Structures Defined by Higher-Order Recursion Schemes

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In a nutshell

(Higher-order) recursion schemes are rewriting systems for simply typed terms.

Simply typed lambda calculus + Recursion
(similar to Statman λY-calculus).

We use schemes to generate infinite trees. For these infinite trees, monadic-second order (MSO) logic or (equivalently the modal μ-calculus) is decidable.

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  generators for languages of finite trees or words
  + only subfamily called the safe schemes
- Higher-order recursion schemes as infinite tree generators
  starting point [Knapik+Niwinski+Urzyczyn01] with only decidability for safe schemes, full decidability [Ong06] ...

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Quest for more an more expressive countable graphs with decidable MSO-theories.

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Linear orders

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- Countable ordinals [Buchi73,Shelah75]
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- Necessary and sufficient conditions for decidability on ω-words [Rabinovich05]

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For uncountable orders, the MSO-theory of the reals is undecidable [Shelah75].

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Around the full binary tree

Full binary tree [Rabin69]

- Configurations graphs of pushdown automata [Muller+Schupp85]
- VR and HR-graphs [Courcelle89,90]
- Prefix-recognizable graphs [Caucal96]

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All these* graphs can be defined in MSO (MSO-interpreted) in trees defined by recursion schemes.

($*$) The only exceptions are the ordinals between ϵ_0 and ω_0.

Outline

- Recursions schemes and further motivation
- Automata models for recursion schemes

Typed terms

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Order of a type : measure of the "nesting" of the type

$\mathrm{Ord}(\mathbf{o}) = 0$

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Examples $\mathbf{o} \to \mathbf{o}$ and $\mathbf{o} \to \mathbf{o} \to \mathbf{o}$ are of order 1 and $(\mathbf{o} \to \mathbf{o}) \to \mathbf{o} \to \mathbf{o}$ is of order 2.

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Typed alphabet N Every $f \in N$ has a type τ (denoted $f : \tau$).

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Notations

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Rewriting relation $\xrightarrow{a}$ for $a \in \Sigma$ over ground terms over N

$$F\, t_1 \ldots t_n \xrightarrow{a} t[x_1/t_1, \ldots, x_n/t_n] \quad \text{if } F\, x_1 \ldots x_n \xrightarrow{a} t \in P$$

Order-1 example

Non-terminals: $S, C, V : \mathbf{o}, F : \mathbf{o} \to \mathbf{o}$

$$\begin{cases} S & \xrightarrow{a} & FC \\ F\,x & \xrightarrow{a} & F\,(B\,x) \\ F\,x & \xrightarrow{b} & x \end{cases} \qquad \begin{array}{lcl} B\,x & \xrightarrow{b} & x \\ C & \xrightarrow{c} & V \end{array}$$

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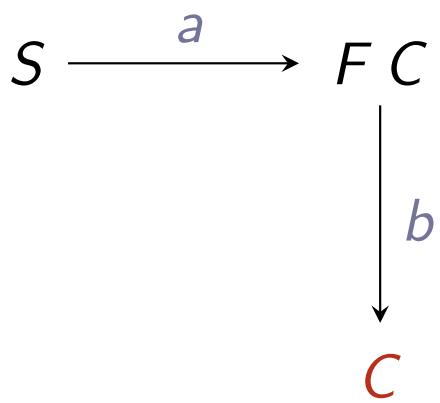

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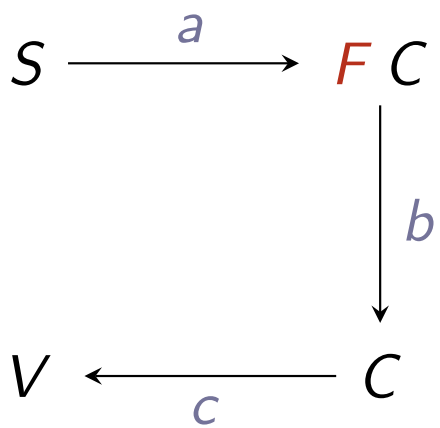

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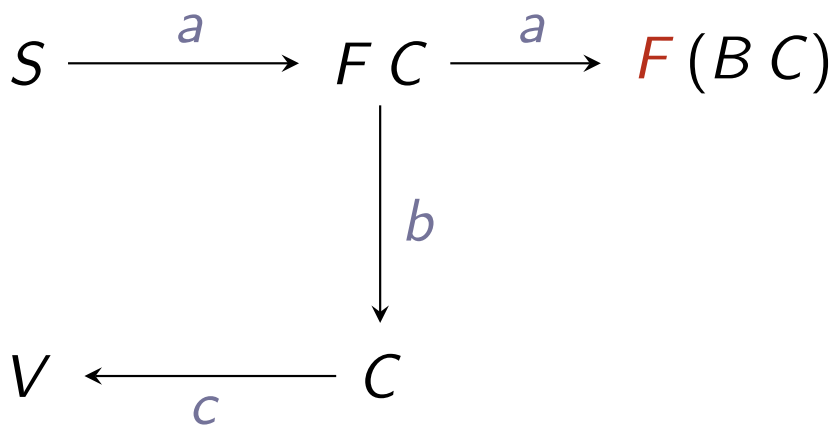

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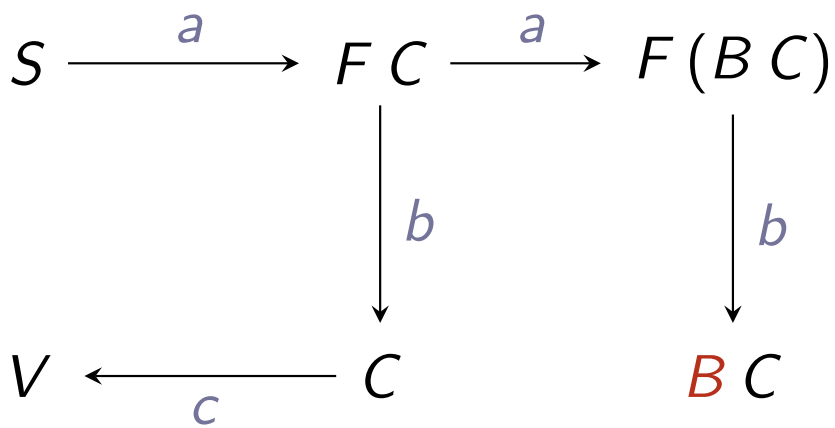

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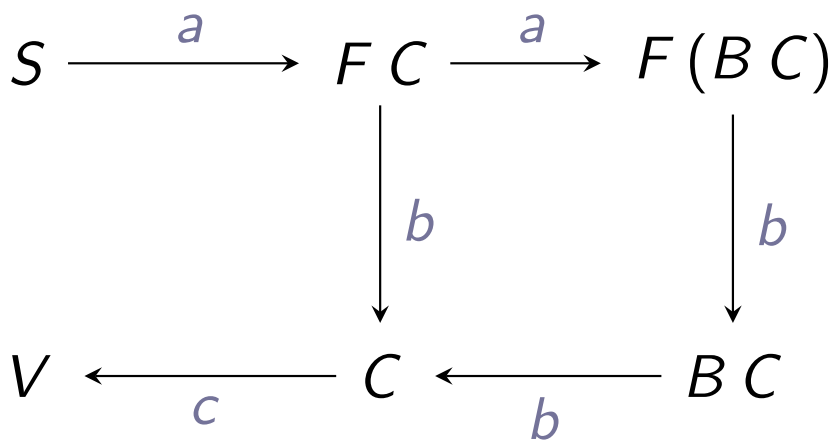

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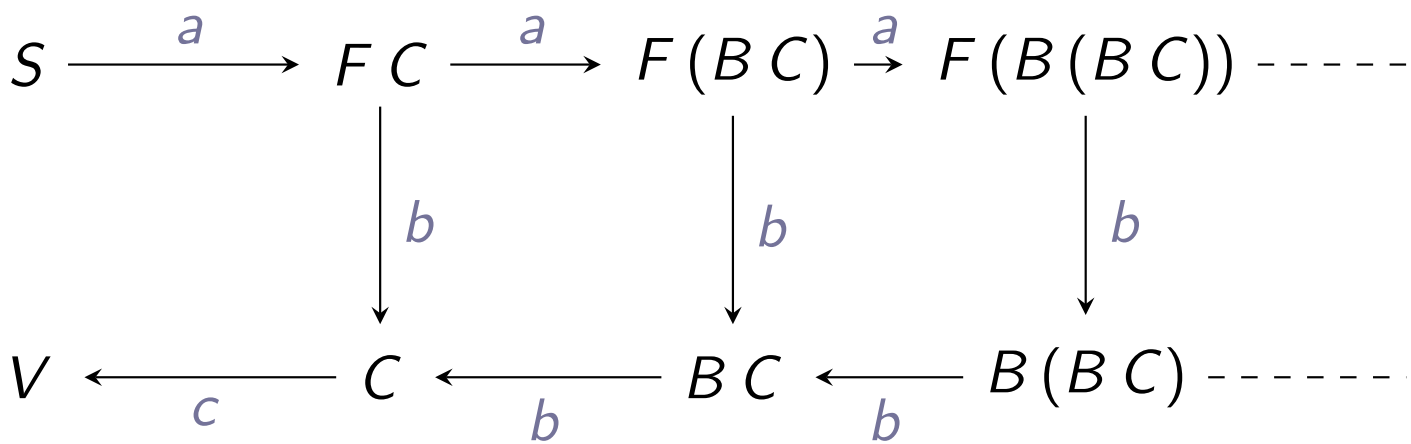

Unfold from the axiom

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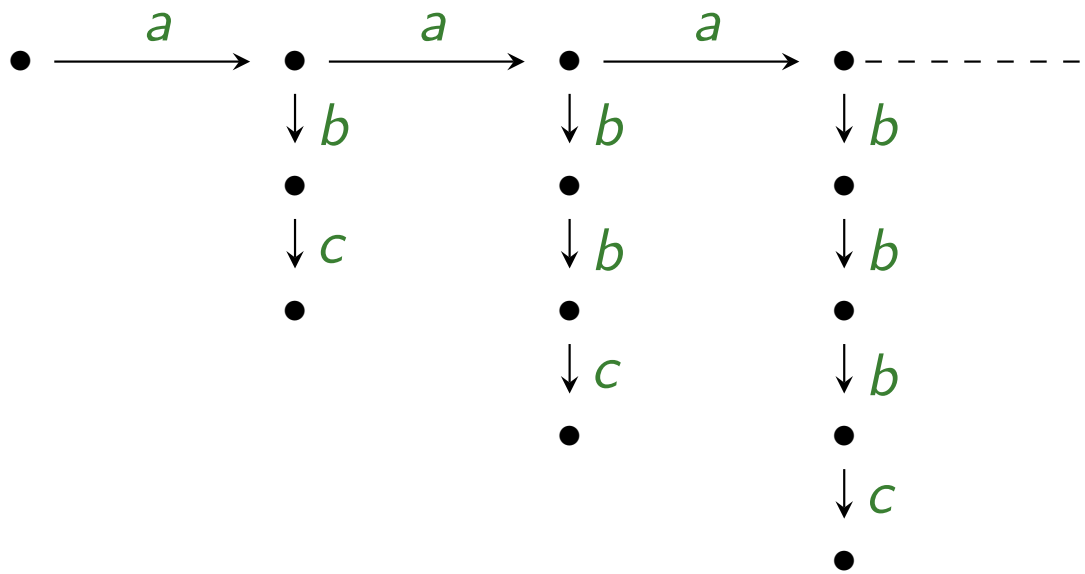

