Fixed Point Argument and Tilings without Long Range Order

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Tilings: local rules define a global order.

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Interesting in different contexts:

- combinatorics
- logic and computability
- physics

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$T \in \mathbb{Z}^2$ is a *period* if $U(x+T) = U(x)$ for all x.

Trivial example 1: one color

$\tau = \{$  $\}$

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There exists only one τ-tiling of $\mathbb{Z}^2$.

Trivial example 2: two colors

$\tau = \{$  ,  $\}$

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There exists two τ-tilings of $\mathbb{Z}^2$.

Trivial example 3: two colors
$\tau =$ all colorings of the 1×1-square

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continuum of tilings

there exist tile sets τ such that

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- there is no τ-tilings

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- there only aperiodic τ-tilings ?

Theorem (Robert Berger 1966): There exists a tile set that allows only aperiodic tilings.

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- There are periodic configurations that are almost tilings (sparse set of tiling errors)
- Tiling is aperiodic, but remote tiles are highly correlated

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B.Durand, A.Shen, A.R. 2008: There exists a tile set τ such that all τ-tilings are strongly aperiodic

Periodic tiling: very strong *remote order*.

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Charles Radin:

There is no Long Range Order if for large shifts $T = (a, b)$, tiles

$$U(x, y) \text{ and } U(x + a, y + b)$$

are *almost independent*.

This work's result:

There exists a set of red and green tiles $\tau = \tau_1 \sqcup \tau_2$ such that for large shifts $T = (a, b)$, colors of tiles

$$U(x, y) \text{ and } U(x + a, y + b)$$

are *almost independent*.

The tool:

a self-similar tile set based on a fixed-point construction (*á la* Kleene)

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J. von Neumann, Self-reproducible Automata (1966)

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Very similar ideas:

Peter Gács, reliable cellular automata (80-th, 90-th)

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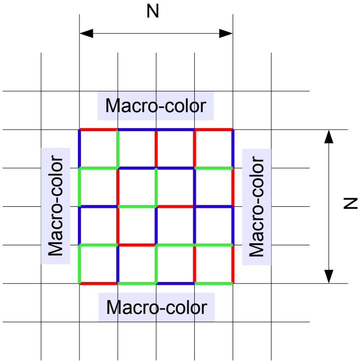
A small miracle: no computability in this statement, but Kleene's recursion theorem helps in the proof!

Macro-tile:



Fix a tile set τ and number $N > 1$.

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A set of tiles τ *simulates* a set of tiles ρ if

- there exists a set M of τ-macro-tiles isomorphic to ρ
- every τ-tiling can be uniquely split by $N \times N$ grid into macro-tiles from M.

Example.

A tile set $\tau_0 = \{$  $\}$

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A tile set τ_1: A tile set that simulates τ_0

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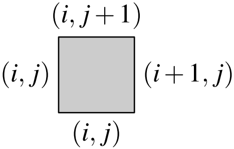

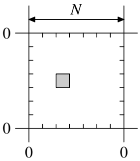
N
0
0
0
0

Self-similar tile set: a tile set that simulates itself.

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The new tool: self-similar tile set *á la* Kleene

Simulating a given tile set ρ by macro-tiles.

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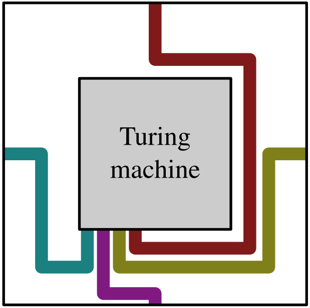
tile set is presented as TM that accepts quadruples of colors that are tiles

Implementation scheme:



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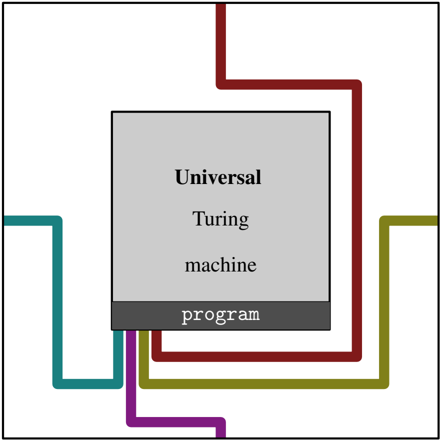

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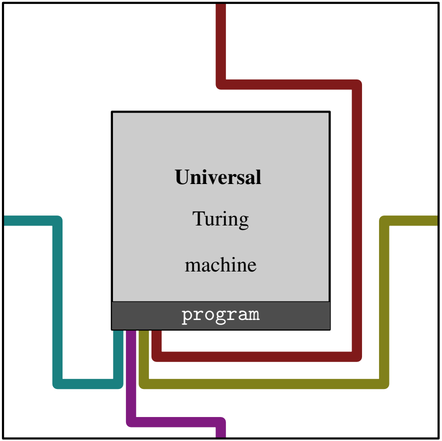

A fixed point: simulating tile set = simulated tile set

Sketch of the fixed point construction: what program should check

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- checking the program

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Can we make a strongly aperiodic tile set tolerant to errors?

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- What about Gibbs measures? Need methods from percolation theory and statistical physics.

Question 2:

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Use other programming models instead of TM ?

Thank you!