Stability of the Sand Piles Model

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Sand Piles Model (SPM)

- Context: Discrete dynamical system, sand piles system
- Sand Piles System

  - Configurations: are sequences of sand piles of decreasing height from left to right

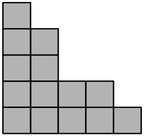
  - Evolution rule: 2 rules

    Falling rule: One grain falls down from one column to the right next column if their height difference is at least 2.

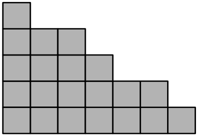
    Adding rule: Adding one grain on one random column such that the obtained one is a configuration

Sand Piles Model

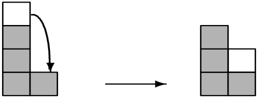
A grain



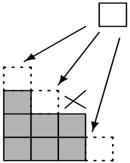
A configuration of SPM



A stable configuration



Falling rule



Adding rule

Problem

1. Stationary
2. Structure
3. Uniqueness
4. Time

Well known results on SPM

- Goles+Kiwi (1993): falling rule: $SPM(n)$ is of a lattice structure, unique fixed point
- Goles+Morvan+Phan (1998): falling rule: Describe all elements of $SPM(n)$, show the recursive structure
- Bak (1973): falling +adding rule: Show observation, investigations, examples and diagrams
- Phan+Tran (2006): falling+adding rule:Simulate mathematically, inverstigate the tranformations between stable configurations, show the lattice structure and measure the time

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Fundamental definitions

- Partition of the positive integer n is a k-tuple of positive integers $a = (a_1, \ldots, a_k)$ such that $\sum_{i=1}^{k} a_i = n$, $a_1 \geq a_2 \geq \cdots \geq a_k$.
- Smooth partition is a partition $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_i - a_{i+1} \leq 1$.
- Strict partition is a partition $(a_1, a_2, \ldots, a_k)$ such that $a_i - a_{i+1} \geq 1$.

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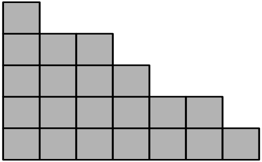
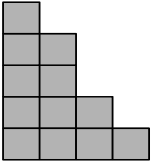
Partitions

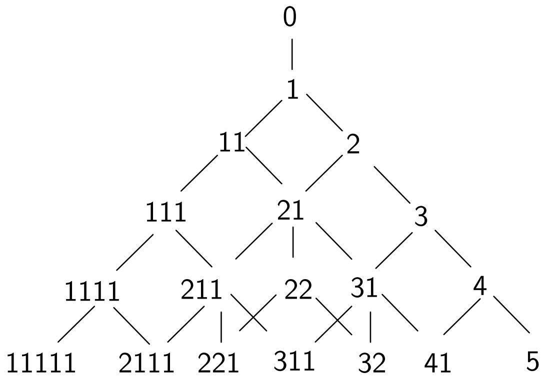


Smooth partition



Strict partition

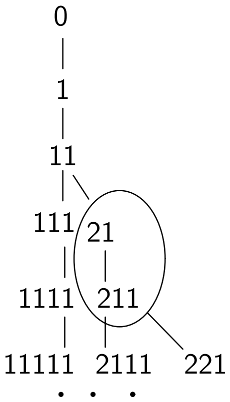
- Young lattice is the lattice of all partitions ordered by containment



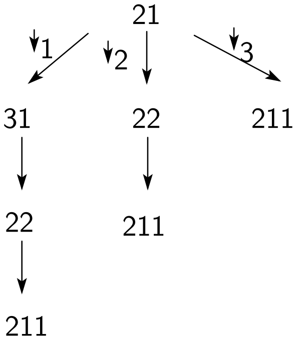
Stable Sand Piles Model

- Configurations:are represented by partitions
  - The initial configuration is (0)
  - Stable configurations (temporary) are smooth partitions
- Evolution rule:
  - Falling rule: the same as the evolution rule of Sand Piles Model.
  - Adding rule: adding one grain on random column of a smooth partition.
- Stable Sand Piles Model (SSPM): Configurations are stable

Stable Sand Piles Model



First elements of *SSPM*



Main results

1. Properties of the *SSPM*
   - Configurations: smooth partitions
   - Structure: sublattice of Young lattice: Theorem 1
2. Avalanche
   - Minimal lenghth: Theorem 2
   - Maximal length: Theorem 3

