Common patterns for order and metric fixed point theorems

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There are plenty of reasons why we can forget the distinction between order and metric fixpoint theorems.

(The usual suspects: A. Einstein or M. Twain)
Order vs. metric fixpoints

(Knaster-Tarski) An order-preserving map on a complete lattice has the least and the greatest fixed point.

(Banach) A contraction on a complete metric space has a unique fixed point.
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OUR GOAL: Show that both are instances of a single theorem with a constructive proof.
Order vs. metric fixpoints

(Knaster-Tarski) An order-preserving map $f : X \to X$ on a complete lattice has the least and the greatest fixed point.

Proof idea: Iterate $f$:

$$\bot, f(\bot), f^2(\bot), f^3(\bot), \ldots$$

and eventually you will reach the least fixed point. Flip the lattice to get the greatest one.
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*(Knaster-Tarski)* An order-preserving map \( f : X \rightarrow X \) on a complete lattice has the least and the greatest fixed point.

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and eventually you will reach the least fixed point. Flip the lattice to get the greatest one.

*(Banach)* A contraction \( f : X \rightarrow X \) on a complete metric space has a unique fixed point.

Proof idea: Iterate \( f \):

\[ x, f(x), f^2(x), f^3(x), \ldots \]

and no matter what \( x \in X \) you started with, eventually you will reach the same fixed point.
(Lawvere 1973) Orders and metric spaces are instances of quantale-enriched categories.

(Edalat & Heckmann 1998) A topology of a complete metric space is homeomorphic to a subspace Scott topology on maximal elements of a continuous directed-complete partial order.
Unification a la Lawvere

A bit of cleaning first!

A metric on a set $X$:

$$d_X : X \times X \to [0, \infty)$$

We use it as:

$$d_X(x, y), d_X(y, z), \ldots$$
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SYMMETRY IS NOT TOO IMPORTANT!

A metric on a set \( X \):

\[
X : X \times X \to [0, \infty]
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\[
X(x, y) = 0 \text{ iff } x = y
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X(x, y) = X(y, x)
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X(x, y) \leq X(x, z) + X(z, y)
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*good bye!*

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A generalized metric on a set $X$:

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**DEFINITION**

$x \leq y$ iff $X(x, y) = 0$. 
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**CONCLUSION:** \( \leq_X \) is a partial order.
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**BETTER CONCLUSION:**

Replace \([0, \infty]\) by \(\{0, \infty\}\) to switch from metrics to orders.
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Unification a la Lawvere: the setup

Let \( \mathcal{Q} \) be a complete lattice with \(+\) and 0.

A \( \mathcal{Q} \)-category is a set \( X \) with a structure \( X : X \times X \to \mathcal{Q} \) satisfying:

\[
X(x, y) = X(y, x) = 0 \text{ implies } x = y, \\
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Unification a la Lawvere: the setup

Let $Q$ be a complete lattice with $+$ and 0.

A $Q$-category is a set $X$ with a structure $X : X \times X \to Q$ satisfying:

$X(x, y) = X(y, x) = 0$ implies $x = y$,
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For $Q = 2$ we recover partial orders.
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For $Q = 2$ we recover partial orders.
For $Q = [0, \infty]$ we recover metric spaces.
Unification a la Lawvere: the setup

Let $\mathcal{Q}$ be a complete lattice with $+$ and $0$.

A $\mathcal{Q}$-category is a set $X$ with a structure $X : X \times X \to \mathcal{Q}$ satisfying:

\[
\begin{align*}
X(x, y) = X(y, x) &= 0 \text{ implies } x = y, \\
X(x, x) &= 0, \\
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\end{align*}
\]

For $\mathcal{Q} = 2$ we recover partial orders.
For $\mathcal{Q} = [0, \infty]$ we recover metric spaces.
But other choices of $\mathcal{Q}$ are possible too.
More on the setup

A \textit{Q-functor} between \textit{Q}-categories is a function $f : X \to Y$ satisfying:

$$Y(fx, fy) \leq X(x, y).$$
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A **Q-functor** between $Q$-categories is a function $f : X \rightarrow Y$ satisfying:

$$Y(fx, fy) \preceq X(x, y).$$

2-functors are order-preserving maps. $[0, \infty]$-functors are non-expansive maps between metric spaces.
More on the setup

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2-functors are order-preserving maps.

$[0, \infty]$-functors are non-expansive maps between metric spaces.