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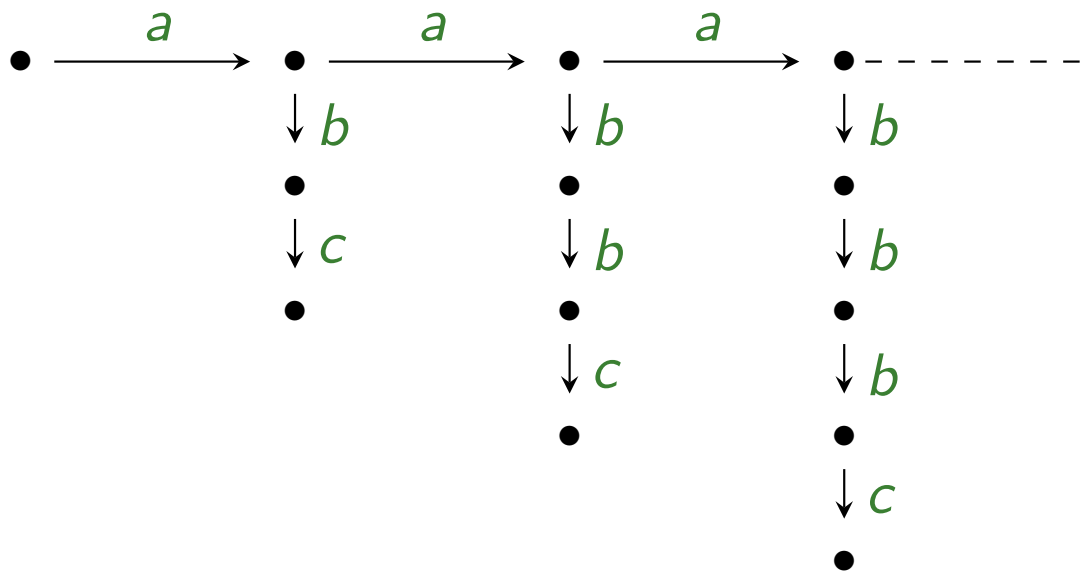

Tree of (the prefixes of) $\{a^n b^n c \mid n \geq 1\}$.

Tree generated by a recursion scheme

Silent productions labelled by ε and "contracted" in the infinite tree

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Syntactic restrictions ensuring that we obtain a deterministic tree.
The productions starting with $F \in N$ are:

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The class of infinite trees generated in this way coincide with the terms generated in the sense of [Knapik-Niwinski-Urzyczyn01].

ε-contraction

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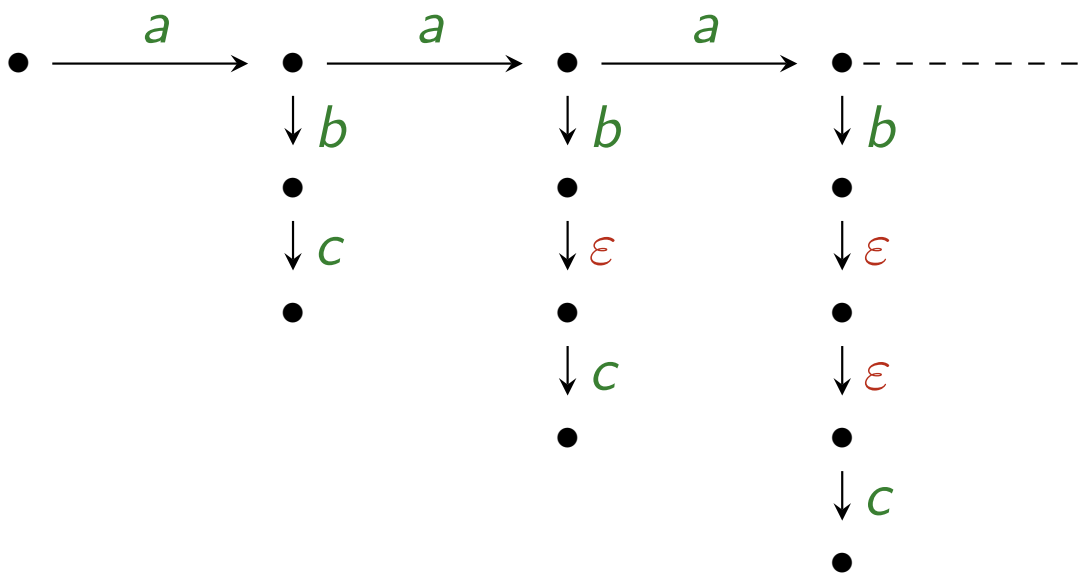

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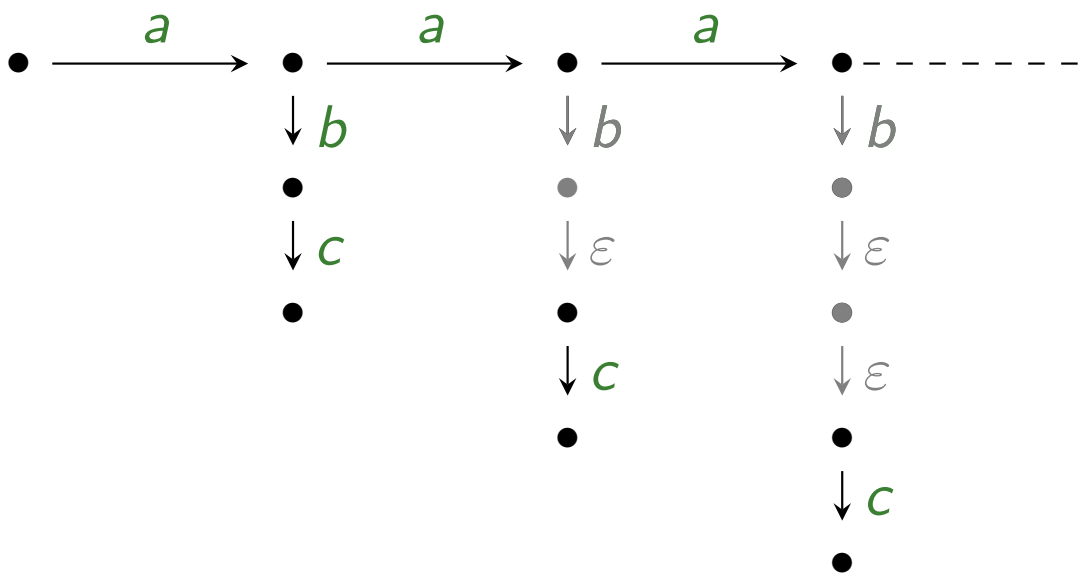

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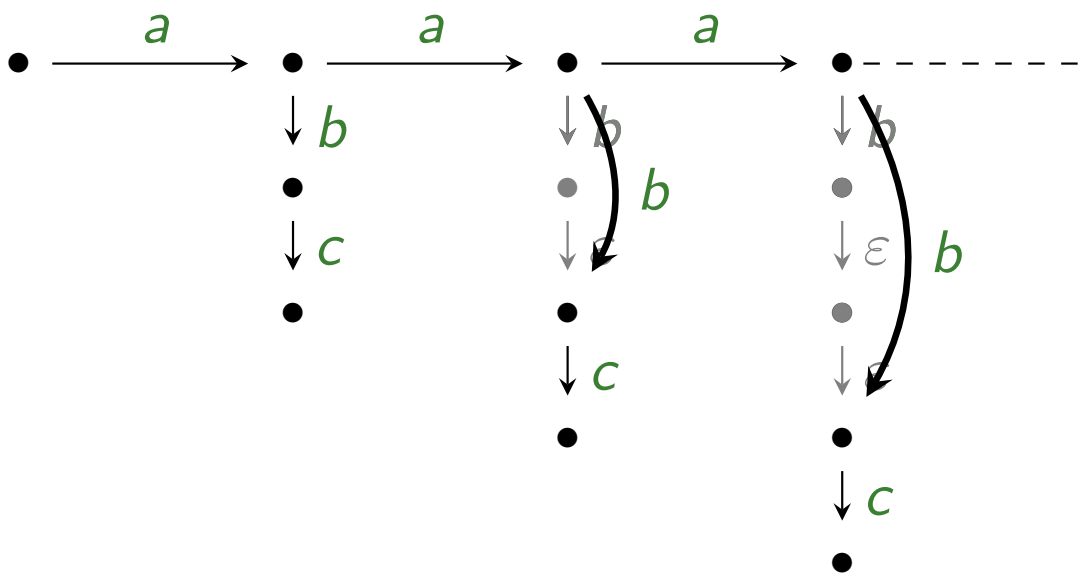

add b-edge when there was a path in $b\varepsilon^*$

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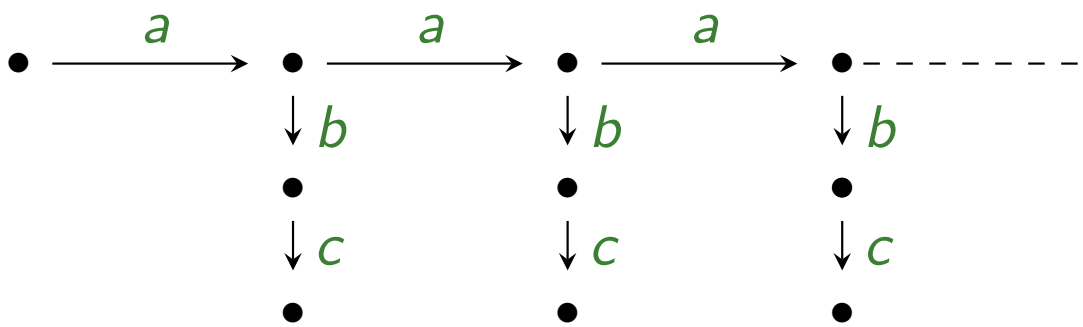

Examples of order 2

Aim Tree of $\{a^n d b^n c^n \mid n \geq 0\}$

$$\begin{cases} S & \xrightarrow{a} & F\,B\,C \\ F\,\phi\,\psi & \xrightarrow{a} & F\,(Cp\,B\,\phi)\,(Cp\,C\,\psi) \\ F\,\phi\,\psi & \xrightarrow{d} & \phi\,(\psi\,V) \end{cases} \qquad \begin{cases} B\,x & \xrightarrow{b} & x \\ C\,x & \xrightarrow{b} & x \\ Cp\,\phi\,\psi\,x & \xrightarrow{\varepsilon} & \phi\,(\psi\,x) \end{cases}$$

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Aim Tree of $\{a^n d\, b^n c^n \mid n \geq 0\}$

$$\begin{cases} S & \xrightarrow{a} & F\, B\, C & \qquad B\, x & \xrightarrow{b} & x \\ F\, \phi\, \psi & \xrightarrow{a} & F\, (Cp\, B\, \phi)\, (Cp\, C\, \psi) & \qquad C\, x & \xrightarrow{b} & x \\ F\, \phi\, \psi & \xrightarrow{d} & \phi\, (\psi\, V) & \qquad Cp\, \phi\, \psi\, x & \xrightarrow{\varepsilon} & \phi\, (\psi\, x) \end{cases}$$

with $x : \mathbf{o}$, $\phi, \psi : \mathbf{o} \to \mathbf{o}$.
$Cp : (\mathbf{o} \to \mathbf{o}) \to (\mathbf{o} \to \mathbf{o}) \to (\mathbf{o} \to \mathbf{o})$
Cp corresponds to the composition of functions
$Cp\, B\, B = B^2$, $Cp\, B\, (Cp\, B\, B) = B^3$, ...
For all $n \geq 0$ and term $t : \mathbf{o}$, $B^n t$ reduces in n b's to t.

