Topological Properties of Event Structures

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Abstract

Motivated by the nice labeling problem for event structures, we study the topological properties of the associated graphs. For each $n \geq 0$, we exhibit a graph G_n that cannot occur on an antichain as a subgraph of the graph of an event structure of degree n. The clique complexes of the graphs G_n are disks (n even) and spheres (n odd) in increasing dimensions. We strengthen the result for event structure of degree 3: cycles of length greater than 3 do not occur on antichains as subgraphs. This amount to saying that the clique complex of the graph of an event structure of degree 3 is acyclic.

Introduction

In this note we present some ideas on the use of algebraic topology finalized to the understanding of mathematical structures modeling concurrency, finitary coherent domains and event structures. These ideas are part of a larger investigation of the finite labeling problem for event structures [2]. The purpose of the note is to show how many natural geometrical questions arise from this working context.

Roughly speaking, the nice labeling problem consists in reconstructing a given finite coherent domain – i.e. a poset which represents the possible of executions of a concurrent system – from the standard ingredients of trace theory [3]. These are an alphabet, a local independence relation, and a prefix closed subset of the free monoid, see [1, 6]. The problem always has a solution, and we are asked to find a solution of minimal cardinality (of the alphabet). The problem is equivalent to a graph coloring problem in that we can associate to a finite coherent domain a graph, of which we are asked to compute its chromatic number. The main technical contribution in [2] is to show that some simple graph cannot occur as a subgraph of the restriction to an antichain of the graph of an event structure of degree 2. We develop this idea for event structures of higher degree an discover a family of graphs that are avoided on antichains. These graphs have a geometrical flavor as they are iteratively constructed by cones and suspensions. This is among the reasons to move from a graph theoretic perspective and to consider instead the clique complex of the graph of an event structure. For degree 3, we show that one dimensional spheres, that is cycles, cannot occur on antichains, unless they are boundaries. This lead to an explicit computation of the homology groups (of antichains of event structure of degree 3) that are shown to be trivial in all the dimension greater than zero.

We conjecture that similar results hold in higher dimensions and degrees. Toward this goal we make explicit the sense for which the homology of antichains makes a functor from a poset of antichains – isomorphic to the poset of upper sets of the event structure – to the category of sequences of abelian groups.

The usual definition of an event structure [10] suggests it is a sort of ordered abstract simplicial complex. We are pursuing this natural idea, event if not in a straightforward way. To our knowledge, this research path has not been followed yet,[1]thus it looks a priority to us to put these ideas forward.

The note is structured as follows. We present in the first section the background for our remarks. This comprises domains and event structures, elements from trace theory, and the nice labeling problem. The reader shall find the definition of the graph of an event structure and the reasons that induce us to study these graphs with fixed clique number. In the second section we shall exhibit some graphs that are avoided on antichains of this class of graphs. In the third section we use the previous considerations to determine the homology groups of the clique complexes of graphs with clique number 3. In the final section we formally define the homology of an event structure, and sketch some conjectures and directions for future researches.

1 Concurrency by Partial Orders and Graphs

1.1 Finitary Coherent Domains

Recall that an element p of a poset $\langle P, \leq \rangle$ is a *complete join prime* if whenever the least upper bound $\bigvee X$ of a possibly infinite set $X \subseteq P$ exists and $p \leq \bigvee X$, then $p \leq x$ for some $x \in X$. For $x \in P$ we let $P(x) = \{\, p \leq x \,|\, p \text{ is a complete join prime} \,\}$.

[1]Event structure semantics is an interleaving semantics, thus the role in our context of higher dimensions is not not to capture true concurrency.

Definition 1.1. A *finitary domain* is a poset $\mathcal{D} = \langle D, \leq \rangle$ such that, for each $d \in D$, the set $P(d)$ is finite and $d = \bigvee P(d)$.

In this paper we are shall be concerned with finite structures, hence the word "finitary" can be safely replaced by "finite". Let us recall that d' is an upper cover of d (denoted by $d \prec d'$) if the open interval $\{ x \mid d < x < d' \}$ is empty. The degree of $d \in \mathcal{D}$, $\deg(d)$, is the number of upper covers of d. The degree of a finitary domain $\mathcal{D}$ is defined by

$$\deg(\mathcal{D}) = \max_{d \in \mathcal{D}} \deg(d)\,.$$

We shall deal with special domains: a finitary domain is *coherent* if whenever $\{ d_i, d_j \}$ are bounded for each $i, j \in \{ 1, 2, 3 \}$, then the set $\{ d_1, d_2, d_3 \}$ is also bounded.

