Initial algebras, strong dinaturality and uniform parameterized fixpoint operators

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This talk

- Simpson, Plotkin gave sufficient conditions for unique existence (in a category) of a uniform parameterized fixpoint operators in terms of existence of bifree algebras of certain functors
- Uniformity is a strong dinaturality condition
- We use a Yoneda-like lemma about initial algebras and strong dinaturality to analyse the fine structure of their proof

Outline

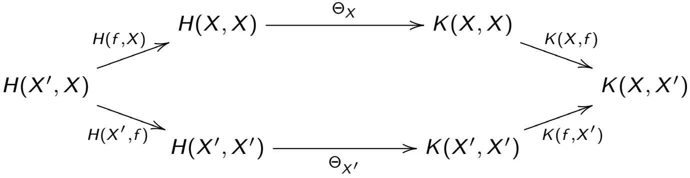
- Strong dinaturality and Yoneda-like lemma for initial algebras
- Uniform parameterized fixpoint operators
  what they are and their unique existence
- Guarded recursion operators (only mention)

From natural to strong dinatural transformations

- Dinaturality and strong dinaturality are
  two generalizations of natural transformations from
  covariant to mixed-variant functors
  with components only defined for the diagonal of the
  domain.
- Correspond to the idea of polymorphic functions with types where the universally quantified type variable may occur both positively and negatively.

Dinatural transformations

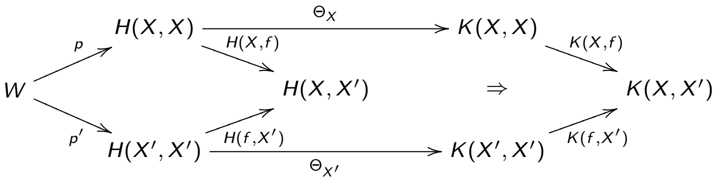
- A *dinatural transformation* between $H, K \in \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{E}$ is given by,
  for any $X \in |\mathbb{C}|$, a map $\Theta_X \in \mathbb{E}(H(X,X), K(X,X))$ such that, for any $f \in \mathbb{C}(X, X')$, the following hexagon commutes in $\mathbb{E}$:



- Dinaturals appear, e.g., in coend and ends.

Strong dinatural transformations

- A *strongly dinatural transformation* between $H, K \in \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{E}$ is given by, for any $X \in |\mathbb{C}|$, a map $\Theta_X \in \mathbb{E}(H(X,X), K(X,X))$ such that, for any map $f \in \mathbb{C}(X, X')$ and any span (W, p, p') on (X, X'), if the square in the following diagram commutes in $\mathbb{E}$, then so does the hexagon:



- If $\mathbb{E}$ has pullbacks (e.g., **Set**), it suffices to require that the outer hexagon commutes for (W, p, p') the chosen pullback of the cospan $(H(X,X'), H(X,f), H(f,X'))$.

No perfect world!

- Dinaturals not generally compose, so mixed-variant functors and dinatural transformations do not form a category.
  But, if $\mathbb{E}$ is Cartesian closed, this "non-category" also is!
- Strong dinaturals compose, mixed-variant functors from $\mathbb{C}$ to $\mathbb{E}$ and strongly dinatural transformations form a category. Denote it $[\mathbb{C}, \mathbb{E}]_{\mathrm{sd}}$.
  But Cartesian closedness of $\mathbb{E}$ does not imply that $[\mathbb{C}, \mathbb{E}]_{\mathrm{sd}}$ is Cartesian closed.

Recall the Yoneda lemma

- Let $\mathbb{C}$ be a locally small category, $C \in |\mathbb{C}|$ an object and $K \in \mathbb{C} \to \mathbf{Set}$ a functor.
- Then
$$\begin{array}{rcl} [\mathbb{C}, \mathbf{Set}](\mathbb{C}(C, -), K) & \cong & K\, C \\ \Theta & \mapsto & \Theta_C\, \mathrm{id}_C \\ \lambda_X\, \lambda k\, K\, k\, x & \leftarrow\!\shortmid & x \end{array}$$
(so $[\mathbb{C}, \mathbf{Set}](\mathbb{C}(C, -), K)$ is, in fact, a set too).
- This isomorphism is natural in C.