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Lattice structure of *SSPM*

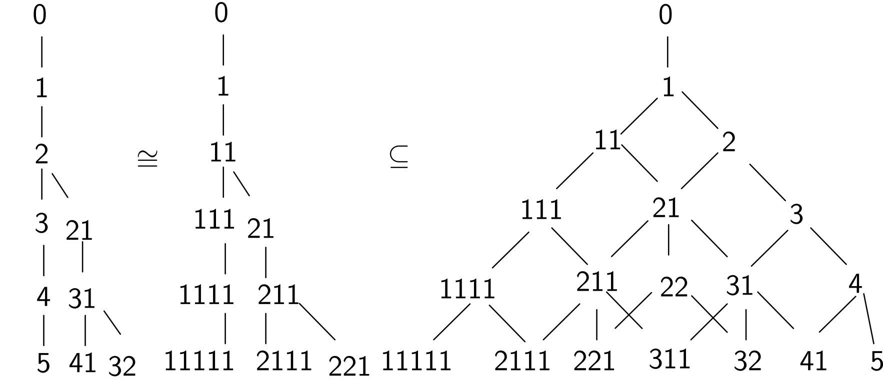
Theorem 1

SSPM is sublattice of the Young lattice and isomorphism to the lattice of strict partitions.

Proof.

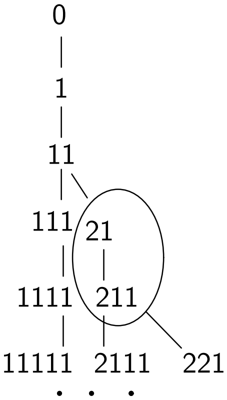
- Prove that the ordered set *SSPM* is isomorphism to the *Strict* by dual mapping.
- Show explicitly $\inf(a, b)$ and $\sup(a, b)$

□



Strict lattice *SSPM* lattice Young lattice

Stable Sand Piles Model



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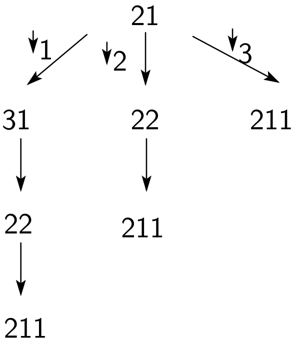


Minimal length

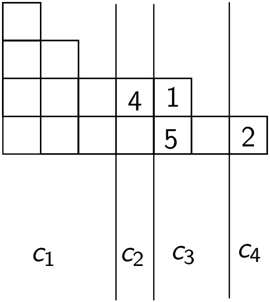
By upside down construction

Theorem 2

1. The minimal length from the initial configuration (0) to a equals $w(a)$.
2. The minimal length from a to b equals $w(a) - w(b)$,

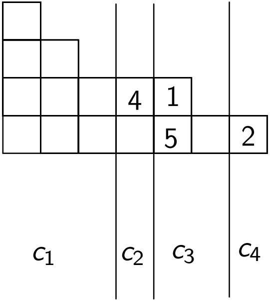
where $w(a) = \sum_{i=1}^{k} a_i$ is the weight of a.

Maximal length



1. Difficulties:
   - We don't how adding will obtain the maximal length
   - The global maximal length is different from the sum of the local maximal lengths
2. Solutions:
   - Split *a* into maximal successive stairs
   - Assign one suitable value (energy) to each grain
   - Calculate the sum of energy of all grains
   - Construct the way to reach the maximal length.

Maximal length



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Maximal length

Theorem 3

Given $a, b \in SSPM$, $b \longrightarrow a$ we have

1. The maximal length from 0 to a is

$$l_m(a) = \frac{a_1(a_1+1)(2a_1-1)}{2} + \frac{(a_{|c_1|+1}+2|c_1|-1)a_{c_1+1}}{2} + \sum_{i=2}^{l} \frac{(a_{|c_1|+\dots+|c_{i-1}|+1}+2|c_{i-1}|-3)a_{|c_1|+\dots+|c_{i-1}|}+1}{2}$$

   Where $c_1, c_2, \dots, c_l$ are maximal successive stairs of a.

2. The maximal length from b to a is: $\sum_{(i,j)\in\Delta(a,b)} e_a(i,j)$,

   where $\Delta(a,b)$ are grains of a not belonging to b.

