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Relational Semantics for Distributive Linear Logic

Abstract

Axioms ruling linear negation have been investigated in the context of the complete semantics for distributive intuitionistic linear logic. Among these are the condition of being a dualizing element and the one of being a cyclic element. The motivation for analyzing other syntactic constraints comes from the observation that groupoids are models for classical linear logic. The analysis proceeds also in the opposite way: given semantic conditions, which could possibly hold in the canonical model of prime filters, equivalent syntactic conditions are found. Last, the relatioships among analyzed axioms are investigated, counterexamples are provided whenever there is no provability dependence.

Keywords. Linear Logic, Quantales, Promonoidal Categories, Kripke Models, Distributive Lattices.

Introduction. We have investigated axioms related to linear negation in the context of intuitionistic generalization of the complete semantics for Distributive Linear Logic presented in [GM]. We show (Section 2.5) that internal monoids in the monoidal category of bimodules between partially ordered sets provide a complete semantics for linear logic based on intuitionism. Such structures can also be thought as promonoidal categories over the monoidal category of truth values, in the sense of [Day]. The basic framework is then a set with a partial order, a ternary relation $|$ and a unitary attribute 1 which both satisfy compatibility constraints with respect to the order, as also shown in [AD]. A monoid in that category satisfies also the associative and unit laws:

$$\exists\zeta(\,\tau \mid \zeta\rho \ \text{ and } \ \zeta \mid \lambda\gamma\,) \ \text{ iff } \ \exists\xi(\,\tau \mid \lambda\xi \ \text{ and } \ \xi \mid \gamma\rho\,)$$

$$\exists\iota(\,\tau \mid \lambda\iota \ \text{ and } \ \iota \in 1\,) \ \text{ iff } \ \tau \preceq \lambda, \qquad \exists\iota(\,\tau \mid \iota\rho \ \text{ and } \ \iota \in 1\,) \ \text{ iff } \ \tau \preceq \rho$$

We give (Section 4) concrete examples of such structures: one of them is suggested by an immediate generalization to intuitionism of the quantale of all relations over a set; another universe can be obtained by putting together all the individuals of a monoid in presheaves.

The analysis, in this context, of axioms ruling linear negation has followed the suggestion in [Yet]: the condition of being a dualizing element and the condition of being a cyclic element

$$^*(p^*) \ = \ (^*p)^*\,, \qquad p^* \ = \ ^*p$$

have been analyzed separately, in (3.1) and (3.2). We found that the first axiom is true if and only if we can exhibit a bijection L in the structure which satisfies

$$\exists\tau(\,\tau \mid \lambda\rho \ \text{ and } \ \tau \notin 0\,) \ \text{ iff } \ L(\rho) \preceq \lambda \ \text{ iff } \ R(\lambda) \preceq \rho$$

where R is the inverse of L. The second axiom holds in a universe iff

$$\exists\tau(\,\tau \mid \lambda\rho \;\text{ and }\; \tau \not\in 0\,) \;\text{ iff }\; \exists\tau(\,\tau \mid \rho\lambda \;\text{ and }\; \tau \not\in 0\,)$$

If the first axiom holds too then the second is true iff $L = R$.

Since the starting point of our work has been the observation (4.2) that groupoids are models not only for linear logic, but also for classical linear logic when a proper choice of linear falsity is made, we have investigated (3.3) also syntactic conditions which are always valid in these particular universes. Those are a non-contradiction principle, which merges cartesian conjunction with linear negation and linear true, and De Morgan laws of complement with respect to linear conjunction and disjunction:

$$p \wedge p^* \;\vdash\; \neg 1\,, \qquad\qquad (p \cdot q)' \;=\; p' + q'$$

In the classical case the non contradiction principle is equivalent to taking the complement of unity as linear false. This choice leads in groupoids to satisfy De Morgan laws which are however not derivable: this is proved in (6.4) by constructing a counterexample.

Quantifier-elimination by means of properties of the free algebra of lower sets has been the general method followed in the analysis from syntax to semantics.

Another part of the work (5) has confronted the opposite problem: given a semantic condition, i.e. one which could be described in terms of $\preceq$, $|$, 1 , 0, we show that, by imposing it in the canonical model of prime filters, an equivalent syntactic condition can be found. For example, defining $\lambda\emptyset\rho$ as $\exists\tau(\,\tau \mid \lambda\rho \;\text{ and }\; \tau \not\in 0\,)$, the analysis in (5.1) of

$$\forall\lambda\exists\rho(\,\lambda\emptyset\rho \;\text{ and }\; \forall\rho'(\,\lambda\emptyset\rho' \;\text{ implies }\; \rho \;\preceq\; \rho'\,)\,)$$

has shown that such a costraint holds in the model of prime filters if and only if the left linear negation ${}^*(\;\;)$ transforms finite conjunctions into disjunctions:

$${}^*(p_1 \wedge \ldots \wedge p_n) \;\;=\;\; {}^*p_1 \vee \ldots \vee {}^*p_n$$

This kind of analysis has been conducted over all the semantical conditions which were obtained as the result of the study of a syntactic axiom. If we obtain, as result, again the same axiom, we can conclude that a theory which satisfies that constraint can be extended to a complete one still satisfying that axiom.

Besides the problem of extending a theory to a complete one, we have been interested in developing a general framework for searching syntactic equivalents of interesting semantic conditions. In this case an analogous way of eliminating quantifiers over prime filters needs extensive use of Extension/Exclusion Lemma which requires the axiom of choice.

The last part of the work (Section 6) has been devoted to understand the relations between the various axioms analysed. At the same time, the construction of counterexamples has been the occasion for a concrete insight into this kind of structures.

1 Distributive logic

In this section we provide tools which will let to construct proper universes for distributive linear logic and hence to investigate related syntactic conditions. As semantic concepts related to linear logic have clear significance in the context of the category of the bimodules over partial ordered sets ($\mathcal{BM}$), we give also a brief presentation of it.

1.1 Universes

A universe U for distributive logic is a couple $< S, \ \preceq >$ where

(1) S is a set of states: $\sigma, \lambda, \rho, \tau, \ldots$

(2) $\preceq$ is an order relation between states.

A morphism between two universes U_1, U_2 is a function

$$f : U_1 \rightarrow U_2$$

order preserving:

$$\sigma \ \preceq \ \sigma' \ \text{ implies } \ f(\sigma) \ \preceq \ f(\sigma')$$

We shall denote the category whose objects are universes and whose morphisms are order preserving functions $\mathcal{U}$. This category is a subcategory of the category of bimodules between partial ordered sets $\mathcal{BM}$ which is defined in the following way:

(1) objects are partial ordered sets,

(2) morphisms between two objects are relations satisfying the following conditions:

$$\sigma R \tau \quad \Longrightarrow \quad \begin{array}{ll} (1) & \sigma' \leq \sigma \ \Longrightarrow \ \sigma' R \tau \\ (2) & \tau \leq \tau' \ \Longrightarrow \ \sigma R \tau' \end{array}$$

(3) composition is just relational composition with identities the orders of the sets

Every morphism $f : U \rightarrow U'$ of preorders gives the bimodule $\mathcal{B}$ defined as:

$$\sigma \mathcal{B} \sigma' \quad \text{iff} \quad f(\sigma) \ \preceq \ \sigma'$$

Viceversa if a bimodule $\mathcal{B} : U \rightarrow U'$ is representable, i.e. if there exists a function f such that the previous condition holds, then f is a morphism of preorders.

1.2 Theories

A distributive theory T is just a distributive lattice:

(1) a set P of propositions: $p, q, l, r, t, \ldots$

(2) a relation of provability between propositions $\vdash$

(3) operators on propositions

$$\top, \wedge, \bot, \vee$$

which satisfy usual axioms for distributive lattices.

A syntactic interpretation between two theories T_1, T_2 is a morphism of lattices

$$\mu : T_1 \leftarrow T_2$$

Theories and syntactic interpretations form the category $\mathcal{T}$.

1.3 Models

A semantic interpretation (model) of a theory T in a universe U is a relation

$$\models\ \subseteq U \times T$$

satisfying the following conditions:

$$\begin{array}{llcl}
(\preceq) & \sigma_1 \preceq \sigma_2 & \Longrightarrow & \text{if } \ \sigma_2 \models p \ \text{ then } \ \sigma_1 \models p \\
(\vdash) & p_1 \vdash p_2 & \Longrightarrow & \text{if } \ \sigma \models p_1 \ \text{ then } \ \sigma \models p_2 \\
(\top) & \sigma \models \top & & \\
(\wedge) & \sigma \models p_1 \wedge p_2 & \Longleftrightarrow & \sigma \models p_1 \ \text{ and } \ \sigma \models p_2 \\
(\bot) & \sigma \not\models \bot & & \\
(\vee) & \sigma \models p_1 \vee p_2 & \Longleftrightarrow & \sigma \models p_1 \ \text{ or } \ \sigma \models p_2
\end{array}$$

Let be ${}_U\mathcal{I}_T$ the set of all interpretations of a theory T in a universe U. Then $\mathcal{I}$ gives a bimodule

$$\mathcal{U} \mathrel{+\!\!\!\rightarrow} \mathcal{T}^{op}$$

between the category of universes and the opposite of theories, under the actions

$${}_f{\models} \quad := \quad \sigma' \,{}_f{\models} p \ \text{ iff } \ f(\sigma') \models p$$

for a morphism $f : U' \to U$, and

$$\models_\mu \quad := \quad \sigma \models_\mu p' \ \text{ iff } \ \sigma \models \mu(p)$$

for a syntactic interpretation $\mu : T \leftarrow T'$. The conditions for bimodules trivially hold:

$$\begin{array}{lcl@{\qquad\qquad}lcl}
{}_{(fg)}{\models} & = & {}_f({}_g{\models}) & \models_{(\mu\nu)} & = & (\models_\mu)_\nu \\
{}_{Id_U}{\models} & = & \models & \models_{Id_T} & = & \models \\
({}_f{\models})_\mu = {}_f(\models_\mu) & & & & &
\end{array}$$

The bimodule is representable on the left and on the right by a pair of functors, respectively 'zones' and 'points', which are therefore adjoints:

$$\mathcal{U}(U, \bullet T) \quad \cong \quad {}_U\mathcal{I}_T \quad \cong \quad \mathcal{T}^{op}(OU, T)$$