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S

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with $x : \mathbf{o}$, $\phi, \psi : \mathbf{o} \to \mathbf{o}$.

$$\begin{array}{ccccccc} S & \xrightarrow{a} & F\,BC & \xrightarrow{a} & F\,B^2\,C^2 & \xrightarrow{a} & F\,B^3\,C^3 \text{ - - - - - -} \\ & & \downarrow d & & \downarrow d & & \\ & & B\,(C\,V) & & B^2\,(C^2\,V) & & \\ & & \downarrow b & & & & \\ & & \bullet & & & & \\ & & \downarrow c & & & & \\ & & \bullet & & & & \end{array}$$

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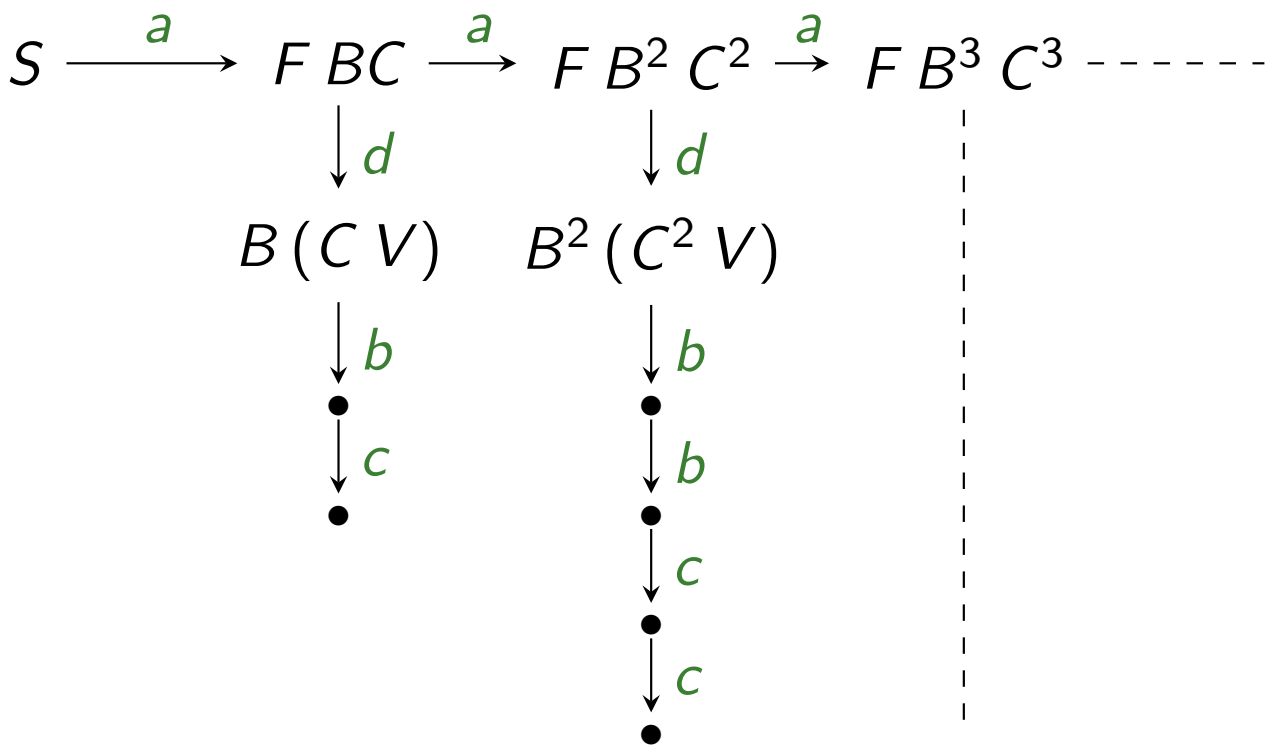

Examples of order-2

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$$\begin{cases} S & \xrightarrow{a} & F\,(Cp\,B\,B) \\ F\,\phi & \xrightarrow{a} & F(Cp\,\phi\,\phi) \\ F\,\phi & \xrightarrow{d} & \phi\,V \end{cases} \qquad \begin{array}{lcl} Bx & \xrightarrow{b} & x \\ Cp\,\phi\,\psi x & \xrightarrow{\varepsilon} & \phi\,(\psi\,x) \end{array}$$

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$$S \xrightarrow{a} F\,B^2 \xrightarrow{a} F\,(B^2 \circ B^2)$$

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$$S \xrightarrow{a} F\,B^2 \xrightarrow{a} F\,B^4$$

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$$S \xrightarrow{a} F\,B^2 \xrightarrow{a} F\,B^4 \xrightarrow{a} F\,B^{2^3} \dashrightarrow$$

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$$S \xrightarrow{a} F\,B^2 \xrightarrow{a} F\,B^4 \xrightarrow{a} F\,B^{2^3} \cdots\cdots\cdots$$

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with $x : \mathbf{o}$, $\phi : \mathbf{o} \to \mathbf{o}$.

$$S \xrightarrow{a} F\,B^2 \xrightarrow{a} F\,B^4 \xrightarrow{a} F\,B^{2^3} \cdots\cdots\cdots$$

$$F\,B^2 \downarrow d\; B^2\,V \downarrow b \bullet \downarrow b \bullet$$

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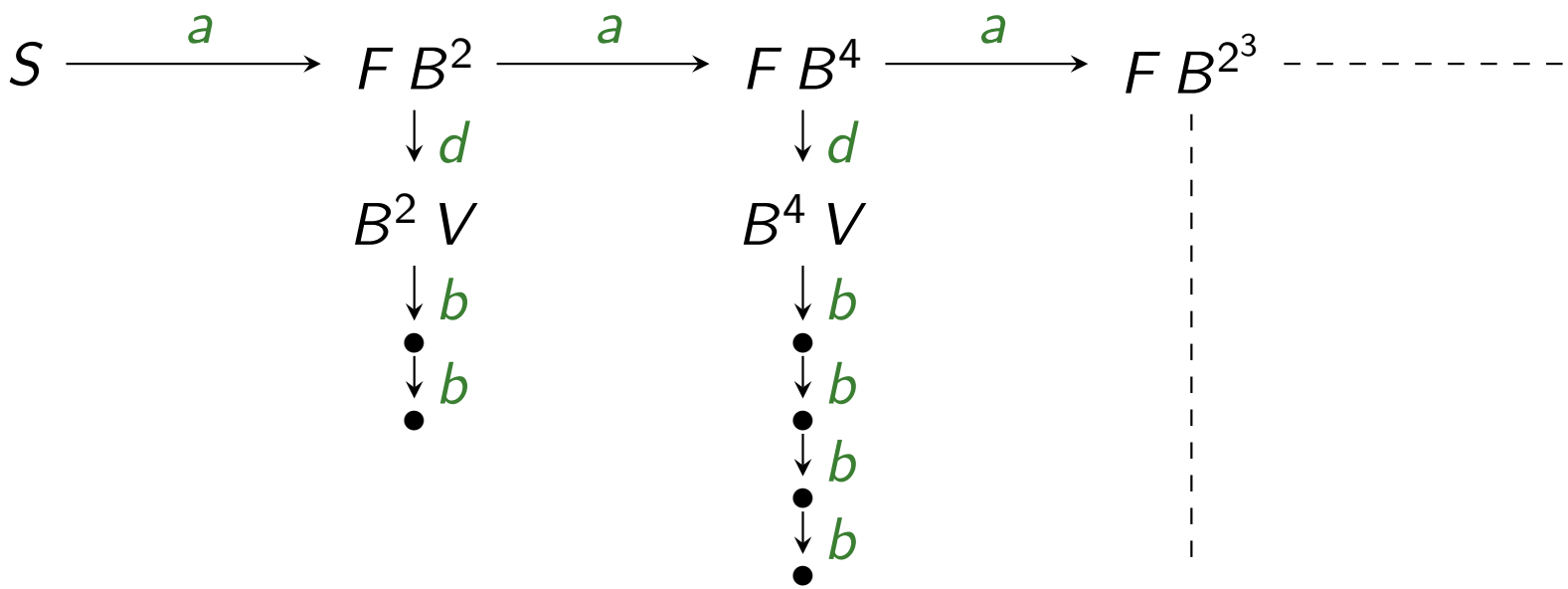

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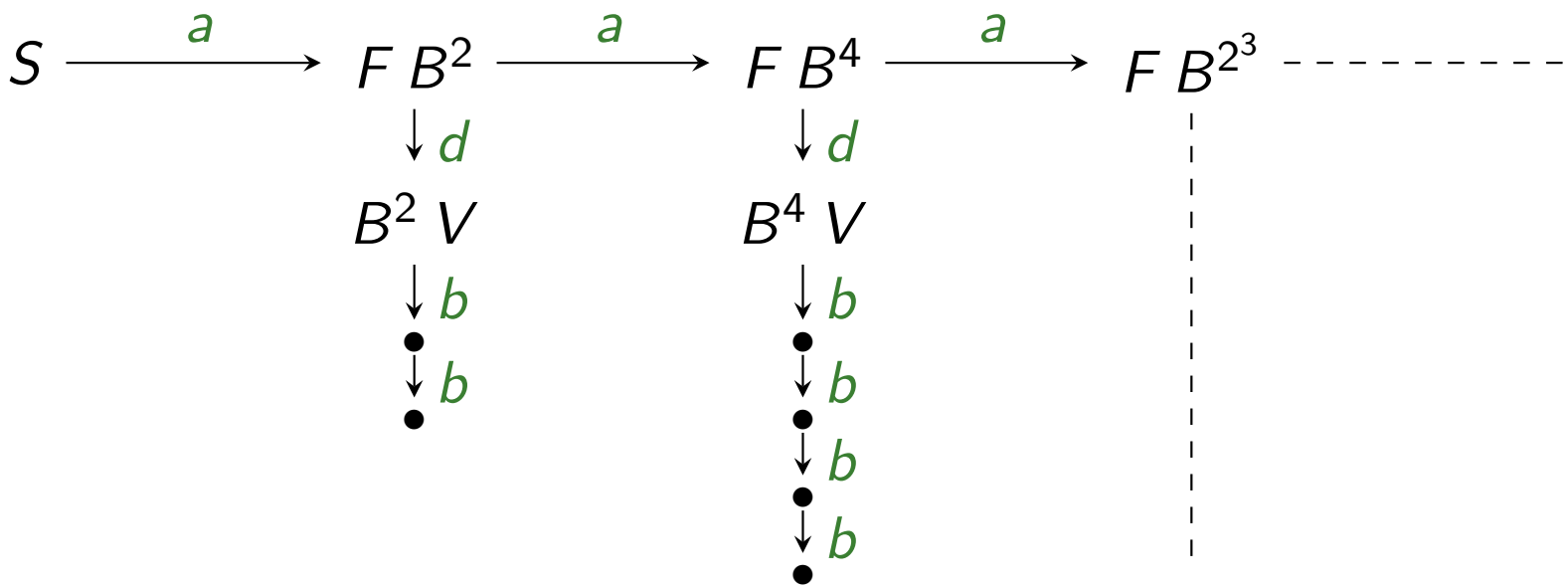

Similarly at order 3, we have the tree of $\{a^n d\, b^{2^{(2^n)}}\}$.