$Q$-functors of type $X \to Y$ form a $Q$-category when considered with the structure:

$$Y^X(f, g) := \sup_{x \in X} Y(fx, gx).$$
More on the setup

Consider a sequence \((x_n)_{n \in \omega}\) such that

*from some \(N\) onwards, elements of the sequence are arbitrarily close to each other.*
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For \(Q = 2\), \((x_n)_{n \in \omega}\) is eventually a chain.
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For $Q = [0, \infty]$, $(x_n)_{n \in \omega}$ is a Cauchy sequence.
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More on the setup

Consider a net $(x_i)_{i \in I}$ such that

from some $N$ onwards, elements of the net are arbitrarily close to each other.

For $Q = 2$, $(x_i)_{i \in I}$ is eventually a directed set.
For $Q = [0, \infty]$, $(x_i)_{i \in I}$ is a Cauchy net.
More on the setup

We encode Cauchy nets/directed sets as maps of type $X^{op} \rightarrow \mathcal{Q}$. 
More on the setup

We **encode** Cauchy nets/directed sets as maps of type $X^{op} \to \mathcal{Q}$.

**DEFINITION:** An *ideal* on $X$ is a map:

$$\phi(z) := \inf_{i \in I} \sup_{k \geq i} X(z, x_k)$$

for some Cauchy net $(x_i)_{i \in I}$. 
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**FACT:** Ideals are $Q$-functors from $X^{op}$ to $Q$. Hence

$$\mathbb{I}X \hookrightarrow \hat{X}, \quad \text{where} \quad \hat{X} := Q^{X^{op}}.$$
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**FACT:** Ideals on $X$ form a $Q$-category:

$$
\mathbb{I}X(\phi, \psi) := \sup_{x \in X} (\psi x - \phi x).
$$
**Definition:** A \( \mathcal{Q} \)-category \( X \) is \( \mathbb{I} \)-complete if there exists a map \( S : \mathbb{I}X \to X \) with

\[
X(S\phi, x) = \mathbb{I}X(\phi, X(-, x))
\]

for all \( \phi \in \mathbb{I}X \) and \( x \in X \).
**DEFINITION:** A $Q$-category $X$ is $\mathbb{I}$-complete if there exists a map $S: \mathbb{I}X \to X$ with

$$X(S\phi, x) = \mathbb{I}X(\phi, X(\cdot, x))$$

for all $\phi \in \mathbb{I}X$ and $x \in X$.

**IMPORTANT:**
Replacing $\mathbb{I}$ by $\overset{\wedge}{(\cdot)}$ we have a notion of $\overset{\wedge}{(\cdot)}$-completeness.
Replacing $\mathbb{I}$ by any suitable $J$ we have a notion of $J$-completeness.
What we gained

\(\mathbb{II}\)-complete 2-categories are directed-complete posets.
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\(\mathbb{I}\)-complete 2-categories are directed-complete posets.
\(\check{\cdot}\)-complete 2-categories are complete lattices.
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\(\mathbb{II}\)-complete 2-categories are directed-complete posets.
\(\mathcal{C}\)-complete 2-categories are complete lattices.
\(\mathbb{II}\)-complete symmetric \([0, \infty]\)-categories are complete metric spaces.
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\( \mathcal{I} \)-complete 2-categories are directed-complete posets.
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Still we have other choices of \( J \) and \( Q \)!
Fixpoints again

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OUR GOAL: Show that both are instances of a single theorem with a constructive proof.
(Knaster-Tarski) A 2-functor on a $(\cdot \cdot)$-complete 2-category has the least and the greatest fixed point.

(Banach) A contraction on a $\mathbb{I}$-complete $[0, \infty]$-category has a unique fixed point.
Fixpoints again

(Knaster-Tarski) A 2-functor on a \((\cdot)\)-complete 2-category has the least and the greatest fixed point.

(Banach) A contraction on a \(\mathbb{I}\)-complete \([0, \infty]\)-category has a unique fixed point.

**BOTH FOLLOW FROM:** A \(Q\)-functor \(f : X \to X\) on a \(J\)-complete \(Q\)-category has a fixed point, providing the direct image \(Q\)-functor

\[
f^* : JX \to JX
\]

\[
f^*(\phi) := \inf_{z \in X} (\phi(z) + X(-, fz))
\]

has a fixed point.
Proof idea

**THEOREM** A $Q$-functor $f : X \to X$ on a $J$-complete $Q$-category has a fixed point, providing that $f^*: JX \to JX$ has a fixed point $\phi$. 
Proof idea

**THEOREM** A \(Q\)-functor \(f : X \to X\) on a \(J\)-complete \(Q\)-category has a fixed point, providing that \(f^* : JX \to JX\) has a fixed point \(\phi\).