Given a universe U let be OU (zones of U) the lattice of all lower subsets; if $f : U_1 \to U_2$ is morphism of universes then $Of : OU_1 \leftarrow OU_2$ is just the inverse image. The inclusion

$$i : \ OU \ \hookrightarrow \ PU$$

has left and right adjoints

$$\Diamond \ \dashv \ i \ \dashv \ \Box$$

which give the best approximations of a classical attribute by means of an attribute. They are defined in the following ways

$$\sigma \in \Diamond p \qquad \text{iff} \qquad \exists\sigma'(\,\sigma \;\preceq\; \sigma'\,,\, \sigma' \in p\,)$$

$$\sigma \in \Box p \qquad \text{iff} \qquad \forall\sigma'(\,\sigma' \;\preceq\; \sigma \Rightarrow \sigma' \in p\,)$$

In the opposite direction the functor 'points' maps a theory T to $\bullet T$, the universe of all prime filters of the lattice T with the ordering given by the relation opposite to inclusion. If $\mu : T_1 \leftarrow T_2$ is a syntactic interpretation then $\bullet\mu : \bullet T_1 \to \bullet T_2$ is the inverse image restricted to prime filters.
Hence a semantic interpretation $\models$ of T in U is just a syntactic interpretation

$$(\;)^v : OU \leftarrow T$$

which assigns to every proposition its 'value' in U,

$$\sigma \in (p)^v \quad \text{iff} \quad \sigma \models p$$

or a morphism of universes

$$(\;)^r : U \to \bullet T$$

i.e. the 'reflection' of a state in the theory:

$$(\sigma)^r = \{\, p \mid \; \sigma \models p \;\;\}$$

The natural isomorphisms above ensure the equivalence of these three points of view.

1.4 Generic model

The universal property of the counity of the adjiunction $O \dashv \bullet$, the Stone map, ensures that $\bullet T$ is a universal model generic with respect to validity. We can state also the completeness theorem using the well-known Extension/Exclusion Lemma (**EEL**), whose proof depends on the Zorn's Lemma.

Lemma 1.1 *Let D be a distributive lattice. If F is a filter and I an ideal in D such that $F \cap I = \emptyset$ then there exists a prime filter π in D such that $\pi \supseteq F$ and $\pi \cap I = \emptyset$.*

Proposition 1.2 *Let be p, q two propositions of the theory T. If $p \;\nvdash\; q$, then there exists a state $\pi \in \bullet T$ such that $\pi \models p$ and $\pi \not\models q$.*

Call $\uparrow p$ and $\downarrow q$ the filter and the ideal generated respectively by p and q. If $p \;\nvdash\; q$ then $\uparrow p \cap \downarrow q = \emptyset$, by the lemma there exists a prime filter π such that $p \in \pi$ and $q \notin \pi$.

2 Distributive linear logic

2.1 Linear operators and axioms

We can enrich the structure of a theory with a binary operator $\cdot$ with right adjoints in each variable:

$$\begin{array}{ll} (\cdot) & l \;\vdash\; l',\;\; r \;\vdash\; r' \;\; \text{implies} \;\; l \cdot r \;\vdash\; l' \cdot r' \\ (\backslash) & l \cdot r \;\vdash\; t \;\; \text{iff} \;\; r \;\vdash\; l\backslash t \\ (/) & l \cdot r \;\vdash\; t \;\; \text{iff} \;\; l \;\vdash\; t/r \end{array}$$

The binary operator $\cdot$ is a linear conjunction if it satisfies also the following conditions of associativity and unity:

$$\begin{array}{ll} (\alpha) & (l \cdot c) \cdot r = l \cdot (c \cdot r) \\ (1) & 1 \cdot p = p = p \cdot 1 \end{array}$$

2.2 Universes, theories and generic model

In a theory of lower subsets OU over a universe U, a binary operator which satisfies the first free axioms is exactly a function preserving arbitrary sups in each place, i.e. a morphism $\cdot : OU \otimes OU \to OU$ in the category $\mathcal{SL}$ of sup-lattices. There is a natural bijection between such morphisms and morphisms $U \otimes U \leftarrow U$ in the category $\mathcal{BM}$ of bimodules over partial ordered sets. The bifunctor $\otimes$ associates to a pair of objects its product in $\mathcal{PO}$ and to a pair of bimodules $\mathcal{B}$, $\mathcal{B}'$ the conjunction of them:

$$< x, x' > \mathcal{B} \otimes \mathcal{B}' < y, y' > \quad \text{iff} \quad x\mathcal{B}y \;\text{ and }\; x'\mathcal{B}'y'$$

With this product the category $\mathcal{BM}$ is a monoidal one, the unity being the one object set with its unique ordering. The bijection maps such a linear conjunction to the bimodule $|$ defined as

$$\tau \mid \lambda\rho \quad \text{iff} \quad \tau \in \Diamond\{\lambda\} \cdot \Diamond\{\rho\}$$

The inverse map gives the product $\cdot$

$$\tau \in l \cdot r \quad \text{iff} \quad \exists \lambda, \rho(\, \tau \mid \lambda\rho \,,\, \lambda \in l \,,\, \rho \in r \,)$$

out of the bimodule $|$, its adjoints being respectively

$$\begin{array}{rcl} l \backslash t & = & \{\, \rho \,|\, \forall \lambda, \tau(\, \lambda \in l \,,\, \tau \mid \lambda\rho \,\Rightarrow\, \tau \in t \,)\} \\ t / r & = & \{\, \lambda \,|\, \forall \tau, \rho(\, \rho \in r \,,\, \tau \mid \lambda\rho \,\Rightarrow\, \tau \in t \,)\} \end{array}$$

The considerations above let us to state a universe for distributive logic with a binary operator with right adjoint in each place is a usual universe with an additional ternary relation $|$ satisfying usual conditions for morphisms in $\mathcal{BM}$:

$$\begin{array}{ll} \tau \mid \lambda\rho & \text{implies} \\ \quad (1) & \tau' \preceq \tau \;\text{ implies }\; \tau' \mid \lambda\rho \\ \quad (2) & \lambda \preceq \lambda' \text{ and } \rho \preceq \rho' \;\text{ implies }\; \tau \mid \lambda'\rho' \end{array}$$

Obviously if we also consider a unity in OU we must add to the universe an attribute 1 stable with respect to the past:

$$\text{if} \quad \iota \in 1 \,,\, \iota' \preceq \iota \quad \text{then} \quad \iota' \in 1$$

We have seen previously that a semantic interpretation of T in U is just a morphism $\mu : T \to OU$ in the category of theories or a morphism $f : \bullet T \leftarrow U$ in the category of universes, this observation will leads us to a choice of morphisms. We shall require arrows in the category of universes to preserve the relations: if $f : U_1 \to U_2$ then

$$\begin{array}{rcl} \tau \mid \lambda\rho & \text{implies} & f(\tau) \mid f(\lambda) f(\rho) \\ \iota \in 1 & \text{implies} & f(\iota) \in 1 \end{array}$$

The requirement to preserve the relations over a morphism of universe f is equivalent to semipreservation of the operators by the inverse image Of. Hence, in the category of theories, we will ask syntactic interpretations only to semipreserve operators:

$$\begin{array}{r}\mu(l)\cdot\mu(r)\ \vdash\ \mu(l\cdot r)\\ 1\ \vdash\ \mu(1)\end{array}$$

In the universe of prime filters over a theory with such a conjunction we have canonical relations defined by:

$$\begin{array}{rcl}\tau \mid \lambda\rho & \text{iff} & \tau \supseteq \lambda\cdot\rho\\ \tau\in 1 & \text{iff} & 1\in\tau\end{array}$$

where $L\cdot R = \{\,l\cdot r\,|\,l\in L\,,\,r\in R\}$ is the usual product in the power set. We have defined hence two categories and the functor 'zones', in order to show that the universe of prime filters is still a generic model we must show the Stone map gives the counity of the adjunction. This is done showing that a value assignment $(\,)^v : OU \leftarrow T$ is a semantic interpretation, i.e. it semipreserves operators, if and only if the reflection $(\,)^r : U \rightarrow \bullet T$ preserves relations:

$$\begin{array}{c}
\forall l, r\ \underline{\qquad (l)^v\cdot(r)^v\ \vdash\ (l\cdot r)^v \qquad}\\
\forall\tau(\underline{\qquad \tau\in(l)^v\cdot(r)^v \qquad}\Rightarrow\underline{\tau\in(l\cdot r)^v})\\
\exists\lambda,\rho(\tau\mid\lambda\rho\,,\,\underline{\lambda\in(l)^v}\,,\,\underline{\rho\in(r)^v})\qquad l\cdot r\in(\tau)^r\\
l\in(\lambda)^r\quad r\in(\rho)^r\\
\hline
\forall\tau,\lambda,\rho(\tau\mid\lambda\rho\Rightarrow\underline{\forall l,r(l\in(\lambda)^r\,,\,r\in(\rho)^r\ \Rightarrow\ l\cdot r\in(\tau)^r)})\\
(\tau)^r\supseteq(\lambda)^r\cdot(\rho)^r
\end{array}$$

similarly,

$$\begin{array}{c}
1\ \vdash\ (1)^v\\
\hline
\forall\iota(\iota\in 1\Rightarrow\underline{\iota\in(1)^v})\\
\underline{1\in(\iota)^r}\\
(\iota)^r\in 1
\end{array}$$

2.3 Models

We are led thus to extend the definition of semantic interpretation in order to respect the choice of morphisms we have done:

$$\begin{array}{lll}
(\cdot) & \tau\mid\lambda\rho\,,\,\lambda\models l\,,\,\rho\models r & \Longrightarrow\ \ \tau\models l\cdot r\\
(1) & \iota\in 1 & \Longrightarrow\ \ \iota\models 1\\
(\backslash) & \rho\models l\backslash t & \Longrightarrow\ \ \forall\lambda,\tau(\,\lambda\models l\,,\,\tau\mid\lambda\rho\ \text{ implies }\ \tau\models t\,)\\
(/) & \lambda\models t/r & \Longrightarrow\ \ \forall\tau,\rho(\,\rho\models r\,,\,\tau\mid\lambda\rho\ \text{ implies }\ \tau\models t\,)
\end{array}$$

Note that the two last conditions assert

$$(l\backslash t)^v\ \vdash\ (l)^v\backslash(t)^v\,,\quad (t/r)^v\ \vdash\ (t)^v/(r)^v$$

which are equivalent to semipreservation of linear conjunction.