Algorithmic properties

Algorithmic properties

Theorem ([Ong06])

For all $n \geq 1$, the model-checking problem for modal mu-calculus is $n - \mathrm{EXPTIME}$ complete for trees generated by order-n recursion schemes.
Hence the trees defined by higher-order recursion schemes have a decidable MSO-theory.

Three proofs

Starting problem

Given a scheme G and an APT $\mathcal{A}$, does $\mathcal{A}$ accept the tree generated by G ?

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  $\Rightarrow$ characterisation of schemes via collapsible automata + induction on the order
- [Kobayashi+Ong09] reduction to a typing problem for G
  $\Rightarrow$ solving the typing problem amount to solving a finite parity game

Software verification by model-checking (borrowed from Kobayashi)

```
Let F x =
      if |x|==0 then close(x)
           else read(x); F(x)
Let S = open("txt"); F("txt")
```

⇓ Abstract the data

```
Let F =
      if * then close
           else read; F
Let S = open; F
```

Software verification by model-checking

$$\begin{cases} O\,k & \xrightarrow{\text{open}} & k \\ R\,k & \xrightarrow{\text{read}} & k \\ C\,k & \xrightarrow{\text{close}} & k \\ F\,k & \xrightarrow{\text{then}} & C\,k \\ F\,k & \xrightarrow{\text{else}} & R\,(F\,k) \\ S & \xrightarrow{\varepsilon} & O\,(F\,V) \end{cases}$$

$\Uparrow$ CPS transformation

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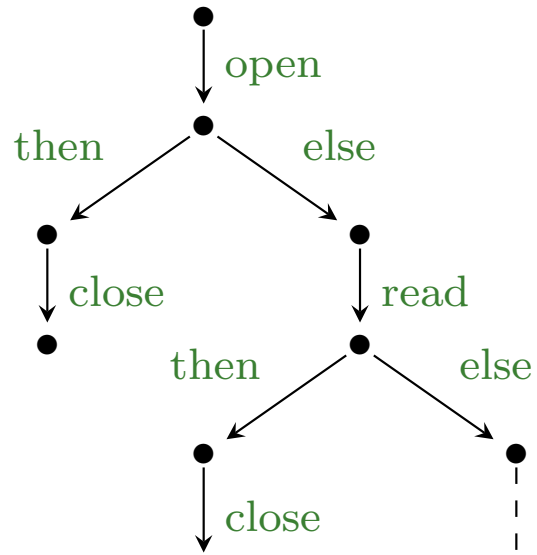

⇑ CPS transformation

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Software verification by model-checking

We can check on the tree that every read follows an open.

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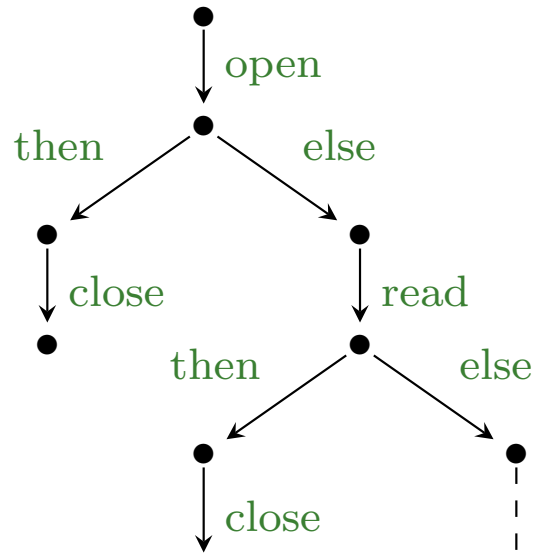

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- still $(n-1) - \mathrm{EXPTIME}$ complete
- yet: 4 states property on an order-4 scheme with 23 productions in 5ms.

Outline

- Recursions schemes and further motivation
- Automata models for recursion schemes

Model for order 1

Theorem ([folklore])

The infinite tree of a word language L is definable by an order 1 recursion scheme iff L is accepted by a deterministic pushdown automaton.

Model for order 1

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The infinite tree of a word language L is definable by an order 1 recursion scheme iff L is accepted by a deterministic pushdown automaton.

Theorem ([Sénizergues00])

The equivalence problem for deterministic pushdown automata is decidable.
Isomorphism is decidable for order 1 schemes.

Model for order 2

Non-determinism is not the way to describe the trees of order-2 schemes.

$\Rightarrow$ There exists a language accepted by non-deterministic pushdown automata (i.e. a context-free language) whose tree has an undecidable MSO-theory.

Model for order 2

Independently introduced by
[Knapik+Niwinski+Urzyczyn+Walukiewicz05] and
[Aeligh+deMiranda+Ong05] for order-2.

Theorem

The infinite tree of a word language L is definable by an order 2 recursion scheme iff L is accepted by a deterministic collapsible automaton of order 2.

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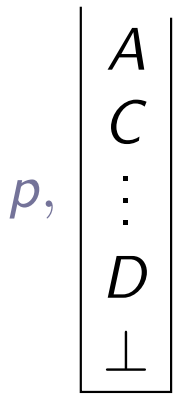
We will focus on level 2.

Pushdown automata

Finite control + stack

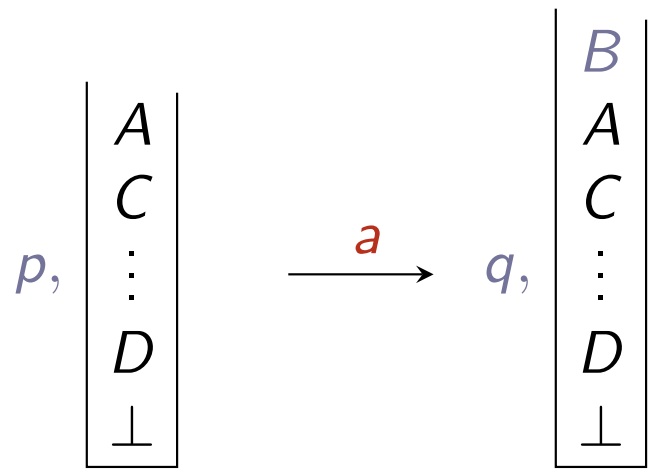


Transitions of the form: $p, A \xrightarrow{a} q, \rho$ with $\rho \in \{\mathrm{push}_B, \mathrm{pop}\}$.

Pushdown automata

Finite control + stack

push_B



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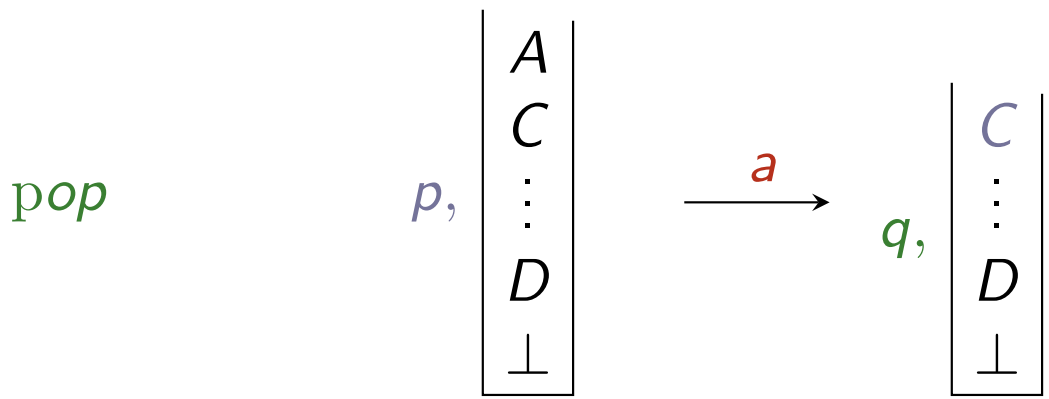

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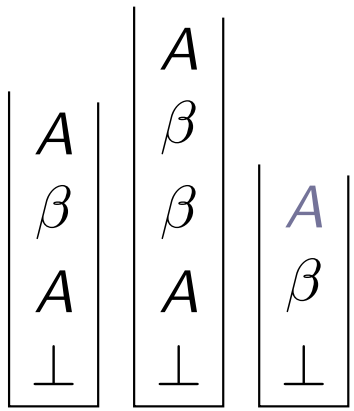
Order-2 pushdown [Maslov76]

Finite control + stack of (ordinary) stacks



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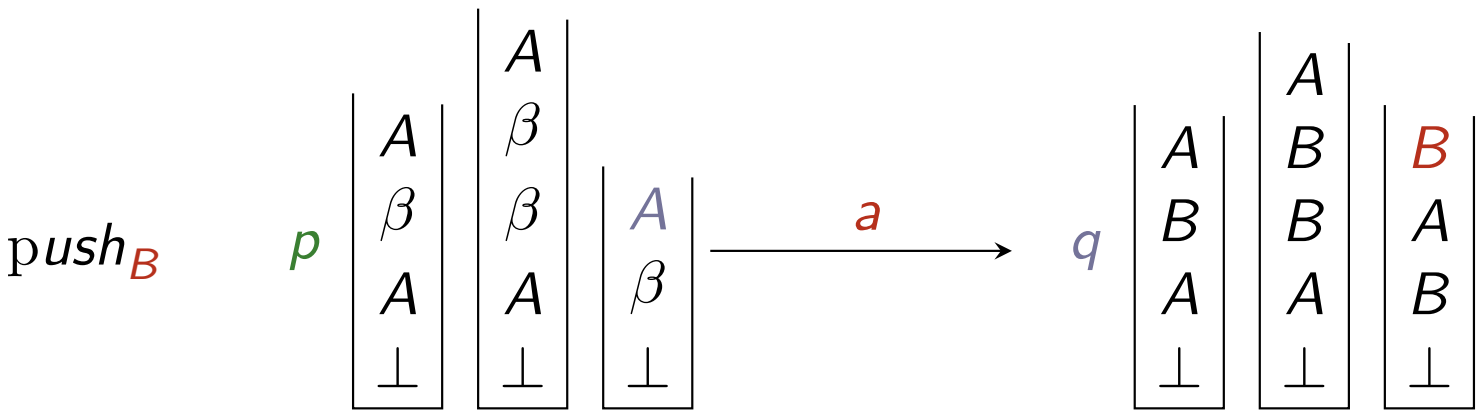

