”Proving” Fixed Points

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In this talk

- A new proof method for order-theoretic fixed point theorems
  $\rightarrow$ A new method to define these fixed points
- A comparison with other traditional proof methods

Fixed points theorems

- There are mainly two kinds of fixed point theorems: metric-theoretic and order-theoretic (see Mr Waszkiewicz's talk this morning).
- We are interested in fixed points of maps defined over partially ordered sets (posets).
- General form
  - Assume a poset satisfying some completeness property.
  - Assume a map satisfying some order property (order preservation, expansion).
  - Then the map has a [least, greatest] fixed point.
- The proof gives in some way a definition of the fixed point.

Example: Tarski's Theorem

Theorem (Tarski 1955)

- *Assume a complete lattice $\mathcal{E}$ (every subset has a least upper bound and a greatest lower bound).*
- *Assume an isotone map η (it preserves order).*
- *Then the map η has a least fixed point and a greatest fixed point.*

Proof.

There are two two standard proofs using two methods:

- the impredicative method, used by Tarski in his original article,
- the iterative method, resorting to ordinals.

Tarski's Theorem: Impredicative Proof Method

Proof.

$$\begin{aligned} \mathrm{lfp}\,\eta &= \wedge\{x \in \mathcal{E} \mid \eta(x) \leq x\} \\ \mathrm{gfp}\,\eta &= \vee\{x \in \mathcal{E} \mid x \leq \eta(x)\} \end{aligned}$$

- Let $S = \{x \in \mathcal{E} \mid \eta(x) \leq x\}$ (set of η-closed points).
- If $x \in S$, then $\eta(x) \in S$.
- $\wedge S \in S$.
- First conclusion: $\wedge S$ is a fixed point.
- All fixed point belongs to S.
- Second conclusion: $\wedge S$ is the least fixed point.
- Use duality for the greatest fixed point.

Tarski's Theorem: Iterative Proof Method

Proof.

$$
\begin{array}{rclcrcl}
\mathrm{lfp}\,\eta & = & \bigvee_\alpha \Delta_\alpha(\eta) & \qquad & \Delta_\alpha(\eta) & = & \eta(\bigvee_{\beta|\beta<\alpha} \Delta_\beta(\eta)) \\
\mathrm{gfp}\,\eta & = & \bigwedge_\alpha \nabla_\alpha(\eta) & \qquad & \nabla_\alpha(\eta) & = & \eta(\bigwedge_{\beta|\beta<\alpha} \nabla_\beta(\eta))
\end{array}
$$

- For all α, $\Delta_\alpha(\eta) \leq \eta(\Delta_\alpha(\eta))$ and $(\Delta_\beta(\eta))_{\beta<\alpha}$ is increasing.
- By Hartogs' lemma: the sequence becomes stationary.
- First conclusion: the limit of the sequence is a fixed point.
- All fixed point is an upper bound of the sequence $(\Delta_\alpha(\eta))_\alpha$.
- Second conclusion: the limit is the least fixed point.
- Use duality for the greatest fixed point.

□

Fixed points: What Kind of Definition?

- The impredicative method is not constructive.
  $\rightarrow$ Specifying a fixed point
- The iterative method is more constructive.
  $\rightarrow$ Iteratively computing an approximation until the limit is reached
- But the iterative method resorts to ordinals, therefore to infinities, possibly in a non-constructive way.
  $\rightarrow$ Heavy machinery (arguably)
- Problem: there is no clear connection between the impredicative method and the iterative method.
  It seems to be an old question in Mathematics.

Contribution: Alternative Method to Prove Fixed Point Theorems

- Equivalently: alternative method to define fixed points
- The deductive method: not impredicative, ordinal-free but still constructive
- The fixed point is inductively proved in an inference system.
  $\rightarrow$ Proving the fixed point
- It is a first step allowing the impredicative method and the iterative method to be connected.