2.4 Specularity

A linear theory is a 2-category with just one object, hence we have to deal with two kind of dualities, one obtained reversing 2-cells, i.e. the usual lattice duality, the second obtained reversing 1-cells. We shall refer to the latter as specularity.

Lemma 2.1 *Let be T a theory with operators $\cdot, \backslash, /$ satisfying the first three axioms. Then T is endowed with another structure (the specular)*

$$\odot \quad , \quad \backslash\!\!\backslash \quad , \quad /\!\!/$$

satisfying same axioms and defined respectively as:

$$\begin{array}{rcl} l\odot r & = & r\cdot l \\ \backslash\!\!\backslash & = & / \\ /\!\!/ & = & \backslash \end{array}$$

Lemma 2.2 *In a theory OU over a universe U the specular linear structure induces a specular ternary relation $\|$ which is related to the original one $\mid$ by the following equivalence:*

$$\tau\|\lambda\rho \quad \textit{iff} \quad \tau \mid \rho\lambda$$

The proof is immediate: $\tau\|\lambda\rho$ iff $\tau \in \Diamond\{\lambda\}\odot\Diamond\{\rho\} = \Diamond\{\rho\}\cdot\Diamond\{\lambda\}$ iff $\tau \mid \rho\lambda$.

Lemma 2.3 *In a universe of prime filters $\bullet T$ the specular relation $\|$ associated to the theory specular of $O\bullet T$, is exactly the ternary canonical relation associated to the theory specular of T.*

Proof.

$$\begin{array}{c} \dfrac{\tau\|\lambda\rho}{\tau \mid \rho\lambda} \\ \hline \tau \supseteq \rho\cdot\lambda \\ \hline \forall l, r(\, l\in\lambda\,,\, r\in\rho \Rightarrow \dfrac{r\cdot l\in\tau}{l\odot r\in\tau}\,) \\ \hline \tau\supseteq\lambda\odot\rho \end{array}$$

□

2.5 Monoidal structure

We will show now that the associativity and unity axioms impose the universe to be a comonoid in the category of bimodules over partial ordered sets. We can revert the order and the direction of the arrow to obtain a monoid in $\mathcal{BM}$ satisfying usual commutativity diagrams.

Proposition 2.4 *The associativity law holds in a theory OU of lower sets iff the following condition holds in U:*

$$\exists\zeta(\,\tau\mid\zeta\rho\,,\,\zeta\mid\lambda\gamma\,) \quad \textit{iff} \quad \exists\xi(\,\tau\mid\lambda\xi\,,\,\xi\mid\gamma\rho\,)$$

Proof. We use as usual lines for equivalences.

$$\dfrac{\forall l,c,r\Big(\dfrac{(l\cdot c)\cdot r \ \vdash\ l\cdot(c\cdot r)}{\forall\tau\Big(\dfrac{\tau\in(l\cdot c)\cdot r}{\exists\zeta,\rho(\tau\mid\zeta\rho\,,\ \dfrac{\zeta\in l\cdot c}{\exists\lambda,\gamma(\zeta\mid\lambda\gamma\,,\ \lambda\in l\,,\ \gamma\in c)}\,,\ \rho\in r)}\Rightarrow\tau\in l\cdot(c\cdot r)\Big)}\Big)}{\forall\tau,\lambda,\gamma,\rho,\zeta((\tau\mid\zeta\rho,\zeta\mid\lambda\gamma)\Rightarrow\dfrac{\forall l,c,r((\lambda\in l,\gamma\in c,\rho\in r)\Rightarrow\tau\in l\cdot(c\cdot r)))}{\dfrac{\tau\in\Diamond\{\lambda\}\cdot(\Diamond\{\gamma\}\cdot\Diamond\{\rho\})}{\exists\xi(\tau\mid\lambda\xi,\xi\mid\gamma\rho)}}}$$

Specular considerations show that

$$l\cdot(c\cdot r)\ \vdash\ (l\cdot c)\cdot r$$

is equivalent to

$$\exists\xi(\,\tau\mid\lambda\xi\,,\ \xi\mid\gamma\rho\,)\quad\text{implies}\quad\exists\zeta(\,\tau\mid\zeta\rho\,,\ \zeta\mid\lambda\gamma\,)$$

□

Proposition 2.5 *The unity laws hold in a theory OU of lower sets iff the following condition holds in U:*

$$\begin{array}{cc}\exists\iota(\,\iota\in 1\,,\ \tau\mid\iota\sigma\,) & \textit{iff}\\ \tau\ \preceq\ \sigma & \textit{iff}\\ \exists\iota'(\,\iota'\in 1\,,\ \tau\mid\sigma\iota'\,) & \end{array}$$

Proof.

$$\dfrac{\forall p\Big(\dfrac{p\cdot 1\ \vdash\ p}{\forall\tau\Big(\dfrac{\tau\in p\cdot 1}{\exists\sigma,\iota(\,\tau\mid\sigma\iota\,,\ \sigma\in p,\iota\in 1\,)}\Rightarrow\tau\in p\Big)}\Big)}{\forall\tau,\sigma,\iota(\,(\,\tau\mid\sigma\iota\,,\ \iota\in 1\,)\Rightarrow\dfrac{\forall p(\,\sigma\in p\Rightarrow\tau\in p\,)}{\dfrac{\tau\in\Diamond\{\sigma\}}{\tau\ \preceq\ \sigma}}\,)}$$

$$\dfrac{\forall p\Big(\dfrac{p\cdot 1\ \vdash\ p}{\forall\tau(\,\tau\in p\Rightarrow\tau\in p\cdot 1\,)}\Big)}{\forall\tau\Big(\dfrac{\dfrac{\tau\in\Diamond\{\tau\}\cdot 1}{\exists\sigma,\iota(\,\tau\mid\sigma\iota,\dfrac{\sigma\in\Diamond\{\tau\}}{\sigma\ \preceq\ \tau}\,,\ \iota\in 1\,)}}{\exists\iota(\,\tau\mid\tau\iota\,,\ \iota\in 1\,)}\Big)}$$

This is all we need for our proof since specular considerations show the equivalence of the specular condition

$$1\cdot p = p$$

with semantic conditions obtained exchanging λ with ρ in the ternary relation. □

This is not sufficient enough to prove completeness of the logic with respect to these structures. We must show that the validity in every model, i.e. in the model of prime filters, is equivalent to state the syntactic conditions hold in the theory. This is done using the **EEL** to eliminate quantifications on prime filters.

Proposition 2.6 *The associativity condition holds in a universe of prime filters $\bullet T$ iff the associativity condition holds in T.*

To prove this proposition we will need few lemmas.

Lemma 2.7 *If b is a base of a filter and c is a complement of an ideal then $b\backslash c$ and c/b are complement of ideals, where $\backslash$ and $/$ are the two adjoints to the operator $\cdot$ in the theory of subset of T.*

Remember that a base of filter b is a subset of T such that finite intersections of elements in b have a lower bound in b. We can combine this lemma with **EEL** to obtain the main result we need:

Lemma 2.8 *Let be b_1, b_2 two bases of filter and ϕ the complement of an ideal. If $b_1 \cdot b_2 \subseteq \phi$ then there exist two prime filters β_1, β_2 such that $b_1 \subseteq \beta_1, b_2 \subseteq \beta_2$ and $\beta_1 \cdot \beta_2 \subseteq \phi$.*

Proof. Because of the Lemma we can immediately assert that condition

$$\exists\zeta(\tau \supseteq \zeta \cdot \rho\,,\, \zeta \supseteq \lambda \cdot \gamma)$$

is equivalent to:

$$\tau \supseteq (\lambda \cdot \gamma) \cdot \rho$$

and similarly for the other part of the condition. We obtain then for semi-associativity

$$\forall\tau, \lambda, \gamma, \rho(\tau \supseteq (\lambda \cdot \gamma) \cdot \rho \quad \text{implies} \quad \tau \supseteq \lambda \cdot (\gamma \cdot \rho))$$

Since the second half of the above implication asserts that

$$\forall l, c, r(\, l \in \lambda\,,\, c \in \gamma\,,\, r \in \rho \Rightarrow l \cdot (c \cdot r) \in \tau\,)$$

we can elaborate all the condition to obtain:

$$\forall\tau, l, c, r(\, \underset{(l \cdot c) \cdot r \in \tau}{\underline{\exists\lambda, \gamma, \rho(\, l \in \lambda\,,\, c \in \gamma\,,\, r \in \rho\,,\, \tau \supseteq (\lambda \cdot \gamma) \cdot \rho\,)}} \Rightarrow l \cdot (c \cdot r) \in \tau\,)$$

Specular calculations show that the other implication gives rise to the other part of the associativity. $\square$

Proposition 2.9 *The unity condition holds in a universe of prime filters $\bullet T$ iff the unity conditions hold in T.*

Proof.

$$\dfrac{\dfrac{\forall\tau, \lambda(\,\exists\iota(\,\tau \supseteq \lambda \cdot \iota\,,\, 1 \in \iota\,) \Rightarrow \dfrac{\tau \supseteq \lambda}{\forall l(\, l \in \lambda \Rightarrow l \in \tau\,)}\,)}{\forall\tau, l(\, \underset{l \cdot 1 \in \tau}{\underline{\exists\iota(\,\tau \supseteq \lambda \cdot \iota\,,\, l \in \lambda\,,\, 1 \in \iota\,)}} \Rightarrow l \in \tau\,)}}{\forall l(\, l \cdot 1\, \vdash\, l\,)}$$

$$\dfrac{\forall\tau\ \dfrac{\dfrac{\exists\iota(\tau \supseteq \tau \cdot \iota\,,\, 1 \in \iota)}{\tau \supseteq \tau \cdot \{1\}}}{\forall t(t \in \tau \Rightarrow t \cdot 1 \in \tau)}}{\forall t(t\, \vdash\, t \cdot 1)}$$

$\square$

3 From Syntax to Semantics

In this section we analyze syntactic conditions which are related to classical linear logic. It is well known that, once we have chosen in an arbitrary way a false among propositions, we can define a negation in the usual way using linear implication:

$$p^* = p\backslash 0$$

Since the context is not commutative we must define a left negation too:

$$^*p = 0/p$$

Such negations have an 'intuitionistic' behavior, i.e. following rules of weak double negation hold:

$$\begin{array}{l} p \vdash {}^*(p^*) \\ p \vdash ({}^*p)^* \end{array}$$

A classical linear negation satisfies also converse conditions:

$$\begin{array}{l} {}^*(p^*) \vdash p \\ ({}^*p)^* \vdash p \end{array}$$

In this case 0 is said to be a dualizing element.