1.2 Local Independence Relations

We see now how finitary domains arise from trace theory [3].

Definition 1.2. A *local independence relation* over an alphabet Σ is a relation $R \subseteq \Sigma^* \times P_2(\Sigma)$.

A relation of the form $wR\{ a, b \}$ informally means that the events a and b are independent (i.e. commute) immediately after the sequence of events encoded in the word w. We denote by $\sim_R$ the least right congruence containing the pairs $wab = wba$ whenever $wR\{ a, b \}$, so that $[w]_R$ denotes the equivalence class of w modulo $\sim_R$. By definition the quotient $\Sigma^*/{\sim_R}$ is a right module, the action on equivalence classes being defined in the natural way: $[w]_R a = [wa]_R$.

We say that the local independence relation R is *stable* if and only if the cube axiom holds for each $\sim_R$-equivalence class q:

$$\begin{tikzcd}
 & qbc \arrow[rr, "a"] & & qabc \\
qc \arrow[ur, "b"] \arrow[rr, "a"] & & qac \arrow[ur, "b"] & \\
 & & & qab \arrow[uu, "c"] \\
q \arrow[uu, "c"] \arrow[rr, "a"] & & qa \arrow[uu, "c"] \arrow[ur, "b"] &
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
 & qbc \arrow[rr, "a"] & & qabc \\
qc \arrow[ur, "b"] & & & \\
 & qb \arrow[uu, "c"] \arrow[rr, "a"] & & qab \arrow[uu, "c"] \\
q \arrow[uu, "c"] \arrow[ur, "b"] \arrow[rr, "a"] & & qa \arrow[ur, "b"] &
\end{tikzcd}$$

This diagram is asserting two implications. The implication from left to right is read as follows: if $qac = qca$ and $qabc = qacb = qcab = qcba$, then $qab = qba$, $qbc = qcb$, and $qbac = qbca$. We leave the reader to make explicit the implication from right to left. We say that R is *coherent* if and only if the implication

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & qbc & & \\
qc \arrow[ur, "b"] \arrow[rr, "a"] & & qac & \\
 & qb \arrow[uu, "c"] \arrow[rr, "a"] & & qab \\
q \arrow[uu, "c"] \arrow[ur, "b"] \arrow[rr, "a"] & & qa \arrow[uu, "c"] \arrow[ur, "b"] &
\end{tikzcd}
\qquad \Longrightarrow \qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & qbc \arrow[rr, "a"] & & qabc \\
qc \arrow[ur, "b"] \arrow[rr, "a"] & & qac \arrow[ur, "b"] & \\
 & qb \arrow[uu, "c"] \arrow[rr, "a"] & & qab \arrow[uu, "c"] \\
 & & qa \arrow[uu, "c"] \arrow[ur, "b"] &
\end{tikzcd}
$$

holds for every $\sim_R$-equivalence class q. Explicitly, if $qab = qba$, $qac = qca$, and $qbc = qcb$, then $qabc = qabc$, $qbac = qbca$, and $qcab = qcba$.

We can define on a right Σ^*-module a preorder by saying $q \leq q'$ if and only if $q' = qw$ for some $w \in \Sigma^*$. The following is the main result of [6].

Theorem 1.3. *If R is stable and coherent, then for any lower set $L \subseteq \Sigma^*/{\sim_R}$, the pair $\langle L, \leq \rangle$ is a finitary coherent domain.*