Outline

- Induction (and coinduction) for inference systems: reminder and presentation of the deductive method
- The deductive method generalized
  - Tarski Theorem revisited – The main idea for generalization
  - Application to other fixed point theorems
    Extension to chain-complete posets
    Bourbaki-Witt's theorem
- Comparison of the methods
- Future work
  - Connection between the methods
  - Coq implementation

Plan

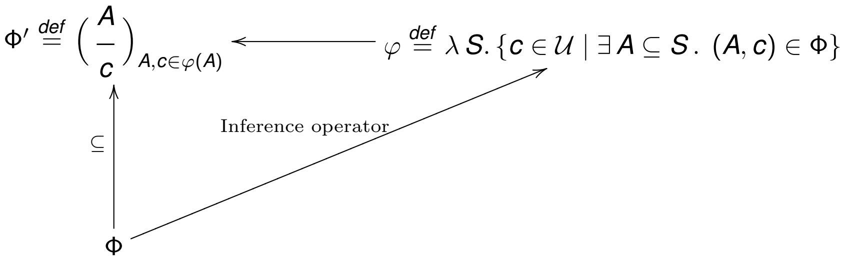
Inference Systems

- A set $\mathcal{U}$ of judgments: the universe
- Inference system over $\mathcal{U}$: a set of deduction rules
- A deduction rule: an ordered pair (A, c), with premises $A \subseteq \mathcal{U}$ and conclusion $c \in \mathcal{U}$

$$\frac{A}{c}$$

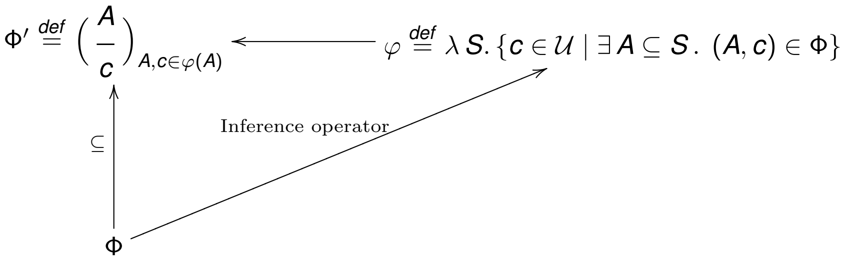
$\rightarrow$ From premises A, deduce conclusion c.

First Interpretation: Fixed Point Approach



- Canonical Galois connection (and even reflection) between inference systems Φ over $\mathcal{U}$ (ordered by inclusion) and isotone operators $\varphi : 2^{\mathcal{U}} \to 2^{\mathcal{U}}$ over $\mathcal{U}$ (ordered point-wise by inclusion)

First Interpretation: Fixed Point Approach



- Application of Tarski's theorem to the powerset $2^{\mathcal{U}}$, a complete lattice, and the inference operator φ, an isotone map
  $\rightarrow$ A least and a greatest fixed point
  Standard approach? Yes.

Second Interpretation: Deductive Method

- Central notion: proofs in an inference system

Rule $\dfrac{\dots \quad a \quad \dots}{c}$ Proof $\dfrac{\dots \quad (\text{proof of } a) \quad \dots}{c}$

- Two interpretations: inductive and coinductive
  - Inductive interpretation: the set $\Delta(\Phi)$ of the conclusions of the well-founded proofs in Φ
  - Coinductive interpretation: the set $\nabla(\Phi)$ of the conclusions of all the proofs in Φ, ill-founded or well-founded

Standard approach? No.

Equivalence theorem

Theorem

$$\operatorname{lfp}\varphi = \Delta(\Phi) \quad \textit{and} \quad \operatorname{gfp}\varphi = \nabla(\Phi).$$

$\rightarrow$ Equivalence between the standard approach using fixed points and the non-standard one using proofs

Equivalence theorem

Proof.