In a second time we have investigated cyclicity of 0:

$$p^* = {}^*p$$

This just means two linear negations coincide.

Since we have observed that, in the power set of the arrows of a small category, the complement of the set of identities is a dualizing element if and only if the category is a groupoid, we have investigated schemas which are always valid in groupoids with this choice of linear false:

$$\begin{array}{rcl} p \wedge p^* \wedge 1 & \vdash & \perp \\ p + q & = & (p' \cdot q')' \end{array}$$

where the operator + is the usual linear disjunction (par) defined as:

$$p + q = {}^*(q^* \cdot p^*) = ({}^*q \cdot {}^*p)^*$$

3.1 Dualizing element

We proceed by chains of equivalences which lead to discover biimplications between a syntactic condition, which is quantified over propositions, and a semantic condition, quantified over states.

We can start the analysis of conditions defining a dualizing element observing that it is sufficient to examine

$$\forall p(\, 0/(p\backslash 0) \vdash p\,)$$

(and its specular) since condition $p \vdash 0/(p\backslash 0)$ always holds because of the adjunction.

$$\forall p, \lambda\left(\frac{\lambda \in 0/(p\backslash 0) \Rightarrow \lambda \in p}{\dfrac{\dfrac{\lambda \notin p}{p \subseteq \{\lambda\}'}}{p \vdash \boxdot\{\lambda\}'} \Rightarrow \lambda \notin 0/(p\backslash 0)} \right)$$

Since the function $0/(\_\backslash 0)$ is order preserving, $\boxdot\{\lambda\}'$ is the universal solution for quantification over p, hence :

$$\dfrac{\forall\lambda\left(\dfrac{\lambda \notin 0/(\boxdot\{\lambda\}'\backslash 0)}{\neg \dfrac{\lambda \in 0/(\boxdot\{\lambda\}'\backslash 0)}{\forall\tau,\rho(\,\tau \mid \lambda\rho\,,\ \dfrac{\rho \in \boxdot\{\lambda\}'\backslash 0}{\forall\tau',\lambda'(\,\tau' \mid \lambda'\rho\,,\ \dfrac{\lambda' \in \boxdot\{\lambda\}'}{\lambda \not\preceq \lambda'} \Rightarrow \tau' \in 0\,)} \Rightarrow \tau \in 0\,)} } \right)}{\forall\lambda\exists\rho(\,\exists\tau(\,\tau \mid \lambda\rho\,,\ \tau \notin 0\,)\,,\ \forall\lambda'(\,\exists\tau(\,\tau \mid \lambda'\rho\,,\ \tau \notin 0\,) \Rightarrow \lambda \preceq \lambda'\,)\,)}$$

Since the other part of the condition for a dualizing element is specular to the one we have just analyzed, it leads to the specular semantic condition:

$$\forall\rho\exists\lambda(\,\exists\tau(\,\tau \mid \lambda\rho\,,\ \tau \notin 0\,)\,,\ \forall\rho'(\,\exists\tau(\,\tau \mid \lambda\rho'\,,\ \tau \notin 0\,) \Rightarrow \rho \preceq \rho'\,)\,)$$

Define a relation $\emptyset$ as:

$$\lambda\emptyset\rho \quad \text{iff} \quad \exists\tau(\,\tau \mid \lambda\rho\,,\ \tau \notin 0\,)$$

Consider it as a morphism $\emptyset : U^{op} \to U$ in the category $\mathcal{BM}$ of bimodules over partial ordered sets. We have till now obtained that 0 is a dualizing element if and only if the following simplified conditions hold:

$$\forall\lambda\exists\rho(\,\lambda\emptyset\rho\,,\ \forall\lambda'(\,\lambda'\emptyset\rho \Rightarrow \lambda' \leq \lambda\,)\,) \tag{1}$$

$$\forall\rho\exists\lambda(\,\lambda\emptyset\rho\,,\ \forall\rho'(\,\lambda\emptyset\rho' \Rightarrow \rho \leq \rho'\,)\,) \tag{2}$$

We will show that every morphism $\mathcal{B} : A \to B$ in the category $\mathcal{BM}$ which satisfies (1) and (2) is an isomorphism in that category. The proof of this will be done in two steps: we first show that the two conditions are equivalent to asserting that $\mathcal{B}$ is representable by an isomorphism of partial orders, and then we show that every morphism in $\mathcal{BM}$ is representable by an isomorphism of partial orders if and only if it is an isomorphism in $\mathcal{BM}$.

Lemma 3.1 *Every morphism $\mathcal{B} : A \to B$ in the category $\mathcal{BM}$ satisfies (1) and (2) if and only if it is representable by an isomorphism of partial orders.*

Proof. Put R_λ for ρ in (1) and L_ρ for λ in (2). Then the two conditions are:

$$\frac{\lambda'\mathcal{B}R_\lambda}{\lambda' \leq_A \lambda} \qquad\qquad \frac{L_\rho\mathcal{B}\rho'}{\rho \leq_B \rho'}$$

Suppose there are two ρ such that the first condition holds. Let them be respectively $R_\lambda^{'}$ and $R_\lambda^{''}$; from $R_\lambda^{'} \leq_B R_\lambda^{'}$ we deduce $L_{R_\lambda^{'}}\mathcal{B}R_\lambda^{'}$ i.e. $L_{R_\lambda^{'}} \leq_A \lambda$ and hence $L_{R_\lambda^{'}}\mathcal{B}R_\lambda^{''}$ i.e $R_\lambda^{'} \leq_B R_\lambda^{''}$. This is sufficient to prove the uniqueness of R_λ and L_ρ. We now show that $R_{()} \cdot L_{()}$ is the identity of A. From $R_\lambda \leq R_\lambda$ we obtain $L_{R_\lambda}\mathcal{B}R_\lambda$ and then $L_{R_\lambda} \leq \lambda$. From analogous

calculations we have that $R_\lambda \leq R_{L_{R_\lambda}}$ and because of $\lambda \mathcal{B} R_\lambda$ we obtain that $\lambda \mathcal{B} R_{L_{R_\lambda}}$ hence $\lambda \leq L_{R_\lambda}$. Specular reasoning leads to the observation that $L_{()} \cdot R_{()}$ is the identity in B.
We can now show that $\mathcal{B}$ is representable on the right by $R_{()}$ and on the left by $L_{()}$. Since $\lambda \mathcal{B} \rho$ is exactly to $L_{R_\lambda} \mathcal{B} \rho$ we obtain:

$$\lambda \mathcal{B} \rho \quad \text{iff} \quad R_\lambda \leq \rho$$

and similarly:

$$\lambda \mathcal{B} \rho \quad \text{iff} \quad \lambda \leq L_\rho$$

Observe that if $\mathcal{B}$ is representable by f then f preserves the order.
Last, suppose that $f : A \to B$ is an isomorphism of partial orders; put $\lambda \mathcal{B} \rho$ iff $f(\lambda) \leq \rho$ then the conditions holds for $R_\lambda = f(\lambda)$ and $L_\rho = f^{-1}(\rho)$. □

Lemma 3.2 *A morphism $\mathcal{B} : A \to B$ in the category $\mathcal{BM}$ is representable by an isomorphism of partial orders if and only if it is isomorphism in its category.*

Proof. Let $\mathcal{B}$ be such an isomorphism. Since $Id \vdash \mathcal{B} \cdot \mathcal{B}^{-1}$ we obtain that $\forall \lambda \exists \rho (\lambda \mathcal{B} \rho, \rho \mathcal{B}^{-1} \lambda)$, let be R_λ such a ρ. Suppose that $\lambda \mathcal{B} \rho$, from $R_\lambda \mathcal{B}^{-1} \lambda$ we obtain $R_\lambda \mathcal{B}^{-1} \cdot \mathcal{B} \rho$ i.e. $R_\lambda \leq \rho$ hence $\lambda \mathcal{B} \rho$ iff $R_\lambda \leq \rho$. We can construct L reasoning in the same way on $\mathcal{B}^{-1}$. We obtain that $L_{R_\lambda} \leq \lambda$ from $R_\lambda \mathcal{B}^{-1} \lambda$ and $\lambda \leq L_{R_\lambda}$ from $\lambda \mathcal{B} R_\lambda$, $R_\lambda \mathcal{B}^{-1} L_{R_\lambda}$.
It is simple to prove that if $f : A \to B$ is an isomorphism of preorder it gives rise to an isomorphism in $\mathcal{BM}$. Just put $\lambda \mathcal{B} \rho$ iff $f(\lambda) \leq \rho$ and $\rho \mathcal{B}' \lambda$ iff$f^{-1}(\rho) \leq \lambda$. Then from $\exists \rho (f(\lambda) \leq \rho, f^{-1}(\rho) \leq \lambda')$ it follows that $\lambda \leq \lambda'$, and conversely if $\lambda \leq \lambda'$ then $f(\lambda) \leq f(\lambda), f^{-1}(f(\lambda)) \leq \lambda$. □

We can end our observations returning to linear false and relation $\emptyset$. What we have proved is the following

Proposition 3.3 *Let be $0 \in OU$ a lower subset of U. 0 is a dualizing element i.e. it satisfies:*

$$0/(p \backslash 0) = p = (0/p) \backslash 0$$

for all $p \in OU$ if and only if the relation $\emptyset$ defined in this way:

$$\lambda \emptyset \rho \quad \textit{iff} \quad \exists \tau (\tau \mid \lambda \rho\, , \, \tau \notin 0)$$

is an isomorphism $\emptyset : U^{op} \to U$ in the category $\mathcal{BM}$ of bimodules over partial ordered sets. This just means that there exists an isomorphism $R : U^{op} \to U$ of preorders such that the representability condition for the relation $\emptyset$ holds:

$$\exists \tau (\tau \mid \lambda \rho\, , \, \tau \notin 0) \quad \textit{iff} \quad R_{(\lambda)} \preceq \rho \quad \textit{iff} \quad L_{(\rho)} \preceq \lambda$$

where L is the inverse of R.