A lower set $L \subseteq \Sigma^*/{\sim_R}$ can be identified with a prefix closed subset of Σ^* which moreover is closed w.r.t. $\sim_R$. We denote the domain arising from the data $\langle \Sigma, R, L \rangle$ by $\mathcal{D}(\Sigma, R, L)$. The reader will have no difficulties in verifying that $\deg(\mathcal{D}(\Sigma, R, L)) \leq \mathrm{card}(\Sigma)$. We are ready to state the **nice labeling problem**:[2] *given a finite coherent domain $\mathcal{D}$ compute the least $n \geq 0$ such that, for some data $\langle \Sigma, R, L \rangle$ with* $\mathrm{card}(\Sigma) = n$, *$\mathcal{D}$ is order isomorphic to $\mathcal{D}(\Sigma, R, L)$.* Since some data $\langle \Sigma, R, L \rangle$ giving rise to $\mathcal{D}$ always exists, we let $\mathrm{nl}(\mathcal{D})$ be such least n. More generally, for $d \geq 0$, we define

$$\mathrm{nl}(d) = \max\{\, \mathrm{nl}(\mathcal{D}) \mid \deg(\mathcal{D}) = d \,\}\,.$$

It was shown in [2] that $\mathrm{nl}(d) = d$ if $d \leq 2$ and that $\mathrm{nl}(d) > d$ otherwise.

1.3 Event Structures

Finitary domains are almost lattices, since the join of two elements might not exist. Nonetheless, they are distributive, meaning also that a Birkhoff-like representation theorem holds: finitary domains are dual to event structures.

Definition 1.4. An *event structure* (with binary concurrency) is a triple $\mathcal{E} = \langle P, \leq, \# \rangle$ such that

- $\langle P, \leq \rangle$ is a poset, such that for each $p \in P$ the lower set $\{\, x \mid x \leq p \,\}$ is finite,

[2]See Corollary 1.7 for the meaning of the word "labeling".

- $\# \subseteq P \times P$ is a symmetric binary relation upper closed w.r.t. the order (i.e. $p\#q$ and $p \leq p'$ implies $p'\#q$). Moreover, if $p\#q$, then $\{ p, q \}$ is an antichain.

The order $\leq$ is known as the *causality* of events, where the binary relation $\#$ is called *conflict*. Given an event structure $\langle P, \leq, \# \rangle$, we define the *concurrency* relation $\frown$ as the complement of $\#$, i.e. $p \frown q$ if and only if it not the case that $p\#q$. The concurrency relation $\frown$ is closed under the order ($x' \leq x \frown y$ implies $x' \frown y$) and every comparable pair is concurrent. Event structures could have been defined by taking the concurrency relation as a primitive notion, and in the following we shall be oblivious of the conflict relation. Given an event structure $\mathcal{E} = \langle P, \leq, \# \rangle$ a lower set of $\mathcal{E}$ is a subset I of P such that $x \leq y \in I$ implies $x \in I$. We let

$$CL(\mathcal{E}) = \{ I \mid I \text{ is a lower set and a clique w.r.t.} \frown \}\,, \quad \mathcal{D}(\mathcal{E}) = \langle CL(\mathcal{E}), \subseteq \rangle\,.$$

Theorem 1.5. *The poset $\mathcal{D}(\mathcal{E})$ is a finitary coherent domain. Moreover, every finite coherent domain $\mathcal{D}$ is order isomorphic to some domain of the form $\mathcal{D}(\mathcal{E})$ for some event structure $\mathcal{E}$.*

The statement is a well known result of concurrency theory, see [10]. We recall that given a domain $\mathcal{D}$, we can define $\mathcal{E}_{\mathcal{D}} = \langle P, \leq, \# \rangle$ with the property that $\mathcal{D}$ is order isomorphic to $\mathcal{D}(\mathcal{E}_{\mathcal{D}})$ as follows: we let P be the set of complete join prime elements of $\mathcal{D}$, $\leq$ is the restriction of the order of $\mathcal{D}$ to P, and we let $p \frown p'$ if and only if the pair $\{ p, p' \}$ is bounded in $\mathcal{D}$.

We shall need one more relation:

$$\begin{aligned} p \overline{\frown} q \text{ if and only if } & \{ p, q \} \text{ is an antichain,} \\ & p' \frown q \text{ for all } p' < p, \text{ and } p \frown q' \text{ for all } q' < q. \end{aligned}$$

Given an event structure $\mathcal{E} = \langle P, \leq, \# \rangle$ we define the undirected graph $\mathcal{G}(\mathcal{E})$ as the pair $\langle P, \overline{\frown} \rangle$.