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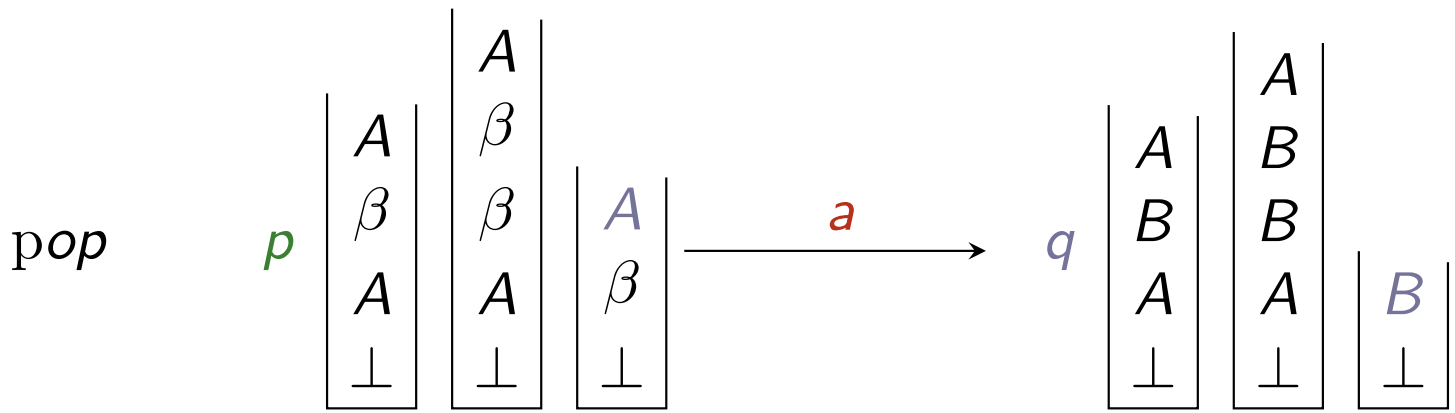

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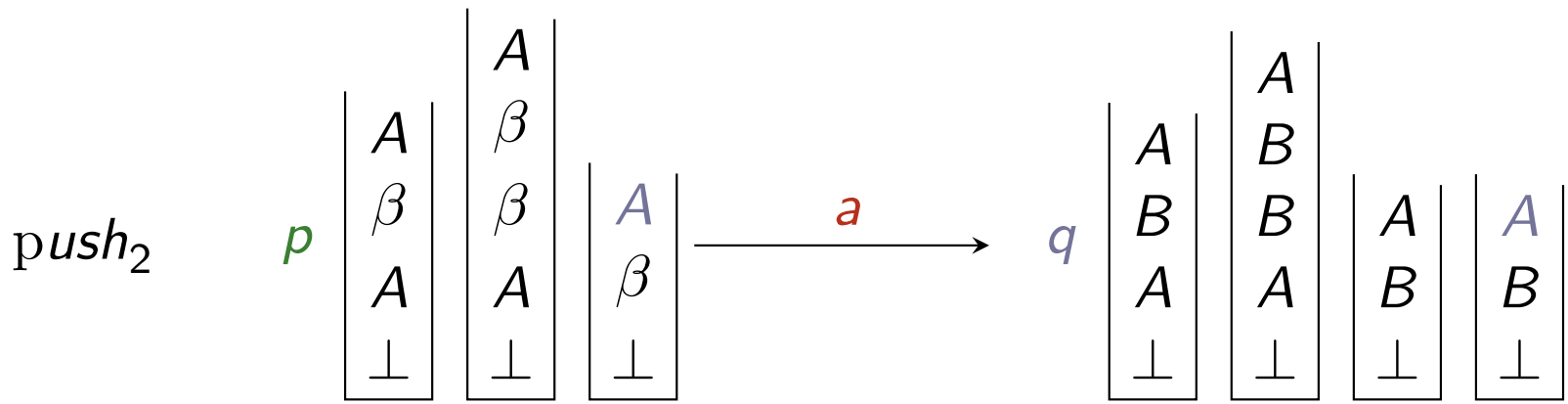

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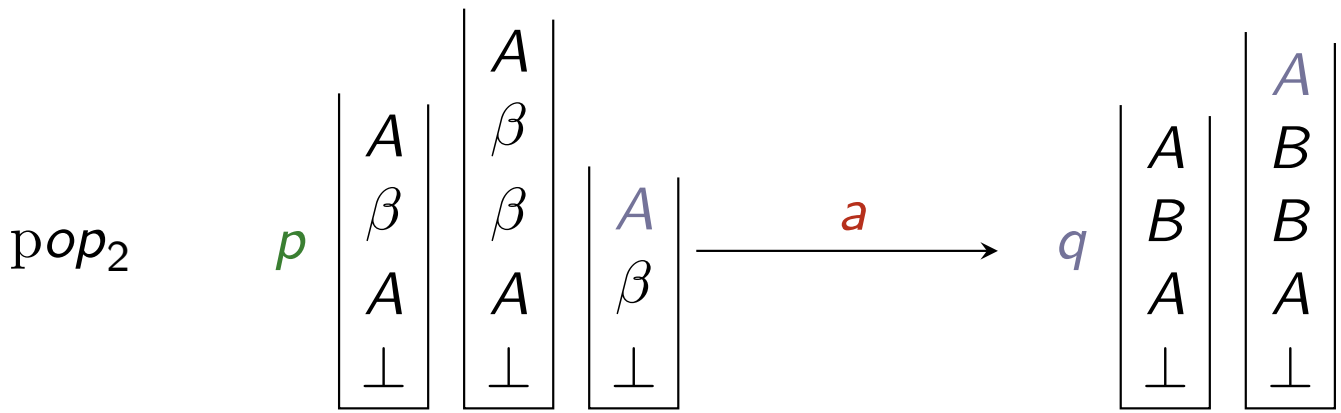
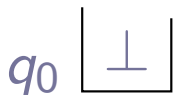


Transitions of the form: $p, A \xrightarrow{a} q, \rho$ with $\rho \in \{push_A, pop, pop_2, push_2\}$.

Example $\{a^n b^n c^n | \mid n \geq 0\}$

Accepting *aabbcc*



Example $\{a^n b^n c^n | \mid n \geq 0\}$

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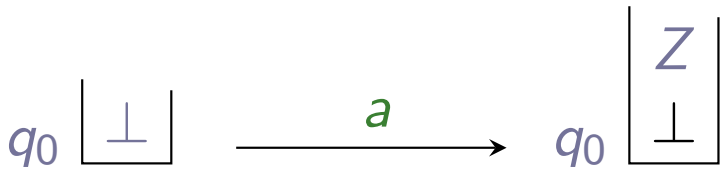

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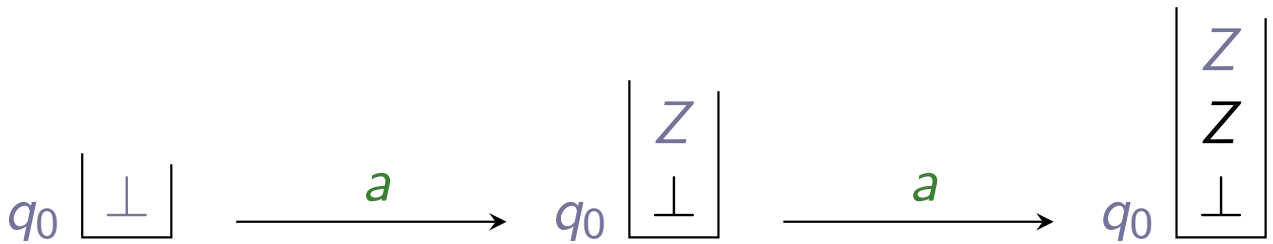
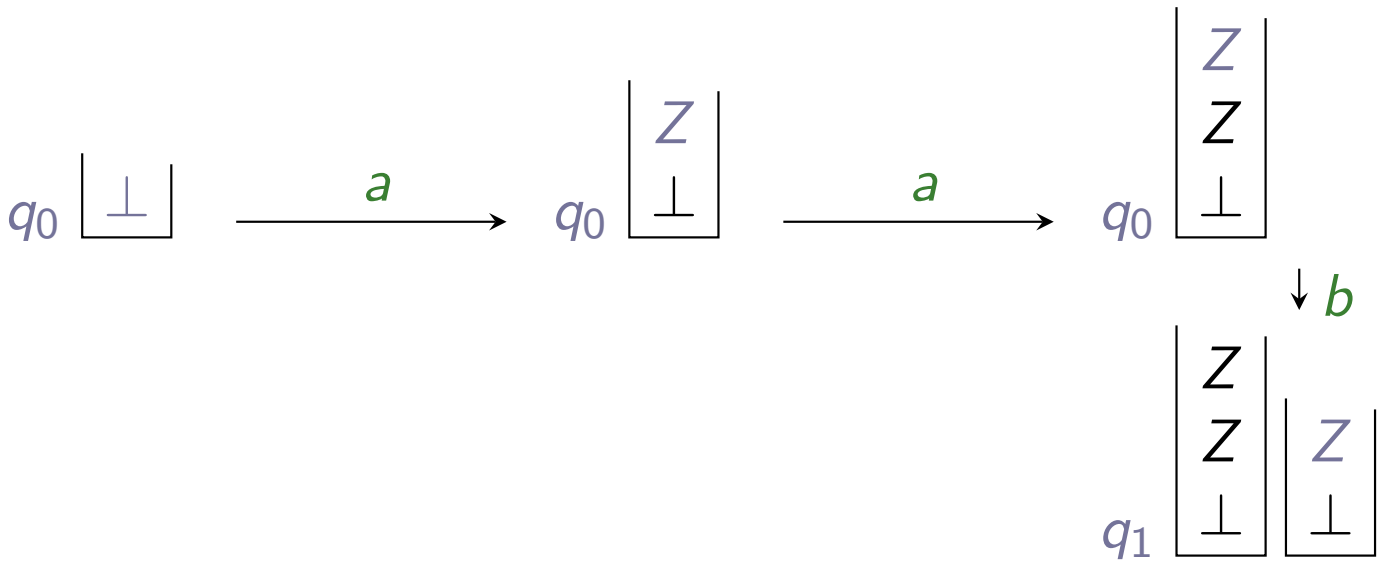