3.2 Cyclic element

We start analyzing half of the condition of cyclicity:

$$\forall l \left(\frac{l \backslash 0 \ \vdash \ 0/l}{\forall \rho \left(\dfrac{\rho \in l \backslash 0}{\dfrac{\Diamond\{\rho\} \ \vdash \ l \backslash 0}{l \ \vdash \ 0/\Diamond\{\rho\}}} \Rightarrow \rho \in 0/l \right)} \right)$$

Observe that, because linear negation reverts the order, the condition obtained till now is satisfied if an only if it is satisfied for the biggest l satisfying the antecedent. Hence we have:

$$\forall \rho (\, \rho \in 0/(0/\Diamond\{\rho\})\,)$$

i.e.

$$\forall \rho, \rho', \tau (\, (\underbrace{\rho' \in 0/\Diamond\{\rho\}}_{\forall \tau' (\, \tau' \mid \rho' \rho \Rightarrow \tau' \in 0\,)}\, , \tau \mid \rho \rho'\,) \Rightarrow \tau \in 0\,)$$

condition which is immediately recognizable to be equivalent to:

$$\forall \lambda, \rho (\, \exists \tau (\, \tau \mid \lambda \rho\, , \, \tau \notin 0\,) \Rightarrow \exists \tau' (\, \tau' \mid \rho \lambda\, , \, \tau' \notin 0\,)\,) \tag{3}$$

The other half of the condition is just the specular to the one analyzed, hence it leads to the the specular of condition (3) which is exactly (3). Hence we have terminated our job, moreover we have produced a semantic proof of the equivalence between the two halves. We can state then:

Proposition 3.4 *Let be $0 \in OU$ a lower set. 0 is a cyclic element i.e. it satisfies:*

$$p \backslash 0 = 0/p$$

for all $p \in OU$ if and only if the relation $\emptyset$ defined above is symmetric:

$$\lambda \emptyset \rho \quad \textit{iff} \quad \rho \emptyset \lambda$$

We can also observe that if 0 is also dualizing then the following corollary holds.

Corollary 3.5 *Let be $0 \in OU$ a dualizing lower subset. Then 0 is also a cyclic element if and only if the isomorphism $R : U^{op} \to U$ of preorders representing the relation $\emptyset$ is an involution.*

Proof. From $L_\lambda \emptyset \lambda$ we obtain for symmetry $\lambda \emptyset L_\lambda$ i.e. $R_\lambda \leq L_\lambda$ and in a similar way we prove $L_\lambda \leq R_\lambda$. □

3.3 Non contradiction principle and De Morgan laws

We turn on the analysis of the condition

$$p \wedge p^* \wedge 1 \ \vdash \ \bot$$

which we call 'non contradiction principle' because, if the theory is provided also of intuitionistic implication (right adjoint to conjunction), as is the case of theories of lower subsets, then the condition is equivalent to:

$$p \wedge p^* \ \vdash \ \neg 1$$

a principle which merges linear with cartesian operators.

$$
\begin{array}{c}
\forall p\left(\dfrac{p\wedge p^*\wedge 1\ \vdash\ \bot}{\forall\iota\left(\dfrac{\iota\in p\wedge p^*\wedge 1}{\iota\in p\,,\ \iota\in p^*\,,\ \iota\in 1}\Rightarrow \iota\in\bot\right)}\right)\\
\hline
\forall\iota(\,\iota\in 1\Rightarrow\forall p(\,\iota\in p\,,\ \iota\in p^*\Rightarrow\iota\in\bot\,)\,)
\end{array}
$$

We can now focus on the second part of the implication in order to eliminate quantification over p.

$$
\begin{array}{c}
\forall p\left(\dfrac{\iota\in p\,,\ \iota\in p^*\Rightarrow\iota\in\bot}{\dfrac{\iota\in p^*}{\dfrac{\Diamond\!\bullet\{\iota\}\ \vdash\ p^*}{\dfrac{p\cdot\Diamond\!\bullet\{\iota\}\ \vdash\ 0}{p\ \vdash\ 0/\Diamond\!\bullet\{\iota\}}}}\ \Rightarrow\ \dfrac{\iota\notin p}{\dfrac{p\subseteq\{\iota\}'}{p\ \vdash\ \bullet\!\Box\{\iota\}'}}}\right)\\
\hline
{}^*\Diamond\!\bullet\{\iota\}\ \vdash\ \bullet\!\Box\{\iota\}'
\end{array}
$$

As usual ${}^*\Diamond\!\bullet\{\iota\}$ is the universal solution for p. We can now explicate the last condition:

$$
\begin{array}{c}
\forall\lambda\left(\dfrac{\lambda\in{}^*\Diamond\!\bullet\{\iota\}}{\forall\tau(\,\tau\mid\lambda\iota\Rightarrow\tau\in 0\,)}\Rightarrow\dfrac{\lambda\in\bullet\!\Box\{\iota\}'}{\iota\not\leq\lambda}\right)\\
\hline
\iota\leq\lambda\Rightarrow\exists\tau(\,\tau\mid\lambda\iota\,,\ \tau\notin 0)\\
\hline
\exists\tau(\,\tau\mid\iota\iota\,,\ \tau\notin 0\,)
\end{array}
$$

We can end stating:

Proposition 3.6 *Let be $0\in OU$ a lower subset. The linear negation defined over 0 satisfies the following principle of non contradiction:*

$$p\wedge p^*\wedge 1\ \vdash\ \bot$$

for all $p\in OU$ if and only if the relation $\emptyset$ is reflexive over 1:

$$\iota\in 1\quad \textit{implies}\quad \iota\emptyset\iota$$

In the case 0 is a dualizing element there are more implications:

Corollary 3.7 *Let be 0 a dualizing linear false. The non-contradiction principle above is satisfied if and only if every $\iota\in 1$ is a fixed point of the isomorphism R representing the relation $\emptyset$.*

Proof. Observe that the condition of the proposition above states that if $\iota\in 1$ then $R_\iota\ \preceq\ \iota$ and $L_\iota\ \preceq\ \iota$. If $\iota\in 1$ then $R_\iota\in 1$, hence $\iota=L_{R_\iota}\ \preceq\ R_\iota\ \preceq\ \iota$. □

Another consequence of the above condition is the following:

Corollary 3.8 *Suppose that every $\iota\in 1$ is a fixed point of the isomorphism R. Then if $\iota\ \preceq\ \iota'$ with $\iota\,,\ \iota'\in 1$ then $\iota=\iota'$.*

Proof. From $\iota\ \preceq\ \iota'$ we obtain $R_{\iota'}\ \preceq\ R_\iota$, i.e. $\iota'\ \preceq\ \iota$ for the corollary. □

In the following analysis we assume 0 to be a dualizing element, so that the linear disjunction + has all well-known properties of classical linear logic. If U is just a set, i.e. it is ordered by identity, we can define another operator, say $\oplus$, defined as

$$p \oplus q \stackrel{\text{def}}{=} (p' \cdot q')'$$

which has its own properties, for example it preserves infs. We investigate the case in which $+$ and $\oplus$ coincide. Such equality takes the form of a De Morgan law:

$$(p+q)' = p' \cdot q'$$

To proceed in the analysis let briefly investigate the operator $+$. Observe that $\lambda \in {}^*r$ iff $R_\lambda \notin r$ and similarly $\rho \in l^*$ iff $L_\rho \notin l$. We can now proceed on calculations about $+$.

$$\begin{array}{c}
\tau \in l + r = {}^*(r^* \cdot l^*) \\
\hline
R_\tau \notin r^* \cdot l^* \\
\hline
\neg \exists \lambda, \rho (R_\tau \mid \rho\lambda \,,\, \dfrac{\rho \in r^*}{L_\rho \notin r} \,,\, \dfrac{\lambda \in l^*}{L_\lambda \notin l}) \\
\hline
\exists \lambda, \rho (R_\tau \mid R_\rho R_\lambda \,,\, \rho \notin r \,,\, \lambda \notin l) \\
\hline
\forall \lambda, \rho (R_\tau \mid R_\rho R_\lambda \Rightarrow \rho \in r \ \text{ or } \ \lambda \in l)
\end{array}$$

We assume now half of the condition proposed since it is sufficient for the whole principle:

$$\forall l, r (\frac{l + r \ \vdash \ (l' \cdot r')'}{\forall \tau (\tau \in l + r \Rightarrow \tau \in (l' \cdot r')')})$$

We invert implication to proceed on the analysis:

$$\begin{array}{c}
\forall l, r, \tau (\dfrac{\dfrac{\tau \notin (l' \cdot r')'}{\tau \in (l' \cdot r')}}{\exists \lambda, \rho (\tau \mid \lambda\rho \,,\, \lambda \in l' \,,\, \rho \in r')} \Rightarrow \tau \notin l + r) \\
\hline
\forall \tau, \lambda, \rho (\tau \mid \lambda\rho \Rightarrow \dfrac{\forall l, r (\dfrac{\lambda \in l'}{l \ \vdash \ \{\lambda\}'} \,,\, \dfrac{\rho \in r'}{r \ \vdash \ \{\rho\}'} \Rightarrow \tau \notin l + r))}{\tau \notin \{\lambda\}' + \{\rho\}'}
\end{array}$$

The second half of the implication, which has been obtained in the usual way through a universal solution for l and r, is just equivalent to $R_\tau \mid R_\rho R_\lambda$:

$$\begin{array}{c}
\neg \dfrac{\tau \in \{\lambda\}' + \{\rho\}'}{\forall \sigma, \sigma' (R_\tau \mid R_\sigma R_{\sigma'} \Rightarrow \sigma \in \{\rho\}' \ \text{ or } \ \sigma' \in \{\lambda\}')} \\
\hline
\dfrac{\exists \sigma, \sigma' (R_\tau \mid R_\sigma R_{\sigma'} \,,\, \sigma \in \{\rho\} \,,\, \sigma' \in \{\lambda\})}{R_\tau \mid R_\rho R_\lambda}
\end{array}$$

