
\begin{aligned}
\mathcal{R}_0 &\triangleq \{(a_0, b_0) \mid a \in \mathbf{c} \ \wedge \ a \ \ \text{li} \ \ b\} \\
&\quad \cup \{(a_k, a_{k+l}) \mid a \in \mathcal{X}, \ k \in \mathbb{Z}, \ l \in \mathbb{N}\} \\
\prec &\triangleq \{(a_{l+k}, b_{m+k}) \mid a_l \ \mathcal{R}_0 \ b_m, \ k \in \mathbb{Z}\} \\
< &\triangleq \textit{transitive closure of } \prec
\end{aligned}
$$

Sketch of Proof

$2 \Rightarrow 3$

Winding to orientability:
Take a weakly separable *SuperPO* $\Pi_\sharp$ of Π, with separator and $\mathbf{c}_0.\mathbf{c}_k \triangleq \mathbf{G}^k\mathbf{c}_0$, and define $\mathcal{U}_k \triangleq \mathcal{U}_k^\sharp \cap \overline{\mathcal{X}}$, where

$$\mathcal{U}_k^\sharp \triangleq \bigcup_{\overline{y}_k \in \mathbf{c}_k, \quad \overline{y}_{k+1} \in \mathbf{c}_{k+1}} [\overline{y}_k, \overline{y}_{k+1}[.$$

- The $\mathcal{U}_k$ are pairwise disjoint and cover $\overline{\mathcal{X}}$
- Set $\Pi_k \triangleq (\mathcal{U}_k, <_k)$; $\mathbf{G}$ induces order isomorphisms $\mathbf{G}_{n,m} : \mathcal{U}_n \rightarrow \mathcal{U}_m$ from Π_n to Π_m.
- Take total ordering Π_0^{tot} on $\mathcal{U}_0$ that embeds $<_0$ (Szpilrajn !)
- Export Π_0^{tot} to $\mathcal{U}_k$ via $\mathbf{G}_{0,k} \Rightarrow$ Done

Sketch of Proof

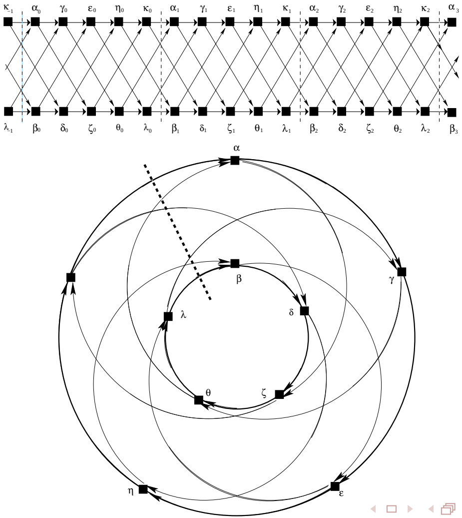
$3 \Rightarrow 1$

Orientability to saturability:

Let $\Gamma_{tot} = (\mathcal{X}, \lhd_{tot})$ be an orientation of Γ; fix $x \in \mathcal{X}$.

- Let $\mathcal{Y}$ be a set disjoint from $\mathcal{X}$, and $\psi : \mathcal{O} \to \mathcal{Y}$ injective
- Set $A_x \triangleq \{O \in \mathcal{O}(\Gamma) \mid x \notin O\}$ and $\mathcal{X}_x \triangleq \mathcal{X} \cup \psi(A_x)$
- Let $\iota_x : \mathcal{X} \to \mathcal{X}_x$ be the insertion of $\mathcal{X}$ into $\mathcal{X}_x$.
- For every cycle $O \in A_x$, set $x_O \triangleq \psi(O)$
- For every edge $[s_i, e_i]$ of O, let
  - $\lhd_x(s_i, x_O, e_i)$ if $\lhd_{tot}(s_i, x, e_i)$, and
  - $\lhd_x(s_i, e_i, x_O)$ otherwise.
- Check that $\Gamma_x = (\mathcal{X}_x, \lhd_x)$ is a superstructure of Γ that has $\mathbf{c}_x \triangleq \psi(\mathcal{O}(\Gamma))$ as a separator.

Winding a shift-invariant PO to a CyO



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Final Remarks

Outlook

- Characterization provides link to
  - concurrent process dynamics
  - Partial order properties
- The difficulty of cyclic orientation is lifted once a separator is known
- Future Work:
  - formalize abstraction mappings
  - Develop morphisms, study category of CyOs vs Posets
  - Investigate functorial properties (adjunctions etc ?)
  - Correlate (?) with topological coverings etc.
  - Things to explore : e.g. cyclic analogs of lattices ...