Lemma 1.6. *A set $\{ x_1, \ldots, x_n \}$ is a clique in the graph $\mathcal{G}(\mathcal{E})$ iff there exists an ideal $I \in CL(\mathcal{E})$ such that the $\{ x_i \} \cup I$, $i = 1, \ldots, n$, are distinct upper covers of I in $\mathcal{D}(\mathcal{E})$.*

For a graph $\mathcal{G}$, let $\gamma(\mathcal{G})$ be the chromatic number of $\mathcal{G}$ and let $\chi(\mathcal{G})$ be its clique number (that is the cardinality of the greatest clique).

Corollary 1.7. *The following relations hold:*

$$\deg(\mathcal{D}(\mathcal{E})) = \chi(\mathcal{G}(\mathcal{E}))\,, \qquad \mathrm{nl}(\mathcal{D}(\mathcal{E})) = \gamma(\mathcal{G}(\mathcal{E})).$$

Thus, for an event structure, we define the degree of $\mathcal{E}$ as the degree of $\mathcal{D}(\mathcal{E})$ or, in an equivalent way, as the clique number of $\mathcal{G}(\mathcal{E})$. The nice labeling problem for $\mathcal{D}$ amounts to find a coloring of $\mathcal{G}(\mathcal{E}_{\mathcal{D}})$ with the smallest number of colors.

The reader who's not motivated by the finite labeling problem might object that studying the concurrency graph of an event structure – i.e. vertexes are elements of P and two elements are related if they are uncomparable and concurrent – might be more interesting. However, it is a trivial observation that every graph can be realized as the concurrency graph of some event structure. On the other hand, not every graph is of the form $\mathcal{G}(\mathcal{E})$: many are the constraints on $\frown$, for example minimal elements form a clique w.r.t. $\frown$. Again, motived by nice labeling problem, it can be shown that some well known graphs with clique number 3 and increasing coloring number do not occur as subgraphs of some graph of the form $\mathcal{G}(\mathcal{E})$. For example, let $\mathcal{M}$ be the Mycielski transform of a graph [9] and let K_3 the total graph on three elements.

Proposition 1.8. *The Mycielski graphs $M^n(K_3)$, $n \geq 2$, do not occur as a subgraphs of some graph of the form $\mathcal{G}(\mathcal{E})$ such that $\chi(\mathcal{G}(\mathcal{E})) = 3$.*