Proof:
1. \(X\) is \(J\)-complete implies \((X, \leq_X)\) is a dcpo.
Proof idea

**THEOREM** A $Q$-functor $f : X \rightarrow X$ on a $J$-complete $Q$-category has a fixed point, providing that $f^* : JX \rightarrow JX$ has a fixed point $\phi$.

Proof:

1. $X$ is $J$-complete implies $(X, \leq_X)$ is a dcpo.
2. $f$ is a $Q$-functor implies $f$ is $\leq_X$-preserving.
Proof idea

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3. $f^*$ has a fixpoint $\phi$, implies $S\phi = Sf^*(\phi) \leq_X f(S\phi)$. 
Proof idea

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Proof:
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2. $f$ is a $Q$-functor implies $f$ is $\leq_X$-preserving.
3. $f^*$ has a fixpoint $\phi$, implies $S\phi = Sf^*(\phi) \leq_X f(S\phi)$.
4. Then we use Patarea’s proof of the fact that an order-preserving map on a dcpo has a least fixed point. QED.
How to obtain classic fixed point theorems

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How to obtain classic fixed point theorems

THEOREM A $Q$-functor $f : X \to X$ on a $J$-complete $Q$-category has a fixed point, providing that $f^* : JX \to JX$ has a fixed point $\phi$.

- (Banach) Take $J = \mathbb{I}$ and $\phi = \inf_n \sup_{m \geq n} X(-, f^m x_0)$. Any choice of $x_0$ gives the same $\phi$, hence the fixed point is unique.
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- **(Knaster-Tarski)** take $J = \hat{X}$. 
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- **(Banach)** Take $J = \mathbb{I}$ and $\phi = \inf_n \sup_{m \geq n} X(\bot, f^m x_0)$. Any choice of $x_0$ gives the same $\phi$, hence the fixed point is unique.

- **(Knaster-Tarski)** take $J = \hat{X}$. Since $X$ is a complete lattice in the induced order, it has $\bot$. Then take $\phi = \inf \sup X(\bot, f^m \bot)$ and get the least point of $f$. 
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- **(Banach)** Take $J = \mathbb{I}$ and $\phi = \inf_n \sup_{m \geq n} X(-, f^m x_0)$. Any choice of $x_0$ gives the same $\phi$, hence the fixed point is unique.
- **(Knaster-Tarski)** take $J = \hat{X}$. Since $X$ is a complete lattice in the induced order, it has $\bot$. Then take $\phi = \inf \sup X(-, f^m \bot)$ and get the least point of $f$. Repeat the same proof for $X^{op}$ to obtain the greatest fixed point of $f$. 
More fixpoints

(Bourbaki-Witt) An expanding map \( f : X \to X \) on a dcpo \( X \) has a fixed point.

(James Caristi, 1976) Let \( f : X \to X \) be an arbitrary map on a complete metric space. If there exists a l.s.c. map \( \varphi : X \to [0, \infty) \) such that:

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(*) \quad X(x, fx) + \varphi(fx) \leq \varphi(x),
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then \( f \) has a fixed point.

Remark: \( f : X \to X \) is expanding iff \( \forall x \in X \) \( x \leq fx \).
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OUR GOAL: Show that both are instances of a single theorem that can have no constructive proof.
Unification a la Edalat & Heckmann


\[
\mathbf{B}X := \{\langle x, r \rangle \mid x \in X \text{ and } r \geq 0\} \subseteq X \times \mathbb{R}_+
\]

\[
\langle x, r \rangle \leq \langle y, s \rangle \iff X(x, y) + s \leq r
\]

\[
X \cong \{\langle x, 0 \rangle \mid x \in X\} (= \max(\mathbf{B}X) \text{ providing } X \text{ is } T_1).
\]
Unification a la Edalat & Heckmann

Edalat and Heckmann’s construction works the same for $\mathcal{Q}$-categories. Therefore:

**THEOREM**

$X$ is an $\mathbb{I}$-complete $\mathcal{Q}$-category iff $(\mathbf{B}X, \leq)$ is a dcpo.
Analysis of Caristi’s Theorem

(Nonsymmetric Caristi) Let \( f : X \to X \) be an arbitrary map on a \( \mathbb{I} \)-complete \([0, \infty]\)-category. If there exists a l.s.c. map \( \varphi : X \to [0, \infty) \) such that:

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Analysis of Caristi’s Theorem

(NonSymmetric Caristi) Let $f : X \rightarrow X$ be an arbitrary map on a $\mathcal{I}$-complete $[0, \infty]$-category. If there exists a l.s.c. map $\varphi : X \rightarrow [0, \infty)$ such that:

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(*) \quad x(x, fx) + \varphi(fx) \leq \varphi(x),
\]

then $f$ has a fixed point.