Example $\{a^n b^n c^n | \mid n \geq 0\}$

Accepting *aabbcc*



Example $\{a^n b^n c^n \mid \mid n \geq 0\}$

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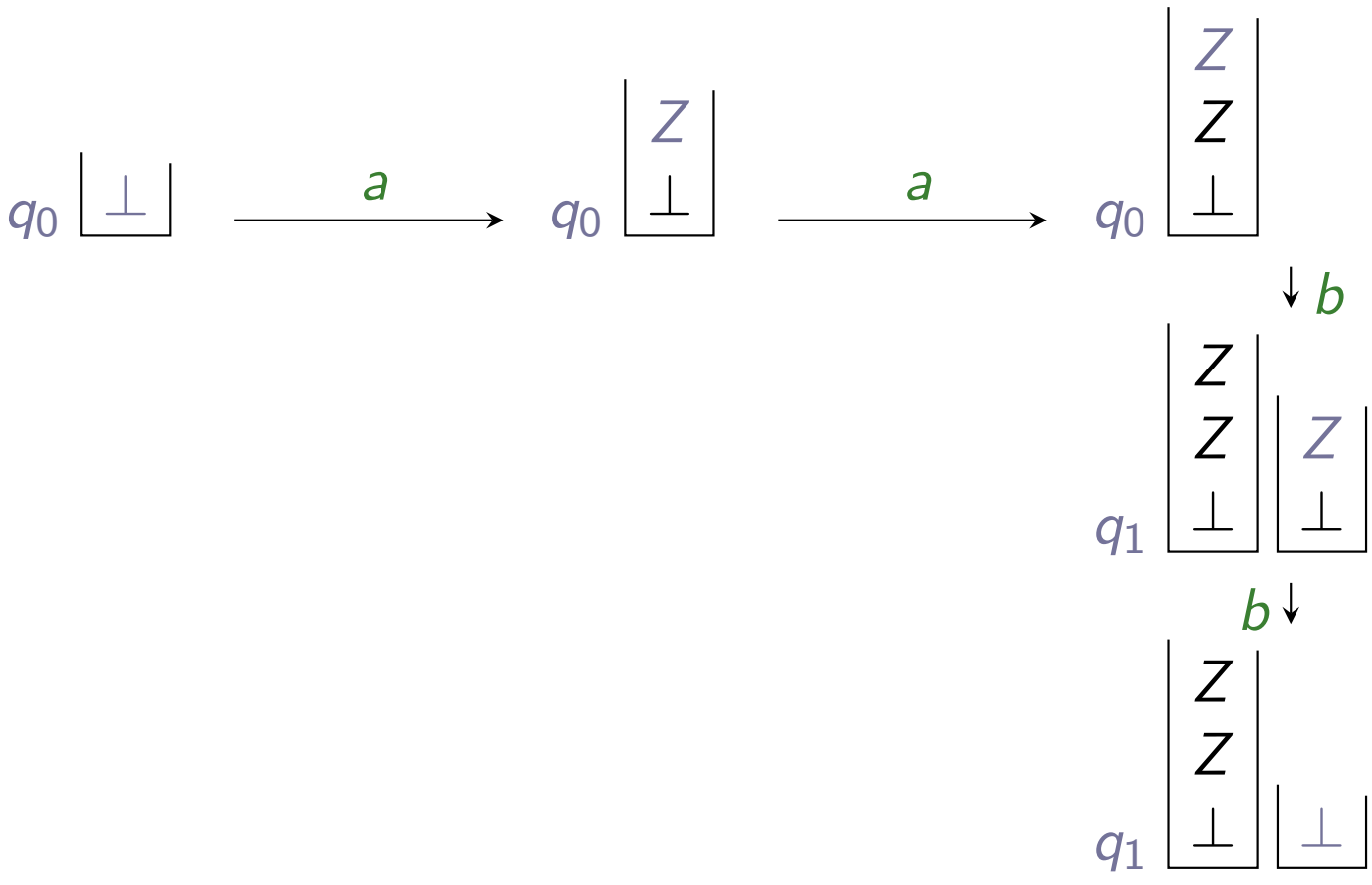

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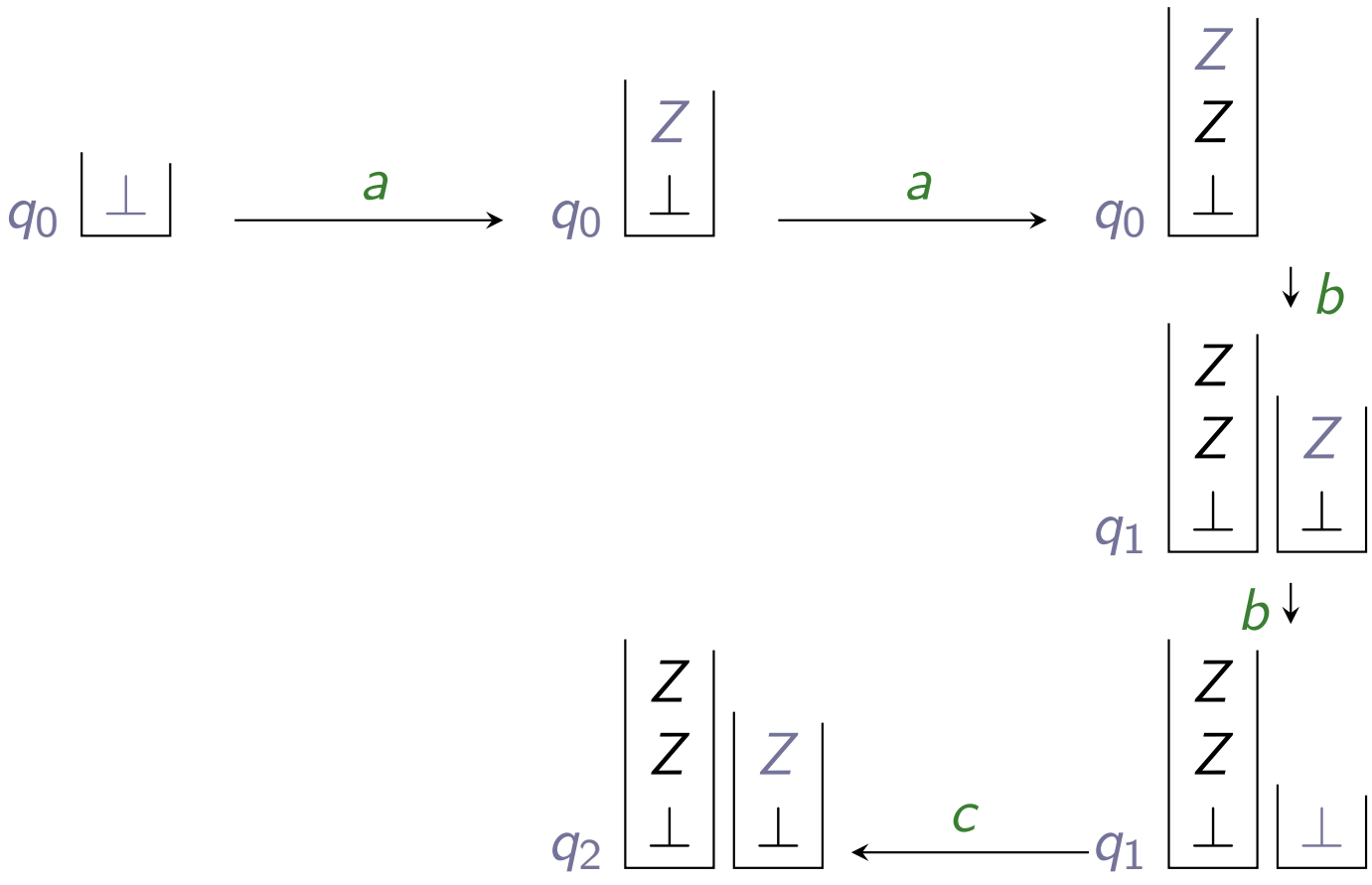

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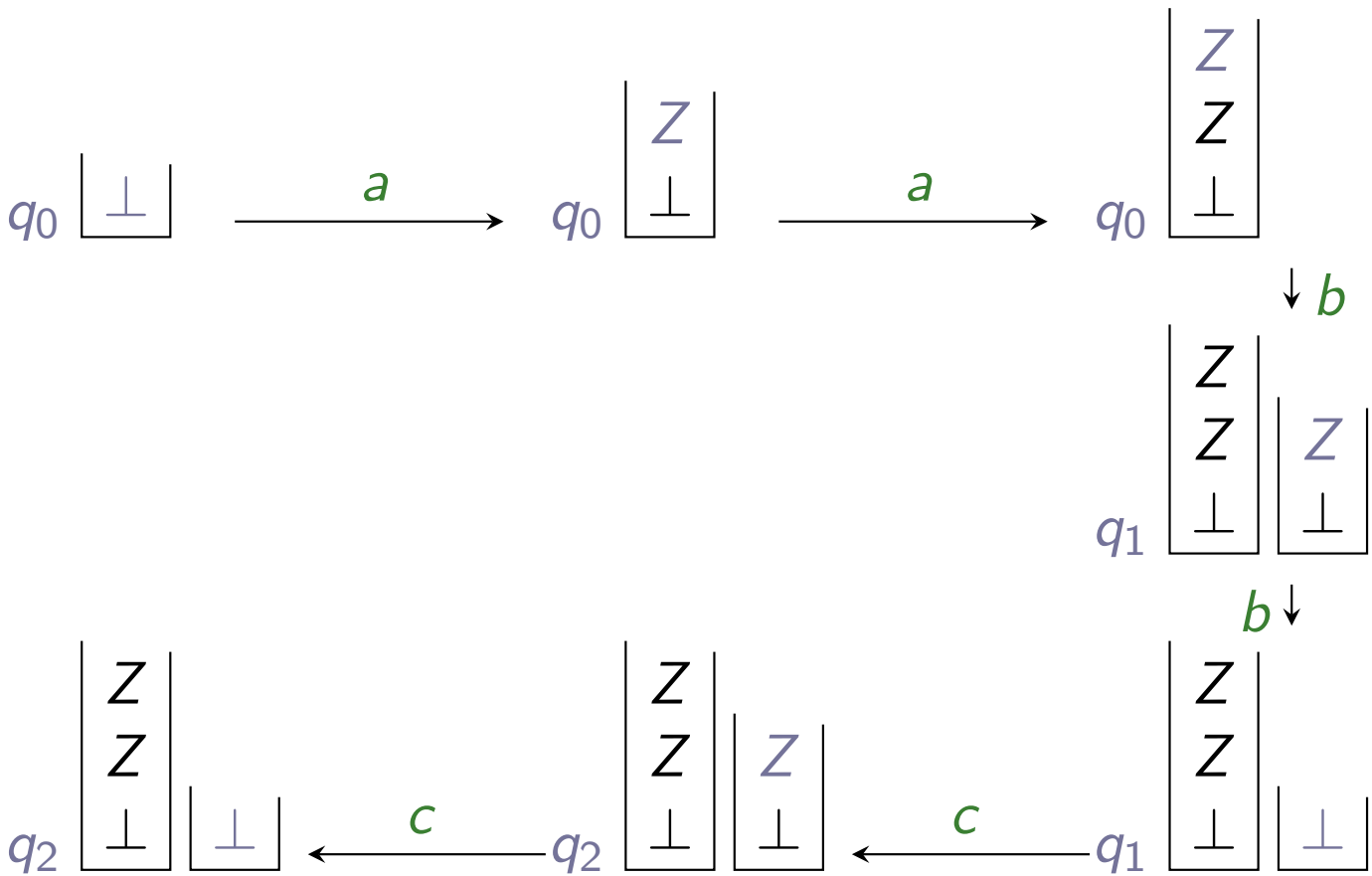