Yoneda lemma for initial algebras

- Let $\mathbb{C}$ be a locally small category, $F \in \mathbb{C} \to \mathbb{C}$ a functor with an initial algebra (which we denote $(\mu F, \mathsf{in}_F)$) and $K \in \mathbb{C} \to \mathbf{Set}$ a functor (whose padding into a mixed-variant functor we denote also by K).
- Then

$$\begin{array}{rcl} [\mathbb{C}, \mathbf{Set}]_{\mathrm{sd}}(\mathbb{C}(F-,-), K) & \cong & K(\mu F) \\ \Theta & \mapsto & \Theta_{\mu F} \mathsf{in}_F \\ \lambda_X \, \lambda k \, K \, (\mathsf{fold}_{F,X} \, k) \, x & \leftarrow\!\shortmid & x \end{array}$$

  (so $[\mathbb{C}, \mathbf{Set}]_{\mathrm{sd}}(\mathbb{C}(F-,-), K)$ is, in fact, a set too).
- This isomorphism is natural in F to the extent that initial algebras exist in $\mathbb{C}$.

Most important special case

- If $KX =_{\mathrm{df}} \mathbb{C}(A, X)$, for $A \in |\mathbb{C}|$ an object (e.g., $A =_{\mathrm{df}} 1$), we get

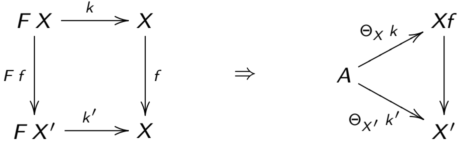
$$\begin{array}{rcl} [\mathbb{C}, \mathbf{Set}]_{\mathrm{sd}}(\mathbb{C}(F-,-), \mathbb{C}(A,-)) & \cong & \mathbb{C}(A, \mu F) \\ \Theta & \mapsto & \Theta_{\mu F} \mathsf{in}_F \\ \lambda_X \, \lambda k \, K\,(\mathsf{fold}_{F,X}\, k) \circ x & \leftarrow\!\dashv & x \end{array}$$

- Compare this to the "impredicative encoding" of inductive types:

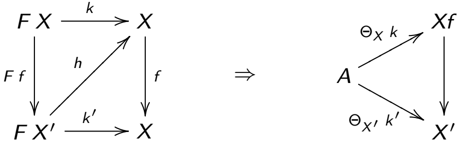
$$\forall X.(F\,X \Rightarrow X) \Rightarrow X \;=\; \mu F$$

Most important special case (ctd)

- A strong dinatural between $\mathbb{C}(-,-)$ and $\mathbb{C}(A,-)$ is, for any X, a function $\Theta_X \in \mathbb{C}(F\,X, X) \to \mathbb{C}(A, X)$, such that, for any $f \in \mathbb{C}(X, X')$, if the square below commutes, so does the triangle:



- The condition for dinaturality is weaker: for any $f \in \mathbb{C}(X, X')$ and $h \in \mathbb{C}(FX', X)$, if the triangles on the left commute, then so does the triangle on the right.



Parameterized fixpoint operators

- Assume given a category $\mathbb{D}$ with finite products.
- A *parameterized fixpoint-like operator* on $\mathbb{D}$ is given by, for any $X, Y \in |\mathbb{D}|$, a function

$$\mathrm{fix}_{X,Y} \in \mathbb{D}(X \times Y, Y) \to \mathbb{D}(X, Y)$$

- A *parameterized fixpoint operator* on $\mathbb{D}$ is a parameterized fixpoint-like operator fix on $\mathbb{D}$ such that
  – for any $f \in \mathbb{D}(X, X')$ and $k' \in \mathbb{D}(X' \times Y, Y)$,

$$\mathrm{fix}\,(k' \circ (f \times \mathrm{id}_Y)) = \mathrm{fix}\,k' \circ f$$

  (naturality);
  – for any $k \in \mathbb{D}(X \times Y, Y)$,

$$\mathrm{fix}\,k = k \circ \langle \mathrm{id}_X, \mathrm{fix}\,k \rangle$$

  (parameterized fixpoint property).