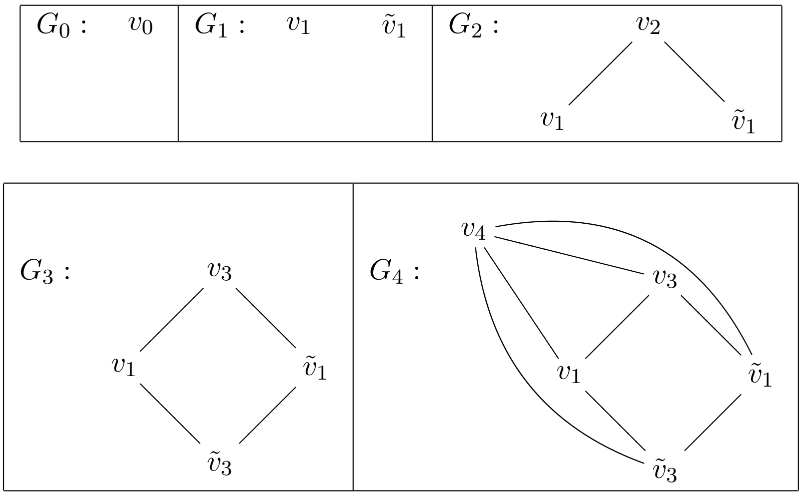
2 Avoided Graphs on Antichains

We are going to study the structure of a graph $\mathcal{G}(\mathcal{E})$ restricted to antichains. An antichain is meant to represent a collection of global states of a system, incompatible among them, that may share local similarities. A global state is characterized by the clique of its enabled unnamed events. Recall that the clique complex $\mathcal{CL}(G)$ of a graph G has as simplices the cliques of a graph. Therefore we shall emphasize that a global state is a simplex in the clique complex of $\mathcal{G}(\mathcal{E})$. Two global states may be similar in that they share some face. An antichain B might be dependent on an antichain A if each event in B depends on an event in A. Our goal is to analyze to what extent the topological properties of antichains are invariant under the dependency relation. These ideas are similar to (and indeed have been suggested by) those in the work [5] on asynchronous computability. A trial to formalize this work in the present context of event structures has failed until now and suggested possible divergences. Thus we let $\mathcal{A}(\mathcal{E})$ the set of antichains of $\langle P_{\mathcal{E}}, \leq_{\mathcal{E}} \rangle$. For $A, B \in \mathcal{A}(\mathcal{E})$, we say that B depends on A, written $B >> A$, if for all $b \in B$ there exists an $a \in A$ such that $a \leq b$. We shall come back to this order on $\mathcal{A}(\mathcal{E})$ in the last section.

2.1 Disks and spheres ...

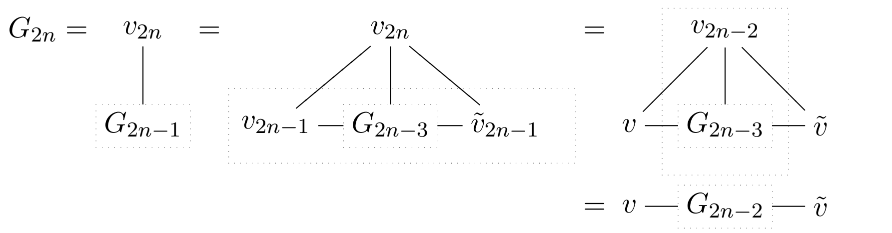
We define a sequence of graphs G_n, for $n \geq -1$. To this goal, for a graph $G = \langle V, E \rangle$, let $v * G = \langle V \uplus \{ v \}, E \cup E' \rangle$ where $E' = \{ \{ v, v' \} \mid v' \in V \}$. Observe that a clique in $v * G$ is either v, or a clique in G, or a clique of the form $\{ v \} \cup S$, with S a clique of G. That is, this operation amounts to *adding a cone* to G in the clique complex of G, $\mathcal{CL}(v * G) = v * \mathcal{CL}(G)$. The *suspension* of G, i.e. adding two cones to G, is defined similarly and it is well defined w.r.t. the clique complex. It will be denoted by $(v, \tilde{v}) * G$.

The graph G_{-1} is the empty graph. If n is even, then $G_n = v_n * G_{n-1}$, and, if n is odd, $G_n = (v_n, \tilde{v}_n) * G_{n-2}$. We sketch the structure of the graphs G_n for $n = 0, \ldots, 4$:



Lemma 2.1. *For each $n \geq 1$, G_n is a suspension of G_{n-2}.*

Proof. The property holds by definition if n is odd. Thus we shall prove that the property holds for G_{2n} with $n \geq 1$. The following diagrams should be self-explanatory.



□

The following proposition immediately follows by the definitions and from the above lemma.

Proposition 2.2. *For each $n \geq 0$, $\mathcal{CL}(G_{2n})$ is a disk in dimension n and $\mathcal{CL}(G_{2n+1})$ is a sphere in dimension n.*

We now give a graph-theoretic characterization of the graphs G_n. To this goal, let P be the following property: *if x, y are distinct nodes of $G = \langle V, E \rangle$ such that $\{ x, y \} \notin E$, then they both form a cone over $G \setminus \{ x, y \}$.*

Lemma 2.3. *The graphs G_n have property P. Moreover, if a graph G has property P, then it contains a copy of G_n as a subgraph, where $n = \mathrm{card}(A) - 1$.*

Proof. Property P clearly holds for G_n if $n \in \{ -1, 0 \}$. Let us suppose that it holds for G_k for $k < n$, and let us prove that it holds for G_n. To this goal let x, y be vertices of G_n. If $x = v_n$ and $y = \tilde{v}_n$, then the property is true. If this is not the case, then $\{ x, y \} \notin E$ implies that x, y are vertices of G_{n-2}, and by induction they are related to all the nodes in $G_{n-2} \setminus \{ x, y \}$. Since they are also both related to v_n and $\tilde{v}_n$, then they are related to all the vertices of $G_n \setminus \{ x, y \}$.