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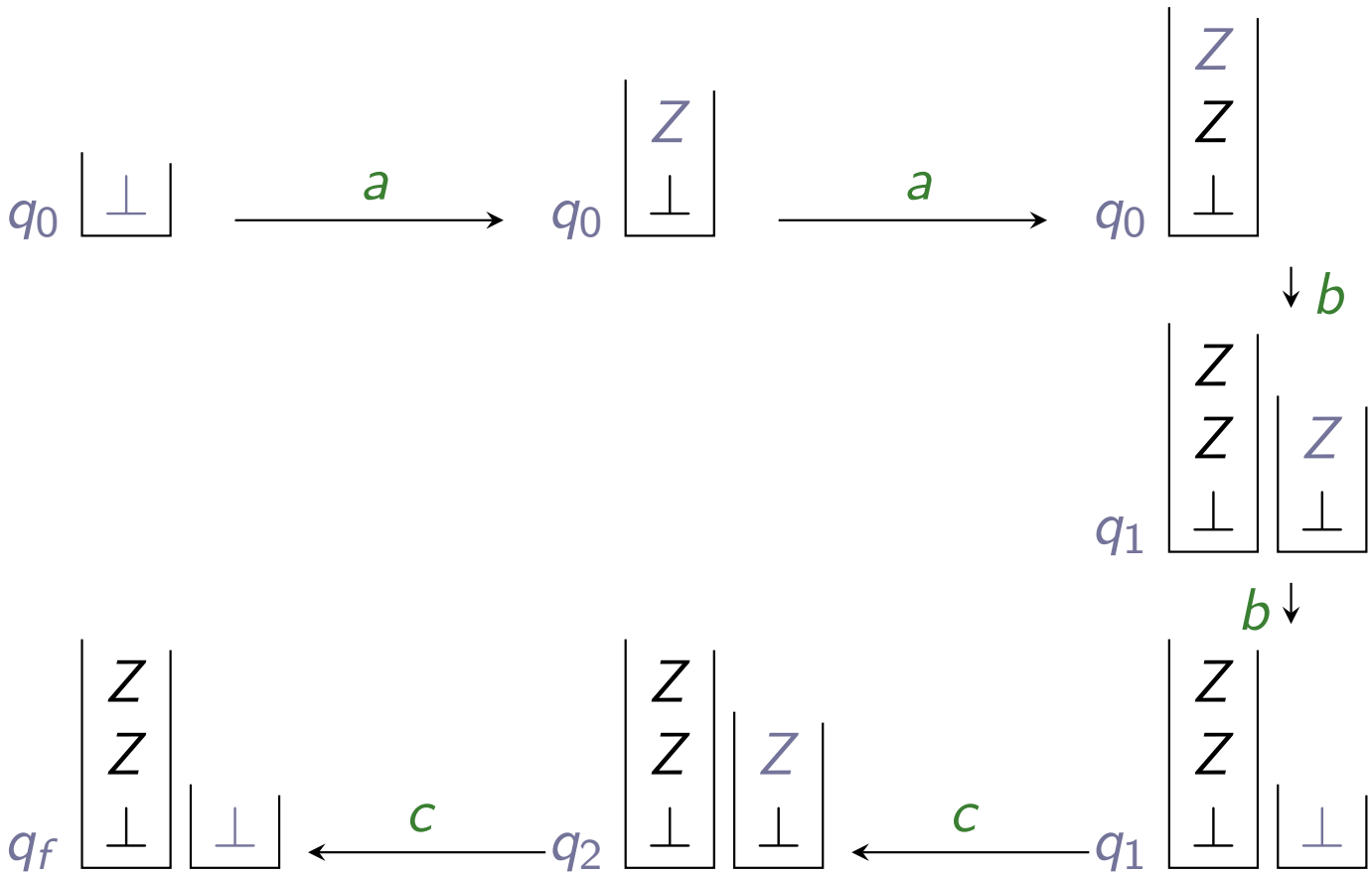
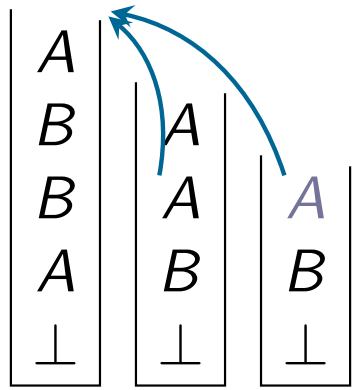


Order-2 collapsible automata

Finite control + stack of (ordinary) stacks with links



Two new operations : $\mathrm{p}ushlk_B + \mathrm{c}ollapse$

Order-2 collapsible automata

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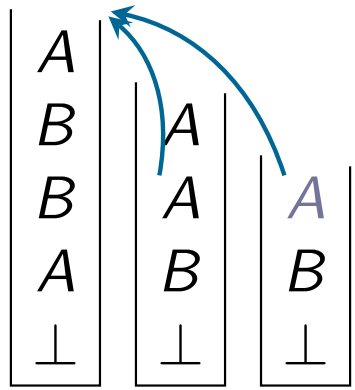

$\mathrm{p}ushlk_B$

Order-2 collapsible automata

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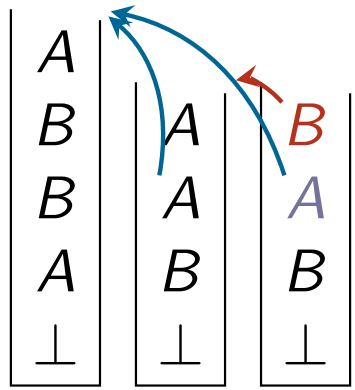
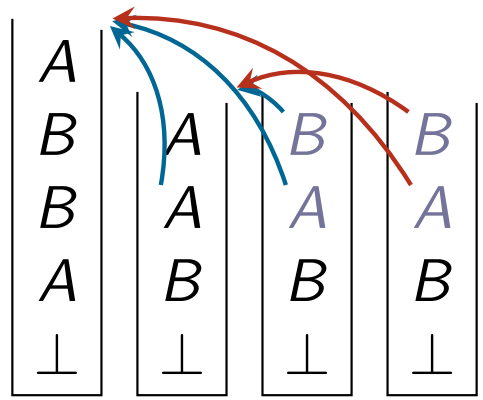


$\mathrm{p}ush_2$

Order-2 collapsible automata

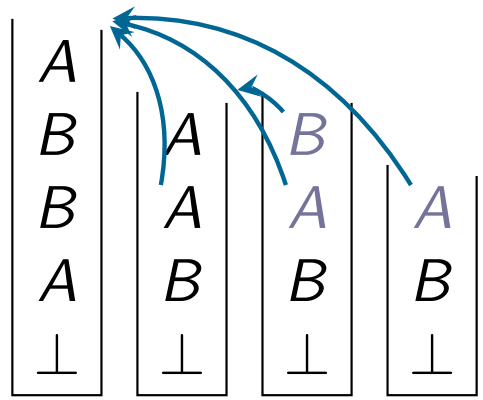
Finite control + stack of (ordinary) stacks with links



pop

Order-2 collapsible automata

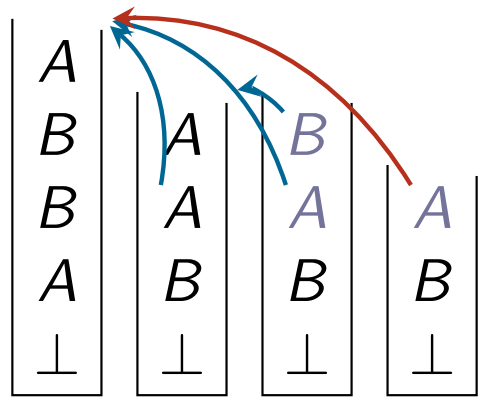
Finite control + stack of (ordinary) stacks with links



collapse

Order-2 collapsible automata

Finite control + stack of (ordinary) stacks with links



collapse

Order-2 collapsible automata

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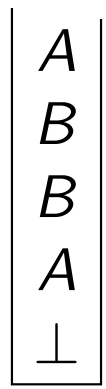

The Urzyczyn language U

Alphabet: $\{+, -, \$\}$.

To a word w over $+, -$, we associate a integer $\| w \|$ defined by:

$$\| w \| = (\text{number of } + \text{ in } w) - (\text{number of } - \text{ in } w).$$

E.g. $\| \varepsilon \| = 0$, $\| + + - - ++ \| = 2$ and $\| + - -- \| = -2$.

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A word is positive if the value of all its prefixes (except ε) is strictly positive.

	+		+		−		+		+		−	
0		1		2		1		2		3		2

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The language U is the set of all words of the form $w\$^n$ where:

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- n is the number of occurrences of $+$ in the longest prefix of w with value $\| w \| - 1$.

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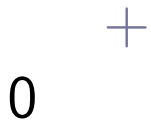
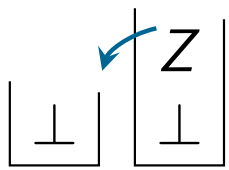
E.g. $++-++-\$\$$ belongs to U.

Order-2 collapsible aut. for U





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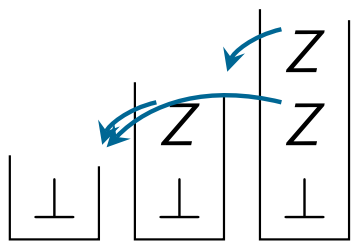

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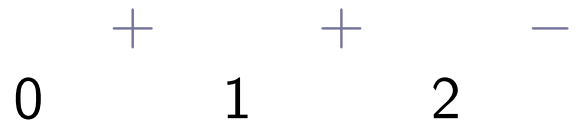

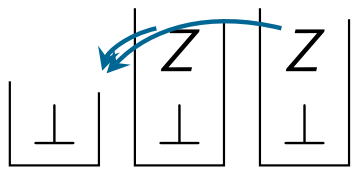

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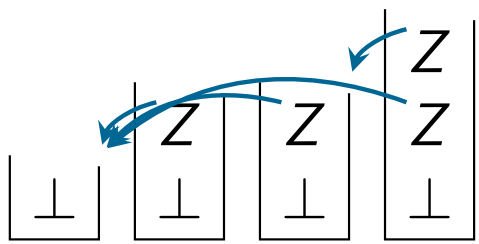

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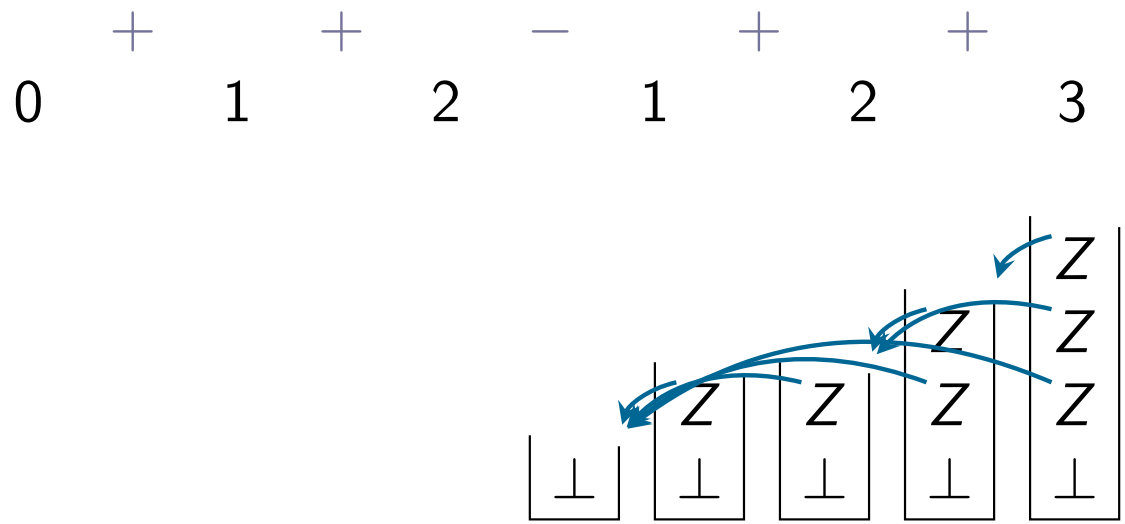

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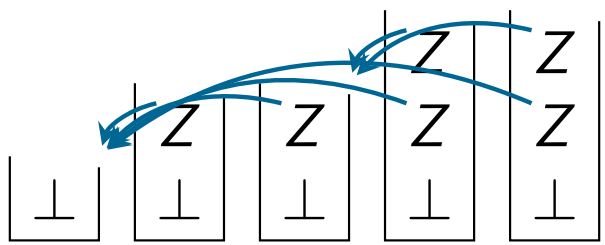