1. $\varphi$ is l.s.c. iff $Z := \{ \langle x, \varphi x \rangle \mid x \in X \} \subseteq BX$ is a dcpo.
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4. Hence (Nonsymmetric Caristi) iff (Bourbaki-Witt).
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2. Moreover, $(*)$ iff $\langle x, \varphi x \rangle \leq \langle Tx, \varphi(Tx) \rangle$ in $BX$.
3. Hence $(*)$ iff the map $\langle x, \varphi x \rangle \mapsto \langle Tx, \varphi(Tx) \rangle$ is expanding.
4. Hence (Nonsymmetric Caristi) iff (Bourbaki-Witt).
5. Moreover, Andrej Bauer proved that (Bourbaki-Witt) has no constructive proof.
6. Hence (Nonsymmetric Caristi) has no constructive proof either.
But…

… maybe (Caristi) has a constructive proof?

NO.
The proof idea is due to Hannes Diener.
Hannes Diener (photo by Andrej Bauer)
Hannes’ proof

Let $a, b \in \mathbb{R}$ be We will show that (Caristi) implies that for any two non-negative reals $a, b$ such that $\neg(a \neq 0 \land b \neq 0)$, we have either $a = 0$ or $b = 0$. 
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Let $a, b \in \mathbb{R}$ be We will show that $(\text{Caristi})$ implies that for any two non-negative reals $a, b$ such that $\neg (a \neq 0 \land b \neq 0)$, we have either $a = 0$ or $b = 0$.

Proof: Let $a, b$ satisfy $\neg (a \neq 0 \land b \neq 0)$. 
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Proof: Let $a, b$ satisfy $\neg(a \neq 0 \land b \neq 0)$. Wlog $a, b \leq \frac{1}{2}$. Define $f : [0, 1] \to [0, 1]$ by

$$f(x) := a(1 - x) - bx + x.$$
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Let \( a, b \in \mathbb{R} \) be We will show that (Caristi) implies that for any two non-negative reals \( a, b \) such that \( \neg (a \neq 0 \land b \neq 0) \), we have either \( a = 0 \) or \( b = 0 \).

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If \( a \neq 0 \), then \( b = 0 \) and the graph of \( f \) lies above the diagonal, and has a unique fixpoint at 1.
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If we could prove \( (\text{Caristi}) \), \( f \) would have a fixpoint \( x_0 \). By considering an appropriate approximation we can decide whether \( x_0 > 0 \) or \( x_0 < 1 \). In the first case it is impossible that \( b \neq 0 \), since then, as mentioned above, \( f \) would have a unique fixpoint at 0; thus \( b = 0 \).
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Conclusion

1. I have argued that theorems of Knaster-Tarski and Banach are in essence 'the same' — by forgetting the distinction between order and metric.
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2. I have argued that theorems of Bourbaki-Witt and Caristi are in essence 'the same' — by switching from a metric space $X$ to its formal ball model $\mathbf{B}X$. 
Conclusion

1. I have argued that theorems of Knaster-Tarski and Banach are in essence ‘the same’ — by forgetting the distinction between order and metric.

2. I have argued that theorems of Bourbaki-Witt and Caristi are in essence ‘the same’ — by switching from a metric space $X$ to its formal ball model $BX$.

3. In fact, (Nonsymmetric Caristi) can be further generalized to become a source theorem for both classic results mentioned in 2.
APPENDIX: Patareaia’s construction

**THEOREM.** A monotone map $f : X \to X$ on a pointed dcpo $X$ has a least fixed point. Proof:
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1. A subset $Y := \{ y \in X \mid y \leq fy \}$
   (a) contains $\bot$, (b) is closed under $f$, (c) is a subdcpo.
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   It satisfies (a)-(c) as well.
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3. Hence $f : C \rightarrow C$ is an order-preserving and expanding map on a pointed dcpo. The set of all such maps $E(X)$ is a dcpo in the pointwise order.
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3. Hence $f : C \to C$ is an order-preserving and expanding map on a pointed dcpo. The set of all such maps $E(X)$ is a dcpo in the pointwise order.
4. But since $f, g \leq f \circ g$ and $f, g \leq g \circ f$ for any maps $f, g$ in $E(X)$, the dcpo $E(X)$ is itself directed.
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5. Therefore $E(X)$ has a top element $\top$. We have $f \circ \top = \top$. 
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5. Therefore \( E(X) \) has a top element \( \top \). We have \( f \circ \top = \top \).
6. Hence \( f(\top(\bot)) = \top(\bot) \), and for any other fixpoint \( x \in X \), the set \( \downarrow x \) satisfies (a)-(c), and thus \( \top(\bot) \in C \subseteq \downarrow x \). QED.