Conway operators

- A *Conway* operator on $\mathbb{D}$ is a parameterized fixpoint operator fix on $\mathbb{D}$ with the further properties that
  – for any $f \in \mathbb{D}(X \times Y, Y')$ and $h \in \mathbb{D}(X \times Y', Y)$,

  $$f \circ \langle \mathrm{id}_X, \mathrm{fix}\,(h \circ \langle \mathrm{fst}, f\rangle)\rangle = \mathrm{fix}\,(f \circ \langle \mathrm{fst}, h\rangle)$$

  (parameterized dinaturality);
  – for any $k \in \mathbb{D}((X \times Y) \times Y, Y)$,

  $$\mathrm{fix}\,(k \circ \langle \mathrm{id}_{X \times Y}, \mathrm{snd}_{X,Y}\rangle) = \mathrm{fix}\,(\mathrm{fix}\,k)$$

  (diagonal property).
- Parameterized dinaturality implies the parameterized fixpoint property, so the latter condition becomes redundant for Conway operators.

Uniformity

- Assume also given
  a category $\mathbb{C}$ with finite products and the same objects as $\mathbb{D}$
  together with an identity-on-objects functor $J \in \mathbb{C} \to \mathbb{D}$ preserving the finite products of $\mathbb{C}$ strictly.
- A parameterized fixpoint-like operator fix on $\mathbb{D}$ is said to be *uniform* wrt. J, if
  – for any $f \in \mathbb{C}(Y, Y')$, $k \in \mathbb{D}(X \times Y, Y)$ and $k' \in \mathbb{D}(X \times Y', Y')$,

$$J\, f \circ k = k' \circ (\mathrm{id}_X \times J\, f) \;\Rightarrow\; J\, f \circ \mathrm{fix}\, k = \mathrm{fix}\, k'$$

.

A specific concrete situation of interest

- Assume $\mathbb{D}$ arising as the coKleisli category of some comonad $(D, \varepsilon, (-)^\dagger)$ on $\mathbb{C}$.
- We can then use as J the right adjoint in its coKleisli setting.
- Example:
  – $\mathbb{C} =_{\mathrm{df}} \mathbf{Cppo}_\perp$ (ω-complete pointed partial orders and strict ω-continuous functions)
  – $D =_{\mathrm{df}} (-)_\perp$ (the lifting functor)
  – $\mathbb{D} \cong \mathbf{Cppo}$ (ω-complete pointed partial orders and all ω-continuous functions)

Uniform fixpoint-like operators, equivalently

- In terms of $\mathbb{C}$, a parameterized fixpoint-like operator is now, for any $X, Y \in |\mathbb{C}|$, a function

$$\mathrm{fix}_{X,Y} \in \mathbb{C}(D\,(X \times Y), Y) \to \mathbb{C}(D\,X, Y)$$

- Uniformity means that,
for any $f \in \mathbb{C}(Y, Y')$, $k \in \mathbb{C}(D\,(X \times Y), Y)$ and
$k' \in \mathbb{C}(D\,(X \times Y'), Y')$,

$$f \circ k = k' \circ D\,(\mathrm{id}_X \times f) \;\;\Rightarrow\;\; f \circ \mathrm{fix}\,k = \mathrm{fix}\,k'$$

- This the strong dinaturality condition of fix!
- Therefore, by Yoneda, if all functors $D\,(X \times -)$ have initial algebras,
a uniform param. fixp.-like operator fix is the same as,
for any $X \in |\mathbb{C}|$, a map

$$\underline{\mathrm{fix}}_X \in \mathbb{C}(DX, \mu(D\,(X \times -)))$$

Uniform param. fixpoint operators, equivalently

- In terms of $\mathbb{C}$, the conditions on param. fixp. oper.s are:
  - for any $f \in \mathbb{C}(D\,X, X')$ and $k' \in \mathbb{C}(D(X' \times Y), Y)$,

$$\mathrm{fix}\,(k' \circ \langle f \circ D\,\mathrm{fst}, \varepsilon_Y \circ D\,\mathrm{snd}\rangle^\dagger) = \mathrm{fix}\,k' \circ f^\dagger$$

  - for any $k \in \mathbb{C}(D\,(X \times Y), Y)$,

$$\mathrm{fix}\,k = k \circ \langle \varepsilon_X, \mathrm{fix}\,k\rangle^\dagger$$