Let us consider a graph G which has property P and let $n = \mathrm{card}(A) - 1$. If G is a total graph, then clearly it contains a copy of G_n. Otherwise, let x, y be unrelated vertices, so that they both form a cone over $G' = G_n \setminus \{ x, y \}$. Since property P is closed under subgraph inclusion, then G' has property P, so that it contains a copy of G_{n-2}. Since x and y are related to all the elements of G_{n-2}, then G contains a copy of G_n. $\square$

2.2 ... are avoided on antichains

If P is a poset, the height $h(p)$ of an element $p \in P$ is the length of the longest chain of the form $p_0 < p_1 < \ldots < p_n = p$. If A is a (finite) antichain, then we define its height by $h(A) = \sum_{a \in A} h(a)$.

Proposition 2.4. *Let $\mathcal{E}$ be an event structure of degree n. If $A \in \mathcal{A}(\mathcal{E})$, then the graph $\langle A, \frown \rangle$ does not contain a subgraph of the form G_n.*

Proof. Let us say that a bad antichain of $\mathcal{E}$ is an antichain of cardinality $n + 1$ wich contains a copy of G_n. Its complexity $\xi(A)$ is the number of unrelated pairs. We shall show that if such an A exists, then there exists another bad antichain A' such that $\xi(A') < \xi(A)$. Since a bad antichain A with $\xi(A) = 0$ is an $n + 1$-clique, this will show that there are no bad antichains in $\mathcal{E}$.

Let us suppose that a bad antichain A exists in $\mathcal{E}$, with $\xi(A) = n > 0$. We can choose such a bad antichain A with $h(A)$ minimal. Since $\xi(A) = n > 0$, we can find distinct $x, y \in A$ that are not in the relation $\frown$. By the property of A, if $z \in A \setminus \{ x, y \}$ then $x \overline{\frown} z \overline{\frown} y$. Since it is not the case that $x \overline{\frown} y$ either we can find an $x' < x$ such that not $x' \frown y$, or we can find an $y' < y$ such that not $x \frown y'$. By symmetry, we can consider the first case only. Let $A' = \{ x' \} \cup (A \setminus \{ x \})$, we pretend that A' is a bad antichain. A' is an antichain. If $z \in A \setminus \{ x \}$, then $z \not\leq x'$, since otherwise $z \leq x$. Also, $x' \not\leq z$: if $z = y$, then $x' \leq y$ implies $x' \frown y$, and otherwise $z \overline{\frown} y$ and $x' < z$ implies $x \frown z$. Thus A' is an antichain and $\mathrm{card}(A') = \mathrm{card}(A)$. Moreover all the edges in A are inherited in A', thus A' is a bad antichain. Since $h(A') < h(A)$, by minimality $\xi(A') < \xi(A)$.[3] $\square$

3 Topological Properties in Degree 3

From now on we shall consider event structures of degree 3. According to Proposition 2.4, if $A \in \mathcal{A}(\mathcal{E})$, then $\langle A, \overline{\frown} \rangle$ does not contain a subgraph G_3, which is a one dimensional sphere. The goal of this section is to prove that one dimensional spheres, i.e. cycles, do not occur as subgraphs of $\langle A, \overline{\frown} \rangle$ unless they are boundaries. In graph theoretic language, this amount to the following Proposition.

Proposition 3.1. *If $\mathcal{E}$ is an event structure of degree 3 and $A \in \mathcal{A}(\mathcal{E})$, then A contains no cycle of the form $p_0 \overline{\frown} p_1 \overline{\frown} \ldots p_{n-1} \overline{\frown} p_n = p_0$ with $n \geq 4$.*

Proof. We shall show that if $A \in \mathcal{A}(\mathcal{E})$ contains a cycle of length $n > 4$, then we can find an antichain $A' \in \mathcal{A}(\mathcal{E})$ containing a cycle of length n' with $n > n' \geq 4$. Since by Proposition 2.4 an $A \in \mathcal{A}(\mathcal{E})$ cannot contain a cycle of length 4, we shall have found a contradiction.