We can end the section stating what we have proved:

Proposition 3.9 *Let be $0 \in OU$ a dualizing lower subset of a universe where the relation $\preceq$ is the identity. Then the linear negation defined over 0 satisfies the following De Morgan law*

$$p + q = (p' \cdot q')'$$

for all $p\,,\, q \in OU$ if and only if the following proposition holds in U:

$$\tau \mid \lambda\rho \quad \textit{implies} \quad R_\tau \mid R_\rho R_\lambda$$

4 Examples

4.1 Universes of pairs

We will focus our attention about a special kind of universes U whose theories OU are relational quantales.
In a closed monoidal category the object A^A is always a monoid with canonical definition of multiplication. In the category $\mathcal{BM}$, since $(\_)^{op} \otimes (\_)$ is right adjoint to the tensor product we have that $A^{op} \otimes A$ is always a monoid in this category (and $A \otimes A^{op}$ a comonoid, i.e. a universe) with canonical compositions defined by:

$$\begin{array}{c} <\tau_1, \tau_2>|<\lambda_1, \lambda_2><\rho_1, \rho_2> \\ \text{iff} \\ \tau_1 \le \lambda_1\,,\ \lambda_2 \le \rho_1\,,\ \rho_2 \le \tau_2 \end{array}$$

$$<\tau_1, \tau_2> \in 1 \quad \text{iff} \quad \tau_1 \le \tau_2$$

In a theory over such a universe the composition is relational composition:

$$\begin{array}{c} <\tau_1, \tau_2> \in L \cdot R \\ \hline \exists \lambda_1, \lambda_2, \rho_1, \rho_2 (\underset{\tau_1 \le \lambda_1\,,\ \lambda_2 \le \rho_1\,,\ \rho_2 \le \tau_2}{\underline{<\tau_1, \tau_2>|<\lambda_1, \lambda_2><\rho_1, \rho_2>}}\,, <\lambda_1, \lambda_2> \in L\,, <\rho_1, \rho_2> \in R) \\ \hline \exists \lambda_2 (\underset{\tau_1 L \lambda_2}{\underline{<\tau_1, \lambda_2> \in L}}\,, \underset{\lambda_2 R \tau_2}{\underline{<\lambda_2, \tau_2> \in R}}) \\ \hline \tau_1 L R \tau_2 \end{array}$$

Similar calculations lead to observe that:

$$\begin{array}{c} \rho_1 (L \backslash T) \rho_2 \\ \text{iff} \\ \forall \lambda (\, \lambda L \rho_1 \Rightarrow \lambda T \rho_2 \,) \end{array}$$

and

$$\begin{array}{c} \lambda_1 (T / R) \lambda_2 \\ \text{iff} \\ \forall \rho (\, \lambda_2 R \rho \Rightarrow \lambda_1 T \rho \,) \end{array}$$

We are going to show that to assign a dualizing attribute 0 in this universe is equivalent to assign an isomorphism of preorder $\between : A \to A$.

Proposition 4.1 *Let be $U \otimes U^{op}$ a universe of pairs and let be $0 \in O(U \otimes U^{op})$ an attribute in this universe. 0 is a dualizing element if and only if the bimodule $\between : U \to U$ defined by:*

$$\lambda \between \rho \quad \textit{iff} \quad <\rho, \lambda> \notin 0$$

is an isomorphism in the category $\mathcal{BM}$, i.e. it is representable by an isomorphism of preorders.

Proof. First of all observe that the relation $\emptyset$ is defined by:

$$\begin{array}{c} <\lambda_1, \lambda_2> \emptyset <\rho_1, \rho_2> \\ \hline \exists \tau_1, \tau_2 (\underset{\tau_1 \le \lambda_1\,,\ \lambda_2 \le \rho_1\,,\ \rho_2 \le \tau_2}{\underline{<\tau_1, \tau_2>|<\lambda_1, \lambda_2><\rho_1, \rho_2>}}\,, <\tau_1, \tau_2> \notin 0\,) \\ \hline <\lambda_1, \rho_2> \notin 0\,,\ \lambda_2 \le \rho_1 \end{array}$$

Hence we analyze the conditions (1) and (2) for $\emptyset$ (see page 12):

$$
\begin{aligned}
&\forall \lambda_1, \lambda_2 \exists \rho_1, \rho_2 (\\
&\qquad < \lambda_1, \rho_2 > \notin 0 \, , \, \lambda_2 \leq \rho_1 \, , \\
&\qquad \forall \lambda_1', \lambda_2' (< \lambda_1', \rho_2 > \notin 0 \, , \, \lambda_2' \leq \rho_1 \Rightarrow \lambda_1 \leq \lambda_1' \, , \, \lambda_2' \leq \lambda_2 \,) \quad)
\end{aligned}
$$

We can analyze the second half of the implication observing the following equivalences:

$$
\forall \lambda_1' (< \lambda_1', \rho_2 > \notin 0 \Rightarrow \frac{\forall \lambda_2' (\, \lambda_2' \leq \rho_1 \Rightarrow \lambda_1 \leq \lambda_1' \, , \, \lambda_2' \leq \lambda_2 \,)}{\lambda_1 \leq \lambda_1' \, , \, \rho_1 \leq \lambda_2})
$$

We have found just now that:

$$
\forall \lambda_1, \lambda_2 \exists \rho_2 (\quad < \lambda_1, \rho_2 > \notin 0 \, , \dfrac{\exists \rho_1 (\, \lambda_2 \leq \rho_1 \, , \, \forall \lambda_1' (< \lambda_1', \rho_2 > \notin 0 \Rightarrow \lambda_1 \leq \lambda_1' \, , \, \rho_1 \leq \lambda_2 \,) \,)}{\forall \lambda_1' (< \lambda_1', \rho_2 > \notin 0 \Rightarrow \dfrac{\lambda_1 \leq \lambda_1' \, , \, \dfrac{\lambda_2 \leq \lambda_2}{true}}{\lambda_1 \leq \lambda_1'})} \quad)
$$

Define a new relation $\bcancel{0} : U \rightarrow U$ as

$$
\lambda \bcancel{0} \rho \quad \text{iff} \quad < \rho, \lambda > \notin 0
$$

We observe that the condition we have found is :

$$
\forall \rho \exists \lambda (\, \lambda \bcancel{0} \rho \, , \, \forall \rho' (\, \lambda \bcancel{0} \rho' \Rightarrow \rho \leq \rho' \,) \,)
$$

and since the analysis of the specular condition on $\emptyset$ leads the specular condition for $\bcancel{0}$ we can state that $\emptyset$ is an isomorphism if and only if $\bcancel{0}$ is. □

We can revert the process and construct a universe for distributive classic-linear logic from a preorder U and an isomorphism of preorders $f : U \rightarrow U$. The isomorphism induces a bimodule, call it $\bcancel{0}$, defined as $\sigma \bcancel{0} \sigma'$ iff $f(\sigma) \leq \sigma'$ which is itself an isomorphism in $\mathcal{BM}$. By the previous arguments we must define 0 as the attribute of all pairs $< \sigma_1, \sigma_2 >$ such that $\sigma_2 \bcancel{0} \sigma_1$ does not hold, i.e. $f(\sigma_2) \not\leq \sigma_1$. The relation $\emptyset$ defined as

$$
\dfrac{< \lambda_1, \lambda_2 > \; \emptyset < \rho_1, \rho_2 >}{\dfrac{\exists \tau_1, \tau_2 (< \tau_1, \tau_2 > | < \lambda_1, \lambda_2 > < \rho_1, \rho_2 > \, , \, < \tau_1, \tau_2 > \notin 0 \,)}{\lambda_2 \leq \rho_1 \, , \, \dfrac{< \lambda_1, \rho_2 > \notin 0}{\dfrac{f(\rho_2) \leq \lambda_1}{f^{-1}(\lambda_1) \leq_{op} \rho_2}}}}
$$

is now clearly representable by the function:

$$
R_{< \lambda_1, \lambda_2 >} = < \lambda_2, f^{-1}(\lambda_1) >
$$

which has as inverse:

$$
L_{< \rho_1, \rho_2 >} = < f(\rho_2), \rho_1 >
$$

We can calculate now linear negations:

$$
\dfrac{\rho_1 (L^*) \rho_2}{\dfrac{L_{< \rho_1, \rho_2 >} \notin L}{\neg (\, f(\rho_2) L \rho_1 \,)}}
$$

and similarly:

$$\frac{\dfrac{\lambda_1({}^*R)\lambda_2}{R_{<\lambda_1,\lambda_2>} \notin R}}{\neg(\,\lambda_2 R f^{-1}(\lambda_1)\,)}$$

Last, note that if $f : U \to U$ is the identity then $R = L$ is the exchange of position between components. If also the order in U is the identity then we obtain the universe for the usual quantale of all relations over a set.

4.2 Small categories

A small category C is a universe putting S to be the set of all arrows of C. The accesibility relation is just the identity, the ternary relation and unary attribute are defined respectively by:

$$\tau \mid \lambda\rho \quad \text{iff} \quad \tau = \lambda \cdot \rho$$

$$\iota \in 1 \quad \text{iff} \quad \exists c \in Obj(C) \;\; \text{such that} \;\; \iota = Id_c$$

Proposition 4.2 *Let be U the set of all arrows of a small category. The set of all arrows which are not identities is a dualizing element if and only if the category is a groupoid.*

Proof. Observe that the relation $\emptyset$ is defined by:

$$\lambda \emptyset \rho \quad \text{iff} \quad \lambda \cdot \rho \in 1$$

Hence if $\emptyset$ is representable with an isomorphism we have that for $\phi : a \to b$, $a, b \in Obj(C)$

$$L_\phi = L_\phi \cdot Id_a = L_\phi \cdot \phi \cdot R_\phi = Id_b \cdot R_\phi = R_\phi$$

hence $L_\phi = \phi^{-1} = R_\phi$. Obviously if C is a groupoid put $L_\phi = R_\phi = \phi^{-1}$. □

4.3 Monoids in presheaves

A monoid in presheaves is just a functor $X : \Sigma \to \mathcal{MON}$ where $\mathcal{MON}$ is the category of monoids in $\mathcal{SET}$ and Σ a small category:

(1) for all objects $\sigma \in \Sigma$ we have a monoid $X(\sigma)$.