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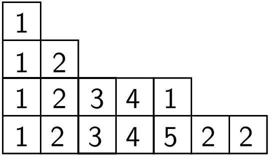
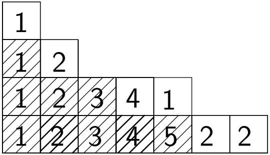
Example

a=(4,3,2,2,2,1,1)



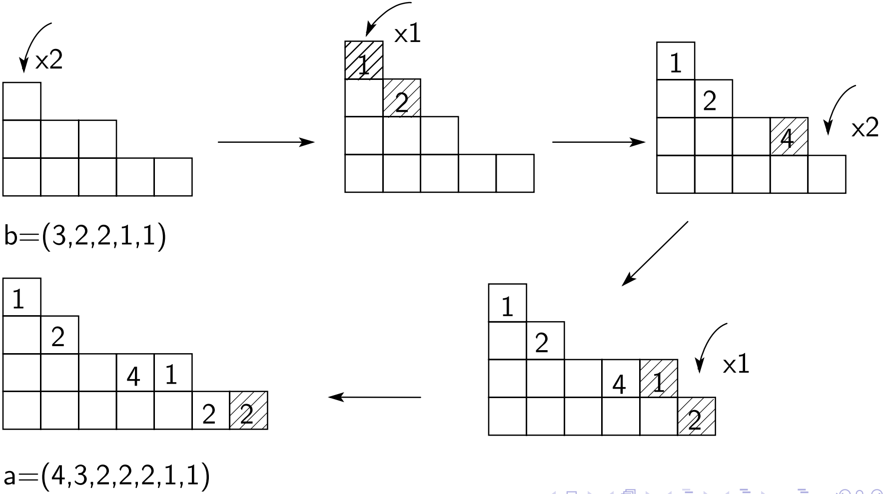
$l_m(a) = 34$

b=(3,2,2,1,1)

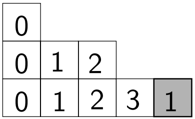


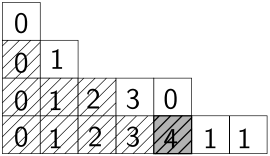
l(a, b)=12

From (3, 2, 2, 1, 1) to (4, 3, 2, 2, 2, 1)



Difference between $l_m(a,b)$ and $l_m(a) - l_m(b)$





Future works

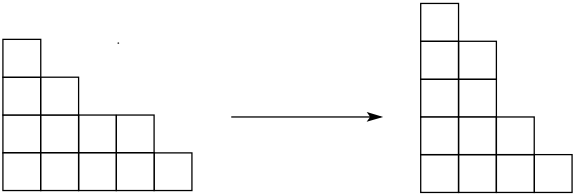
- Stability of Bidimention Sand Piles Model
- Two directions Sand Piles Model
- Other modified Sand Piles Models

Thank you for your attention:-)

Containment order

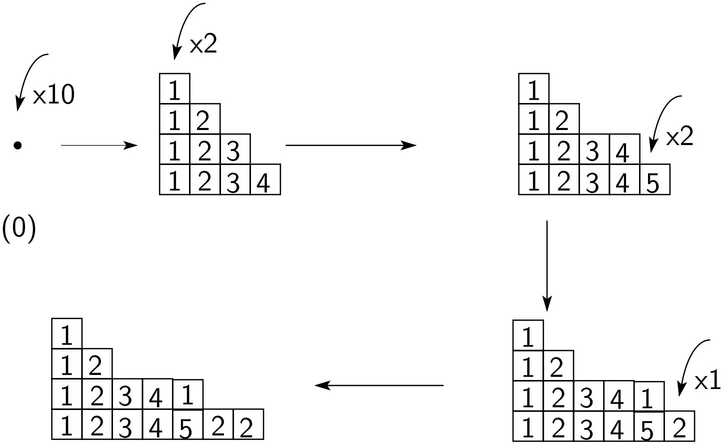
$a \leq b$ if and only if $a_i \geq b_i$ for all $i = 1, 2, \ldots, \min\{l(a), l(b)\}$.

Dual mapping



a=(4,3,2,2,1)   a*=(5,4,2,1)

From (0) to (4, 3, 2, 2, 2, 1)



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