Application of the following reasoning principles

Method	Induction	Coinduction
Impredicative	$\dfrac{\varphi(S) \subseteq S}{\mathrm{lfp}\,\varphi \subseteq S}$	$\dfrac{S \subseteq \varphi(S)}{S \subseteq \mathrm{gfp}\,\varphi}$
Deductive	Well-foundation for proofs	Proof construction by guarded recursive equations

Equivalence theorem

- Inductive case: well-known (Aczel 1977)
- Coinductive case: folklore (Grall 2003, Leroy-Grall 2009)

Plan

Induction (and Coinduction) for Inference Systems

Generalization of the deductive method
- Tarski's theorem revisited
- Applications to other fixed point theorems

Comparison

Future Work

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Main idea

- Assumption: an isotone map η over a complete lattice $(\mathcal{E}, \leq)$
- Question: How to define an inference system Φ over $\mathcal{E}$, or equivalently an inference operator $\varphi : 2^{\mathcal{E}} \to 2^{\mathcal{E}}$, whose inductive and coinductive interpretations produce the least and greatest fixed points of η?

Main idea

- First attempt: $\varphi(S) \stackrel{\textit{def}}{=} \eta(S)$
  $\rightarrow$ Trivially fails: $\operatorname{lfp} \varphi = \emptyset$.

Main idea

- Second attempt: Embedding of the complete lattice in its powerset via a closure operator

$$
\begin{array}{ccccc}
\mathcal{E} & \xrightarrow{\iota} & \{\leq x \mid x \in \mathcal{E}\} & \xrightarrow{\delta} & 2^{\mathcal{E}} \\
\uparrow \eta & & & & \uparrow \varphi \\
\mathcal{E} & \xleftarrow[\iota^{-1}]{} & \{\leq x \mid x \in \mathcal{E}\} & \xleftarrow[\gamma]{} & 2^{\mathcal{E}}
\end{array}
$$

- $\iota(x) \stackrel{def}{=} \leq x$: canonical isomorphism
- $\gamma(S) \stackrel{def}{=} \leq (\vee S)$: closure operator with adjoint embedding δ

$\rightarrow \varphi(S) = \leq \eta(\vee S)$

New Statement

Theorem (Tarski revisited)

- *Assume a complete lattice $\mathcal{E}$.*
- *Assume an isotone map η.*
- *Define an inference system Φ over $\mathcal{E}$ with the following rules, and only these rules:*

$$\frac{S}{c} \quad \left(S \subseteq \mathcal{E}, c \leq \eta(\vee S)\right).$$

- *Then the map η has a least fixed point and a greatest fixed point satisfying:*

$$\leq (\mathrm{lfp}\,\eta) \;=\; \Delta(\Phi) \quad and \quad \leq (\mathrm{gfp}\,\eta) \;=\; \nabla(\Phi).$$

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Variations

- Chain-complete posets instead of complete lattices
  $\rightarrow$ Chains (included the empty one) are assumed to have a least upper bound.
- Isotony or expansion for the map
- The inference system is restricted.
  - Premises: chain
    Indispensable assumption
  - Conclusion: greatest possible conclusion
    Assumption only needed for Bourbaki-Witt's Theorem

Two Extensions

Theorem (Extension to Chain-Complete Posets)

- *Assume a chain-complete poset $\mathcal{E}$.*
- *Assume an isotone map η.*
- *Define an inference system Φ over $\mathcal{E}$ with the following rules, and only these rules:*

$$\frac{C}{\eta(\vee C)} \quad (C \subseteq \mathcal{E}, C \ \mathit{chain}).$$

- *Then η has a least fixed point $\operatorname{lfp}\eta$ satisfying:*

$$\leq (\operatorname{lfp}\eta) \quad = \quad \leq \Delta(\Phi).$$

Two Extensions

Proof.

- $\Delta(\Phi)$ is a chain. By induction over well-founded proofs.
- Conclusion follows.

□

Two Extensions

Theorem (Bourbaki-Witt's Theorem)

- *Assume a chain-complete poset $\mathcal{E}$.*
- *Assume an expansive map η: any point is η-consistent $(\forall x \in \mathcal{E} .\ x \leq \eta(x))$.*
- *Define an inference system Φ over $\mathcal{E}$ with the following rules, and only these rules:*

$$\frac{C}{\eta(\vee C)} \quad (C \subseteq \mathcal{E}, C \ chain).$$

- *Then η has a fixed point $\mathrm{fp}\,\eta$ satisfying:*

$$\leq (\mathrm{fp}\,\eta) \ = \ \leq \Delta(\Phi).$$

Two Extensions

Proof.