(2) if $u : \sigma \to \sigma'$ is an arrow in Σ we have that $X(u) : X(\sigma) \to X(\sigma')$ is a morphism of monoids.

(3) X preserves compositions:

$$Id_{X(\sigma)} = X(Id_\sigma)$$
$$X(u)X(k) = X(uk)$$

Given such a monoid we can construct the associated universe. Let be

$$U' = \sum_{\sigma \in Obj(\Sigma)} X(\sigma)$$

and put

$$a \leq b \text{ iff}$$
$$\exists \alpha, \beta, u : \alpha \leftarrow \beta(a \in X(\alpha) \, , \, b \in X(\beta) \, , \, a = X(u)(b) \,)$$

The relation we have defined is reflexive and transitive: from $a = Id_{X(\alpha)}(a)$ we obtain $a = X(Id_\alpha)(a)$ and hence $a \leq a$; from $a = X(u)(b)$ and $b = X(k)(c)$ we obtain $a = X(u)(X(k)(c))$ i.e. $a = X(uk)(c)$, hence $a \leq c$. We can use the quotient of U' with respect to the symmetric relation:

$$a \equiv a' \quad \text{iff} \quad a \leq a' \text{ and } a' \leq a$$

to obtain the object U whose elements are equivalence classes and the order is defined on representatives:

$$[a] \ \preceq \ [b] \quad \text{iff} \quad a \leq b$$

The unitary attribute will be the set of all unities:

$$[i] \in 1 \quad \text{iff} \quad \exists c \in Obj(C)(\, i = id_{X(c)} \,)$$

The ternary relation is:

$$[t]|[l][r]$$
$$\text{iff}$$
$$\exists \tau, \lambda, \rho, \quad u : \tau \leftarrow \lambda, k : \tau \leftarrow \rho($$
$$t = X(u)(l) \cdot X(k)(r) \, , \, t \in X(\tau) \, , \, l \in X(\lambda) \, , \, r \in X(\rho) \quad)$$

We can use this construction to obtain classical universes where the complement of 1 is a dualizing element. Let be G a group (a groupoid with only one object $*$) and $X : G \rightarrow \mathcal{GRP}$ a representation of G as a group of automorphisms of another group $X(*)$. Since every arrow of G has an inverse we have that $a \leq a'$ iff $a' \leq a$ in U' and the relation $\preceq$ is the identity in U. Actually equivalence classes are orbits under the action of G. We have an involution R in U defined with inverses of $X(*)$:

$$R_{[\alpha]} = [\alpha^{-1}]$$

i.e. $[\alpha]^{-1}$ for simplicity. The function is well defined since if $\alpha = X(g)(\alpha')$ then $\alpha^{-1} = (X(g)(\alpha'))^{-1} = X(g)(\alpha'^{-1})$. Since the relation $[\lambda] \, \emptyset [\rho]$ offers

$$\iota = X(g)(\lambda) \cdot X(g')(\rho)$$

for some $g, g' \in G$, we obtain $X(g')(\rho) = X(g)(\lambda^{-1})$ and $\rho = X(g'^{-1}g)(\lambda^{-1})$ and $[\rho] = [\lambda]^{-1}$. The viceversa is immediate since $\iota = Id(\lambda) \cdot Id(\lambda^{-1})$. The function R then represents the relation $\emptyset$ and obviously it is a bijection which has as inverse itself. In this universe all syntactic conditions we have analyzed holds since we have just seen that 1^c is a dualizing element; observe that from $\tau = X(g)(\lambda) \cdot X(g')(\rho)$ then $\tau^{-1} = X(g')(\rho^{-1}) \cdot X(g)(\lambda^{-1})$, i.e. $\tau \mid \lambda\rho$ implies $R_\tau \mid R_\rho R_\lambda$. This condition, associated to the 'De Morgan Laws' implies all the others (see section 6).

This example generalize the one of no. 4.2: we can repeat all the construction substituting the categories $\mathcal{MON}$ and $\mathcal{GRP}$ with $\mathcal{CAT}$ (the category of small categories) and $\mathcal{GRPD}$

(groupoids) respectively. Example no. 4.2 can be obtained putting the group with just one element in place of G.

At last we can observe that this kind of universe are not representative of all universes. Observe that the following condition:

$$\iota \in 1 \Rightarrow \iota \mid \iota\iota$$

which is equivalent to syntactic condition:

$$p \wedge q \wedge 1 \vdash p \cdot q$$

always holds in these universes. Suppose now that the condition holds in a universe of pairs. From $< \iota_1, \iota_2 > \in 1$, i.e. $\iota_1 \leq \iota_2$, we obtain $< \iota_1, \iota_2 > | < \iota_1, \iota_2 > < \iota_1, \iota_2 >$, in particular $\iota_2 \leq \iota_1$, hence the order is the identity. Chose an order which is not the identity to show that the above condition does not always hold.

5 From semantics to syntax

In this section we assume semantic conditions to be valid in the generic model of a theory. By means of equivalences between formulas which lead to replace quantification on prime filters with quantification on propositions, we show validity of those conditions is equivalent to syntactic schemas to be true in the theory. This work has two purposes:

- Conditions found in correspondence with syntactic schemas in section 3 are analysed and it is shown their equivalence in the model of prime filters with related schemas in the theory: in this way we ensure that every theory in which such schemas hold can be extended to a complete one in which those schemas are still true.
- The process of discovery of equivalent condition between syntax and semantics can be started from semantics.

5.1 Representability of $\emptyset$

We can now focus about the relation $\emptyset$ in the universe of prime filters in order to investigate syntactic equivalents to representability condition for $\emptyset$, and, in a second time, representability by means of an isomorphism. Observe that the relation $\emptyset$ has canonical equivalents in the universe of prime filters.

$$\begin{array}{c}
\lambda \emptyset \rho \\
\hline
\exists \tau (\, \tau \supseteq \lambda \cdot \rho \,, \, 0 \notin \tau \,) \\
\hline
\forall l, r (\, \overline{l \in \lambda \,, \, r \in \rho \Rightarrow l \cdot r \, \nvdash \, 0} \,) \\
\overline{l \in \lambda \,, \, \overline{l \cdot r \, \vdash \, 0} \Rightarrow r \notin \rho} \\
r \, \vdash \, l \backslash 0 \\
\hline
\forall l (\, l \in \lambda \Rightarrow l^* \notin \rho \,)
\end{array}$$

We'll use the following equivalent condition of the one we have obtained:

$$\forall l (\, l^* \in \rho \Rightarrow l \notin \lambda \,)$$

which has as equivalents:

$$\lambda \cap \rho^{*^-} = \emptyset$$
$$\rho^{*^-} \subseteq \lambda^c$$
$$\lambda \subseteq (\rho^c)^{*^-}$$

where the operator $(\_)^{*^-}$ is just the inverse image of the right negation:

$$P^{*^-} = \{\, p \,|\, p^* \in P \,\}$$

and it maps an ideal to a filter and viceversa.
We can also decide to eliminate quantification over l instead of r in the previous calculations; in this case we obtain specular conditions:

$${}^{*^-}\lambda \cap \rho = \emptyset$$
$${}^{*^-}\lambda \subseteq \rho^c$$
$$\rho \subseteq {}^{*^-}(\lambda^c)$$

being the operator ${}^{*^-}(\_)$ the inverse image of the left negation.

We can now proceed on analyzing conditions for right representability of $\emptyset$:

$$\forall\lambda\exists\rho(\,\lambda\emptyset\rho\,,\,\forall\rho'(\,\lambda\emptyset\rho' \Rightarrow \rho \,\preceq\, \rho'\,)\,)$$

We analyze the second half of the condition:

$$\begin{array}{c}
\forall\rho'(\,\lambda\emptyset\rho' \Rightarrow \dfrac{\rho \supseteq \rho'}{\forall r(\,r \in \rho' \Rightarrow r \in \rho\,)}\,)\\
\hline
\forall r(\,\exists\rho'(\,\dfrac{\lambda\emptyset\rho'}{{}^{*^-}\lambda \cap \rho' = \emptyset}\,,\, r \in \rho'\,) \Rightarrow r \in \rho\,)\\
\hline
r \notin {}^{*^-}\lambda\\
\hline
r \notin \rho \Rightarrow r \in {}^{*^-}\lambda
\end{array}$$

We obtain second part of the condition is:

$$\rho^c \subseteq {}^{*^-}\lambda$$

and since the first part is just the opposite relation:

$$\rho^c \supseteq {}^{*^-}\lambda$$

global condition is equivalent to:

$$\forall\lambda\exists\rho(\,\rho^c = {}^{*^-}\lambda\,)$$

i.e. for all λ ${}^{*^-}\lambda$ is a prime ideal. Remember that ${}^{*^-}\lambda$ is an ideal since ${}^*(\,) : T^{op} \to T$ preserves finite infs of T^{op} and the preimage of a filter of T is a filter in T^{op}, i.e. an ideal. Hence it is sufficient to require ${}^{*^-}\lambda$ satisfies conditions for prime ideals:

$$\forall\lambda(\,\top \notin {}^{*^-}\lambda\,)$$
$$\forall\lambda(\,p \wedge q \in {}^{*^-}\lambda \Rightarrow p \in {}^{*^-}\lambda \;\text{ or }\; q \in {}^{*^-}\lambda\,)$$

The first condition asserts that for all λ ${}^*\top \notin \lambda$ i.e. ${}^*\top = \bot$; the second asserts:

$$\frac{\forall\lambda(\, {}^*(p \wedge q) \in \lambda \Rightarrow \dfrac{{}^*p \in \lambda \ \text{ or } \ {}^*q \in \lambda}{{}^*p \vee {}^*q \in \lambda}\,)}{{}^*(p \wedge q) \ \vdash \ {}^*p \vee {}^*q}$$

Since the opposite relation always holds, we can state the following

Proposition 5.1 *The relation $\emptyset$ is representable on the right in a universe of prime filters $\bullet T$ if and only if left negation ${}^*(\_)$ satisfies in the theory T the De Morgan laws:*

$$\begin{aligned} {}^*\top &= \bot \\ {}^*(p \wedge q) &= {}^*p \vee {}^*q \end{aligned}$$

We can use specular considerations to state a dual proposition.