Thus let $n > 4$ and among all the cycles $p_0 \overline{\frown} p_1 \overline{\frown} p_2 \ldots p_n \overline{\frown} p_{n+1} = p_0$ lying on antichain, chose a cycle C of minimal height. If $p_0 \overline{\frown} p_2$, then $p_0 p_2 \ldots p_n$ is a cycle of shorter length lying on an antichain, and we have reached our goal. Otherwise $p_0 \overline{\frown} p_2$ does not hold, and either we can find $p'_0 < p_0$ such that $p'_0 \not\frown p_0$, or we can find $p'_2 < p_2$ such that $p_0 \not\frown p'_2$. By symmetry, we can assume the first case holds. As in the proof of Proposition 2.4 $\{ p'_0, p_1, p_2, p_3 \}$ form an antichain, and $p'_0 p_1 p_2 p_3$ is a path. By minimality of C, $p'_0 p_1, \ldots p_{n-1} p'_0$ is not an antichain and thus the set $\{ j \in \{ 4, \ldots, n-1 \} \mid p_j \geq p'_0 \}$ is not empty. Let i be the minimum in this set, and observe that $p_{i-1} \overline{\frown} p_i$ and $p'_0 \leq p_i$ but $p'_0 \not\leq p_{i-1}$ implies $p_{i-1} \overline{\frown} p'_0$. Thus $p'_0 p_1 p_2 p_3 \ldots p_{i-1} p'_0$ is an antichain and a cycle of lenght at least 4 and strictly less than n. $\square$

[3] More precisley we have $x' \overline{\frown} y$ and $h(A') = h(A) - 1$.

The Proposition can be used to prove that every graph $\langle A, \frown \rangle$, $A \in \mathcal{A}(\mathcal{E})$, can be colored with at most three colors. We shall skip on this point and proceed instead to a somewhat straightforward computation of the homology groups of antichains of event structures of degree 3. More precisely, if $A \in \mathcal{A}(\mathcal{E})$, then we let $H_n(A)$ be the n-th homology group of the clique complex of the graph $\langle A, \frown \rangle$.

Corollary 3.2. *Let* $\deg(\mathcal{E}) = 3$ *and* $A \in \mathcal{A}(\mathcal{E})$. *Then*

$$\begin{aligned} H_0(A) &= \textit{an arbitrary finitely freely generated abelian group} \\ H_n(A) &= 0, \quad \textit{for } n \geq 1. \end{aligned}$$

Proof. We observe firstly that it is not difficult to construct an event structure $\mathcal{E}$ and an $A \in \mathcal{A}(\mathcal{E})$ with an arbitrary number of connected components. Therefore we shall be interested to the groups $H_n(A)$ with $n > 0$.

Since $\mathcal{E}$ contains no clique of cardinality greater than 3, the groups $C_n(A)$[4]are trivial (hence $H_n(A) = 0$) for $n > 2$. On the other hand, let $\gamma = \sum_i \alpha_i \gamma_i$ be a chain in dimension 2, where $\alpha_i \in \mathbb{Z}$ and the γ_i are 2-dimensional oriented simplices, that is, each γ_i is a clique of cardinality 3 together with an orientation. If $\delta_2(\gamma) = 0$ and $\gamma \neq 0$, then we can find distinct i, j such that γ_i and γ_j share a common 2-face, but this implies an occurrence of G_3 as a subgraph of A. Therefore $\ker \delta_2 = 0$ and $H_2(A) = 0$.

Finally, let $\gamma = \sum_{i=1}^{n} \alpha_i \gamma_i$ be a chain in dimension 1 such that $\alpha_i \neq 0$ and $\delta_1(\gamma) = 0$. If $n > 3$, then there is a cycle of length greater than 3. Thus $n = 3$, and γ is the boundary of some 2-dimensional simplex. □

4 The Homology of Event Structures

In the previous section we have isolated a class of graphs – the cycles – that are some kind of topological transformation of the graph G_3. We have proved then that a graph in this class does not occur as a subgraph of an antichain of an event structure of degree 3. Given the results of Section 2 it is tempting to conjecture that analogous properties hold in higher dimensions and degrees. W.r.t. a given class R_n, the conjecture could take the following form: *if* $G \in R_n$, *then* G *does not occur as a subgraph of an antichain* $A \in \mathcal{A}(\mathcal{E})$ *with* E *of degree* $2n + 1$. The class R_n might consists of those graphs whose clique complex geometric realization is homeomorphic (or homotopic) to a n-sphere. It could also be the class of graphs that are contracticble transformations [7] of the graph G_{2n+1}. A computational approach suggests