- $\Delta(\Phi)$ is a chain. Intricate proof by induction over well-founded proofs.
- Conclusion follows.

□

Plan

Deductive Method vs Impredicative Method

Example: Bourbaki-Witt's Theorem

- The fixed point is defined from the inductive set generated by an inference system Φ.
- The inductive set $\Delta(\Phi)$ is also the intersection of the φ-closed sets, where φ is the inference operator associated to Φ.

$$\Delta(\Phi) = \bigcap \left\{ S \mid \forall \,\mathrm{chain}\; C \subseteq S \,.\; \eta(\vee C) \in S \right\}$$

Deductive Method vs Impredicative Method

Example: Bourbaki-Witt's Theorem

- In the original proof given by Bourbaki, the fixed point is defined from a set equal to the intersection I of all admissible subsets.

$$I = \bigcap \left\{ S \mid \begin{array}{lc} & (\bot \in S) \\ \wedge & (\eta(S) \subseteq S) \\ \wedge & (\forall \text{chain } C \subseteq S.\ \vee C \in S) \end{array} \right\}$$

- It turns out that an admissible subset is also φ-closed. $\rightarrow$ Very close notions (definition and properties)

Deductive Method vs Iterative Method

Tarski's theorem

- Transfinite sequence $(\Delta_\alpha(\eta))_\alpha$ of iterates
- Inference system Φ containing the following rules, and only these rules:

$$\frac{S}{s} \quad (S \subseteq \mathcal{E}, s \leq \eta(\vee S))$$

- Characterizations of the least fixed point:

$$\leq (\mathrm{lfp}\,\eta) = \Delta(\Phi) \qquad \mathrm{lfp}\,\eta = \bigvee_\alpha \Delta_\alpha(\eta)$$

Deductive Method vs Iterative Method

Tarski's theorem

- $\Delta_\alpha(\Phi)$: set of all *x* that are conclusion of a proof with height less or equal to α
- Comparison

$$\leq \Delta_\alpha(\eta) = \Delta_\alpha(\Phi).$$

Deductive Method vs Iterative Method

Two other theorems

- Transfinite sequence $(\Delta_\alpha(\eta))_\alpha$ of iterates
- Inference system Φ containing the following rules, and only these rules:

$$\frac{C}{\eta(\bigvee C)} \quad \big(C \subseteq \mathcal{E}, C \text{ chain}\big).$$

- Characterizations of the (least) fixed point:

$$\leq (\mathrm{fp}\,\eta) = \leq \Delta(\Phi) \qquad \mathrm{fp}\,\eta = \bigvee_\alpha \Delta_\alpha(\eta)$$

Deductive Method vs Iterative Method

Two other theorems

- $\Delta_\alpha(\Phi)$: set of all *x* that are conclusion of a proof with height equal to α
- Comparison

$$\{\Delta_\alpha(\eta)\} = \Delta_\alpha(\Phi)$$

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Induction (and Coinduction) for Inference Systems

Generalization of the deductive method
- Tarski's theorem revisited
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Comparison

Future Work

Contribution: the Deductive Method

- A new method: proving fixed points
- An alternative to the impredicative method and the iterative method
  - Specifying fixed points
  - Computing fixed points
- A sketch of a comparison
- Future? Two issues

First Issue: Connection between the Three Methods

- Thesis: the deductive method is central.
- Via the inference operator: connection to the impredicative method
- Via the well-order canonically associated to the well-foundation of proofs: connection to the iterative method

Second Issue: Implementation in Coq

- Coq: a proof assistant using a calculus of inductive and coinductive constructions, an extension of type theory
- The deductive method seems to be the best solution.
  - Iterative method: ordinals are needed.
    $\rightarrow$ Expensive (set theory) or restrictive (constructive ordinals)
  - Impredicative method: no direct support contrary to the deductive method
  - Deductive method: direct support, reasoning principles available (induction over well-founded proofs)
- Main issue: possibility to extract a program computing the fixed point from the proof that the fixed point satisfies its specification, following the Curry-Howard correspondence
  $\rightarrow$ Problem: classical logic is needed.