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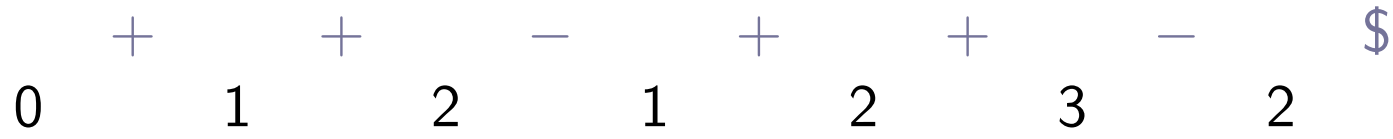

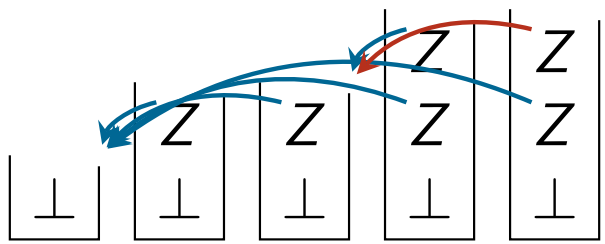

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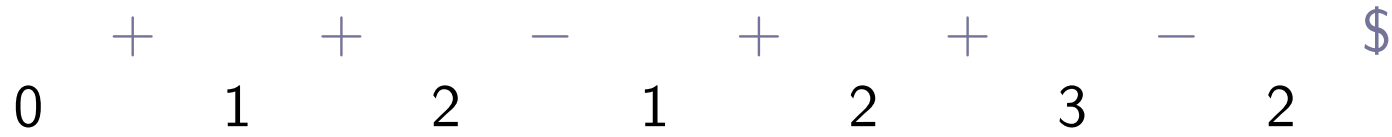

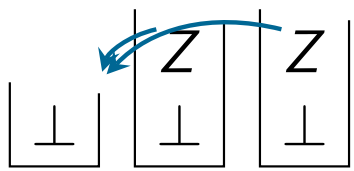

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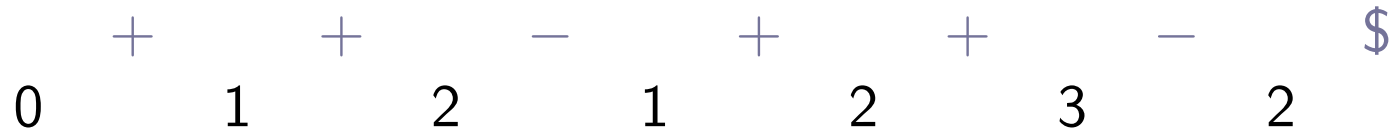

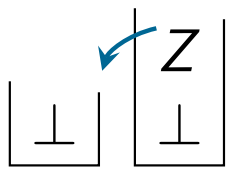

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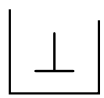

Coming back to the main theorem

Theorem

The infinite tree of a word language L is definable by an order 2 recursion scheme iff L is accepted by a deterministic collapsible automaton of order 2.

Simulating schemes by automata

$$\begin{cases} S & \xrightarrow{a} & FC \\ F\,x & \xrightarrow{a} & F\,(B\,x) \\ F\,y & \xrightarrow{b} & y \end{cases} \qquad \begin{array}{lcl} B\,z & \xrightarrow{b} & z \\ C & \xrightarrow{c} & V \end{array}$$

with $x, y, z : \mathbf{o}$.

Alphabet : $\{S, F\,C, z, V, F(B\,x), B\,x, y\}$.

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$$\boxed{S}$$

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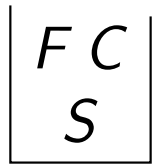

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$$\boxed{\begin{array}{c} F\,(B\,x) \\ F\,C \\ S \end{array}}$$

Simulating schemes by automata

$$\begin{cases} S & \xrightarrow{a} & F\,C & \qquad & B\,z & \xrightarrow{b} & z \\ F\,x & \xrightarrow{a} & F\,(B\,x) & & C & \xrightarrow{c} & V \\ F\,y & \xrightarrow{b} & y & & & & \end{cases}$$

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$$S \xrightarrow{a} F\,C \xrightarrow{a} F\,(B\,C) \xrightarrow{b} B\,C \xrightarrow{b} C \xrightarrow{c} V$$

$$\left|\begin{array}{c} y \\ F\,(B\,x) \\ F\,C \\ S \end{array}\right|$$

Simulating schemes by automata

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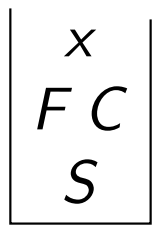

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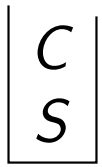

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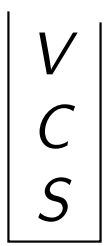

... at order 2

$$\begin{cases} S & \xrightarrow{a} & F\ B\ C \\ F\ \phi\ \psi & \xrightarrow{d} & \phi\ (\psi\ V) \\ B\ x & \xrightarrow{b} & x \\ C\ x & \xrightarrow{b} & x \end{cases}$$

with $x : \mathbf{o}$, $\phi, \psi : \mathbf{o} \to \mathbf{o}$.

... at order 2

$$\begin{cases} S & \xrightarrow{a} & F\,B\,C \\ F\,\phi\,\psi & \xrightarrow{d} & \phi\,(\psi\,V) \\ B\,x & \xrightarrow{b} & x \\ C\,x & \xrightarrow{b} & x \end{cases}$$

with $x : \mathbf{o}$, $\phi, \psi : \mathbf{o} \to \mathbf{o}$.

Alphabet : $\{S, F\,B\,C, B, C, \phi\,(\psi\,V), (\psi\,V), x\}$

$$S \xrightarrow{a} F\,B\,C \xrightarrow{a} B(CV) \xrightarrow{b} C\,V \xrightarrow{c} V$$

$\boxed{S}$

... at order 2

$$\begin{cases} S & \xrightarrow{a} & F\ B\ C \\ F\ \phi\ \psi & \xrightarrow{d} & \phi\ (\psi\ V) \\ B\ x & \xrightarrow{b} & x \\ C\ x & \xrightarrow{b} & x \end{cases}$$

with $x : \mathbf{o}$, $\phi, \psi : \mathbf{o} \to \mathbf{o}$.

Alphabet : $\{S, F\ B\ C, B, C, \phi\ (\psi\ V), (\psi\ V), x\}$

$$S \xrightarrow{a} F\ B\ C \xrightarrow{a} B(CV) \xrightarrow{b} C\ V \xrightarrow{c} V$$

$$\boxed{\begin{array}{c} F\ B\ C \\ S \end{array}}$$

... at order 2

$$\begin{cases} S & \xrightarrow{a} & F\,B\,C \\ F\,\phi\,\psi & \xrightarrow{d} & \phi\,(\psi\,V) \\ B\,x & \xrightarrow{b} & x \\ C\,x & \xrightarrow{b} & x \end{cases}$$

with $x : \mathbf{o}$, $\phi, \psi : \mathbf{o} \to \mathbf{o}$.

Alphabet : $\{S, F\,B\,C, B, C, \phi\,(\psi\,V), (\psi\,V), x\}$

$$S \xrightarrow{a} F\,B\,C \xrightarrow{a} B(CV) \xrightarrow{b} C\,V \xrightarrow{c} V$$

$$\boxed{\begin{array}{c} \phi\,(\psi\,V) \\ F\,B\,C \\ S \end{array}}$$

... at order 2

$$\begin{cases} S & \xrightarrow{a} & F\,B\,C \\ F\,\phi\,\psi & \xrightarrow{d} & \phi\,(\psi\,V) \\ B\,x & \xrightarrow{b} & x \\ C\,x & \xrightarrow{b} & x \end{cases}$$

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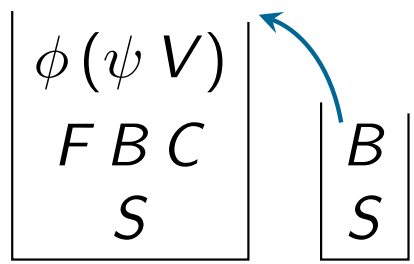
$$S \xrightarrow{a} F\,B\,C \xrightarrow{a} B(CV) \xrightarrow{b} C\,V \xrightarrow{c} V$$



... at order 2

$$\begin{cases} S & \xrightarrow{a} & F\,B\,C \\ F\,\phi\,\psi & \xrightarrow{d} & \phi\,(\psi\,V) \\ B\,x & \xrightarrow{b} & x \\ C\,x & \xrightarrow{b} & x \end{cases}$$

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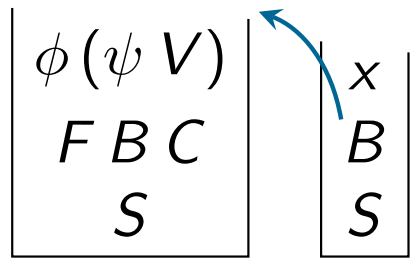

... at order 2

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Alphabet : $\{S, F\ B\ C, B, C, \phi\ (\psi\ V), (\psi\ V), x\}$

$$S \xrightarrow{a} F\ B\ C \xrightarrow{a} B(CV) \xrightarrow{b} C\ V \xrightarrow{c} V$$

$$\boxed{\begin{array}{c} \psi\ V \\ F\ B\ C \\ S \end{array}}$$

... at order 2

$$\begin{cases} S & \xrightarrow{a} & F\,B\,C \\ F\,\phi\,\psi & \xrightarrow{d} & \phi\,(\psi\,V) \\ B\,x & \xrightarrow{b} & x \\ C\,x & \xrightarrow{b} & x \end{cases}$$

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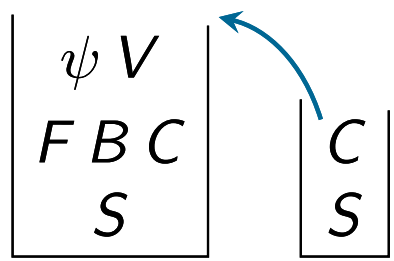
$$S \xrightarrow{a} F\,B\,C \xrightarrow{a} B(CV) \xrightarrow{b} C\,V \xrightarrow{c} V$$



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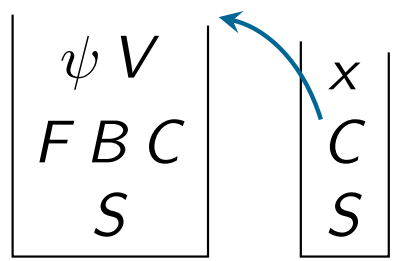
$$S \xrightarrow{a} F\ B\ C \xrightarrow{a} B(CV) \xrightarrow{b} C\ V \xrightarrow{c} V$$



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$$S \xrightarrow{a} F\ B\ C \xrightarrow{a} B(CV) \xrightarrow{b} C\ V \xrightarrow{c} V$$

$$\boxed{\begin{array}{c} V \\ F\ B\ C \\ S \end{array}}$$

Open questions

Expressivity

- What graphs, linear orders can be MSO-defined on schemes ?
- Strictness of the hierarchy
- Pumping lemma for collapsible automata
- Decidability of isomorphism

Connection with programming languages

- What ML-like language and with which evaluation policy can be "compiled" into collapsible automata
- Tools
  $\Rightarrow$ efficient algorithms for collapsible automata