[4] $C_n(A)$ is the group freely generated by simplices, up to a choice of an orientation, cf. [8], of the clique complex of $\langle A, \frown \rangle$.

instead to investigate the homology groups $H_n(A)$ and ask whether $H_k(A) = 0$ if $A \in \mathcal{A}(\mathcal{E})$, $\deg(\mathcal{E}) = 2n+1$, and $k \geq n$. We shall develop some consideration in this direction. The reader may have noticed that part of the proof of Proposition 3.1 amounts to pushing down a cycle from an antichain B to an antichain A whenever $B >> A$. In the rest of the paper we argue that this is possible in higher dimensions as well, since it is a consequence of the functorial properties of the correspondence which takes an antichain (an element of a partially ordered set) to its graph, and then to its simplicial complex, and finally to the sequence of its homology groups.

Recall from Section 2 the definition of the poset $\langle \mathcal{A}(\mathcal{E}), >> \rangle$.[5]For B, A such that $B >> A$, define the relation $R^{B,A} : B \longrightarrow A$ as the order restricted to B and A; that is, for $b \in B$ and $a \in A$, $bR^{B,A}a$ if and only if $b \geq a$. Observe that $R^{A,A} = Id_A$, but that only the inclusion $R^{C,B} \circ R^{B,A} \subseteq R^{C,A}$ holds. The lax-functor that we have defined lands in a 2-category richer than the one of sets and relations. Every antichain carries the structure of a graph with its associated clique complex and, as we shall see, the cliques are sent to cliques.

Lemma 4.1. *Let γ be a clique in $\langle B, \frown \rangle$, let $B >> A$ and define $\Phi^{B,A}(\gamma) = \{ a \in A \,|\, \exists b \in \gamma,\, bR^{B,A}a \}$. Then $\Phi^{B,A}(\gamma)$ is a clique in $\langle A, \frown \rangle$.*

The proof is straightforward. The above observation allows to define a the functorial action $H^{B,A}_* : H_*(B) \longrightarrow H_*(A)$ for $B >> A$. The relevant observation is that the mapping $\Phi^{B,A}$ is an acyclic carrier, cf. [8, §13]. To realize the maps H_*, for each pair B, A with $B >> A$, choose a function $f^{B,A} : B \longrightarrow A$ such that $f^{B,A}(b) \in \Phi^{B,A}(\{ b \})$ (such a function exists because of the definition of the relation $>>$).

Lemma 4.2. *The map $f^{B,A}$ is simplicial from the clique complex of $\langle B, \frown \rangle$ to the clique complex of $\langle A, \frown \rangle$. Moreover the induced map $f^{B,A}_\sharp : C_n(B) \longrightarrow C_n(A)$ is carried by $\Phi^{B,A}$.*

As a consequence of the acyclic carrier theorem, the choice of different maps $f^{B,A}$ does not affect the induced map at the level of homology groups. This implies that H_* is a functor. Indeed $f^{B,A}(f^{C,B}(c)) \in \Phi^{C,A}(\{ c \})$ so that $f^{B,A}_\sharp \circ f^{C,B}_\sharp$ and $f^{C,A}_\sharp$ are both carried by $\Phi^{C,A}$ and there exists a homotopy between them.

[5]This poset is isomorphic to the lattice of upper sets of $P_\mathcal{E}$. It is interesting to restrict consideration to maximal antichains – an antichain A is a maximal antichain if for each $y \in P \setminus A$ there exists an $x \in A$ such that y and x are comparable. Maximal antichains form a sublattice (without bottom and top) of upper subsets and lower sets of $P_\mathcal{E}$ [4].

Proposition 4.3. *There is a well defined homology functor H_* from the poset $\langle \mathcal{A}(\mathcal{E}), >\!\!> \rangle$ to the category of infinite sequences of abelian groups.*

The above Proposition is certainly a simple consequence of existing theory, but certainly it is an unavoidable step toward further understanding of the topological properties of event structures.

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