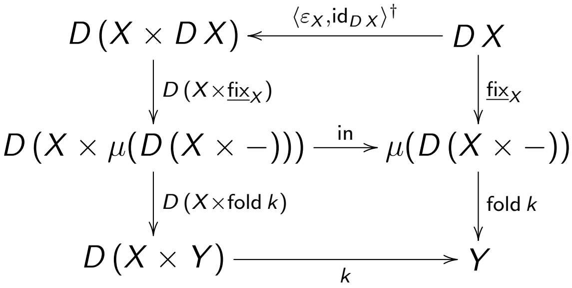
- If all functors $D\,(X \times -) \in \mathbb{C} \to \mathbb{C}$ have initial algebras, a uniform param. fixpoint operator fix is the same as, for any $X \in |\mathbb{C}|$, a map $\underline{\mathrm{fix}}_X \in \mathbb{C}(DX, \mu(D\,(X \times -)))$ s.t.
  - for any $f \in \mathbb{C}(D\,X, X')$,

$$\mu(\langle f \circ D\,\mathrm{fst}, \varepsilon_- \circ D\,\mathrm{snd}\rangle^\dagger) \circ \underline{\mathrm{fix}}_X = \underline{\mathrm{fix}}_{X'} \circ f^\dagger$$

  - for any $X \in |\mathbb{C}|$,

$$\underline{\mathrm{fix}}_X = \mathrm{in}_{D\,(X \times -)} \circ \langle \varepsilon_X, \underline{\mathrm{fix}}_X\rangle^\dagger$$

Some intuition (?)



Conway operators, equivalently

- In terms of $\mathbb{C}$, the conditions on Conway operators are:
  - for any $f \in \mathbb{C}(D(X \times Y), Y')$, $h \in \mathbb{C}(D(X \times Y'), Y)$,

    $$f \circ \langle \varepsilon_X, \mathrm{fix}(h \circ \langle \varepsilon_X \circ D\,\mathrm{fst}, f\rangle^\dagger)\rangle^\dagger = \mathrm{fix}(f \circ \langle \varepsilon_X \circ D\,\mathrm{fst}, h\rangle^\dagger)$$

  - for any $k \in \mathbb{C}(D((X \times Y) \times Y), Y)$,

    $$\mathrm{fix}(k \circ D\,\langle \mathrm{id}_{X \times Y}, \mathrm{snd}_{X,Y}\rangle) = \mathrm{fix}(\mathrm{fix}\,k)$$

- If all functors $D(X \times -)$, $D(X \times D(X \times -))$, $D((X \times -) \times -) \in \mathbb{C} \to \mathbb{C}$ have initial algebras, a uniform Conway operator is the same as, for any $X \in |\mathbb{C}|$, a map $\underline{\mathrm{fix}}_X \in \mathbb{C}(DX, \mu(D(X \times -)))$ satisfying the conditions of the previous slide, but also:
  - for any $X \in |\mathbb{C}|$,

    $$\mathrm{in} \circ \langle \varepsilon_X, \mathrm{fold}(\langle \varepsilon_X \circ D\,\mathrm{fst}, \mathrm{in}\rangle^\dagger) \circ \underline{\mathrm{fix}}\rangle^\dagger = \mathrm{fold}(\mathrm{in} \circ \langle \varepsilon_X \circ D\,\mathrm{fst}, \mathrm{id}\rangle^\dagger) \circ \underline{\mathrm{fix}}$$

  - for any $X \in |\mathbb{C}|$,

    $$\mathrm{fold}(\mathrm{in} \circ D\,\langle \mathrm{id}, \mathrm{snd}\rangle) \circ \underline{\mathrm{fix}} = \mathrm{fold}(\mathrm{fold}\,\mathrm{in} \circ \underline{\mathrm{fix}}) \circ \underline{\mathrm{fix}}$$

Unique existence conditions

- If every functor $D(X \times -) \in \mathbb{C} \to \mathbb{C}$ has a bifree algebra, then $\mathbb{D}$ has a unique uniform wrt. J parameterized fixpoint operator.
- If all functors $D\,(X \times -), D\,(X \times D\,(X \times -)), D\,((X \times -) \times -) \in \mathbb{C} \to \mathbb{C}$ have bifree algebras, then $\mathbb{D}$ has a unique uniform wrt. J Conway operator.

Conclusion

- Same technique applies to guarded recursion operators.
- The Yoneda-like lemma stages the invocations of the initial algebra resp. bifree algebra existence assumptions in Simpson and Plotkin's theorems:
  initial algebra existence – equivalent formulation
  bifree algebra existence – the equivalent map exists uniquely