Proposition 5.2 *The relation $\emptyset$ is representable on the left in a universe of prime filters $\bullet T$ if and only if right negation $(\_)^*$ satisfies in the theory T the De Morgan laws:*

$$\begin{aligned} \top^* &= \bot \\ (p \wedge q)^* &= p^* \vee q^* \end{aligned}$$

We can now examine the question about representability by an isomorphism. First note that in the universe of prime filters $\rho = R_\lambda$ if and only if ρ satisfies representability conditions with respect to λ i.e. it satisfies:

$$\lambda \emptyset \rho\,,\ \forall\rho'(\,\lambda\emptyset\rho' \Rightarrow \rho \ \preceq \ \rho'\,)$$

and using previous calculations we obtain:

$$\rho^c = {}^{*^-}\lambda$$

and similarly

$$\lambda = L_\rho \quad \text{iff} \quad \lambda^c = \rho^{*^-}$$

Since R and L represent the same relation L is right adjoint to R so that following conditions hold:

$$L_{R_\lambda} \ \preceq \ \lambda \quad , \quad R_{L_\rho} \ \preceq \ \rho$$

Hence we need to investigate opposite relations

$$\lambda \ \preceq \ L_{R_\lambda} \quad , \quad \rho \ \preceq \ R_{L_\rho}$$

on prime filters.

$$\dfrac{\forall\lambda,\rho(\ \dfrac{\rho = R_\lambda}{\rho^c = {}^{*^-}\lambda} \ \Rightarrow \ \dfrac{\lambda \ \preceq \ L_\rho}{\lambda \supseteq (\rho^{*^-})^c = (\rho^c)^{*^-}}\ )}{\forall\lambda(\dfrac{\lambda \supseteq ({}^{*^-}\lambda)^{*^-}}{\forall l(\, \dfrac{l \in ({}^{*^-}\lambda)^{*^-}}{{}^*(l^*) \in \lambda} \Rightarrow l \in \lambda\,)})}$$

$$\overline{\forall l \quad {}^*(l^*) \ \vdash \ l}$$

We have demonstrated the following

Proposition 5.3 *In the universe of prime filters $\bullet T$ of a theory T the relation $\emptyset$ is bi-representable by a couple $< R, L >$ which are respectively a section and a retraction if and only if negations preserve meets and the left negation is a retraction of the right negation:*

$$^*(p^*) = p$$

The specular proposition is:

Proposition 5.4 *In the universe of prime filters $\bullet T$ of a theory T the relation $\emptyset$ is bi-representable by a couple $< R, L >$ which are respectively a retraction and a section if and only if negations preserve meets and the right negation is a retraction of the left negation:*

$$(^*p)^* = p$$

5.2 Symmetry of $\emptyset$

$$
\begin{array}{c}
\forall\lambda,\rho(\,\lambda\emptyset\rho \Rightarrow \underline{\qquad\rho\emptyset\lambda\qquad}\,) \\
\underline{\lambda^{*^-} \subseteq \rho^c} \\
\forall l(\,l^* \in \lambda \Rightarrow l \notin \rho\,) \\
\hline
\forall l,\rho(\,\underline{\exists\lambda(\,l^* \in \lambda\,,\,\lambda \cap \rho^{*^-} = \emptyset\,)} \Rightarrow l \notin \rho) \\
\underline{l^* \notin \rho^{*^-}} \\
\underline{l^{**} \notin \rho} \\
l \in \rho \Rightarrow l^{**} \in \rho \\
\hline
\forall l \quad l \vdash l^{**}
\end{array}
$$

Observe that the last condition is just equivalent to the cyclicity condition for the two negations. Hence we can state:

Proposition 5.5 *In the universe of prime filters $\bullet T$ of a theory T the relation $\emptyset$ is symmetric if and only if left negation coincides with right negation:*

$$p^* = {}^*p$$

i.e. 0 *is a cyclic element.*

5.3 Reflexivity of $\emptyset$ over 1

$$
\begin{array}{c}
\forall\iota(\,\iota \in 1 \Rightarrow \underline{\qquad\iota\emptyset\iota\qquad}\,) \\
\underline{\iota^{*^-} \subseteq \iota^c} \\
\forall p(\,p^* \in \iota \Rightarrow p \notin \iota\,) \\
\hline
\forall p(\,\exists\iota(\,\underline{1 \in \iota\,,\,p \in \iota\,,\,p^* \in \iota}\,) \Rightarrow false\,) \\
1 \wedge p \wedge p^* \in \iota \\
\hline
1 \wedge p \wedge p^* \vdash \bot
\end{array}
$$

The proposition we have proved is:

Proposition 5.6 *In the universe of prime filters $\bullet T$ of a theory the relation $\emptyset$ is reflexive over* 1 *if and only if non contradiction principle*

$$1 \wedge p \wedge p^* \vdash \bot$$

holds in the theory.

5.4 De Morgan Laws

In section 3 we have investigated a particular De Morgan law which requires classical complement. The semantic condition we found was

$$\tau \mid \lambda\rho \Rightarrow R_\tau \mid R_\rho R_\lambda \tag{4}$$

Condition 4 implies accessibility relation to be the identity, which is proved in the following Lemma; it is known that an Heyting algebra where every prime filter is a maximal filter is a Boolean algebra. Because of these considerations, we are going to analyze condition 4 in the context of universes of prime filters of Boolean theories.

Lemma 5.7 *If condition 4 holds in a universe for distributive classical linear logic then the accessibility relation is the identity.*

Proof. From $\tau \mid \lambda\rho$ we can deduce $L_\rho \mid L_\tau\lambda$ because associativity: observe that from $\omega \mid L_\tau\tau$, $\omega \notin 0$ we deduce that $\exists\zeta$ such that $\omega \mid \zeta\rho$ and $\zeta \mid L_\tau\tau$; but this implies that $\zeta \emptyset \rho$ hence $L_\rho \preceq \zeta$ and $L_\rho \mid L_\tau\tau$.
So if $\iota \in 1$ we deduce $\iota \preceq L_\iota$ from $\iota \mid \iota\iota'$, $\iota' \in 1$, $L'_\iota \mid L_\iota\iota$ and $\iota' \mid R_\iota\iota$ using 4: $\iota \preceq \iota' \preceq R_\iota$ and $\iota \preceq L_\iota$. Hence if $\tau \preceq \tau'$ then there exists $\iota \in 1$ such that $\tau \mid \tau'\iota$ and $\tau \mid \tau' L_\iota$ and using 4 we obtain $R_\tau \mid \iota R_{\tau'}$ so that $R_\tau \preceq R_{\tau'}$ i.e. $\tau' \preceq \tau$. □

We are ready to analyze condition 4 in the spectrum of a Boolean theory $\bullet T$.

$$\forall \tau, \lambda, \rho(\, \tau \supseteq \lambda \cdot \rho \Rightarrow \frac{R_\tau \supseteq R_\rho \cdot R_\lambda}{\forall p, q(\, p \in R_\rho\,,\, q \in R_\lambda \Rightarrow p \cdot q \in R_\tau\,)}\,)$$

We can read the previous condition as:

$$\forall \tau, p, q(\, \exists \lambda, \rho(\, \tau \supseteq \lambda \cdot \rho\,,\, \frac{p \in R_\rho}{\dfrac{p \in {}^{*^-}(\rho^c)}{\dfrac{{}^*p \notin \rho}{({}^*p)' \in \rho}}}\,,\, \frac{q \in R_\lambda}{({}^*q)' \in \lambda}\,) \Rightarrow \frac{p \cdot q \in R_\tau}{({}^*(p \cdot q))' \in \tau}\,)$$

As usual we apply the Extension/Exclusion Lemma:

$$\frac{\forall p, q, \tau(\, \dfrac{\exists \lambda, \rho(\, ({}^*p)' \in \rho\,,\, ({}^*q)' \in \lambda\,,\, \tau \supseteq \lambda \cdot \rho\,)}{({}^*q)' \cdot ({}^*p)' \in \tau} \Rightarrow ({}^*(p \cdot q))' \in \tau\,)}{\forall p, q \quad ({}^*q)' \cdot ({}^*p)' \vdash ({}^*(p \cdot q))'}$$

We have demonstrated the following

Proposition 5.8 *Semantic condition 4 holds in the universe of prime filters $\bullet T$ of a Boolean theory T if and only if the following schema holds in the theory:*

$$^*(p \cdot q) \vdash ((^*q)' \cdot (^*p)')'$$

We can immediately observe that the schema is equivalent to the De Morgan law

$$q + p \vdash (q' \cdot p')'$$

the one we have started from in section 3.

6 Derivability and independence of axioms

In this section we propose syntactic proofs for implications between syntactic conditions we have noticed primarily through semantic considerations. In a second time we construct universes where couples of syntactic conditions are respectively validated and falsified; therefore we conclude that latter syntactic conditions are not derivable from the former.

Proposition 6.1 *Ciclicity of linear false and non contradiction principle*

$$\begin{array}{rcl} p^* & \vdash & ^*p \\ p \wedge p^* \wedge 1 & \vdash & \perp \end{array}$$

can be derived from De Morgan law

$$(p+q)' = p' \cdot q'$$

using properties of classic logic and classic-linear logic.

Proof. Observe that from unicity of neuter element we can obtain $1' = 0$; using the unicity of the complement and De Morgan properties of linear negation we recover $(p^*)' = (p')^*$ and similarly for the other negation. Hence from $p' \vdash p'$ we obtain $1 \vdash p'^* + p' = p^{*\prime} + p'$ i.e. $(p^{*\prime} + p')' = p^* \cdot p \vdash 0$, hence $p^* \vdash {}^*p$.
The non contradiction principle is clearly equivalent to:

$$p \vdash 1 \Rightarrow p \wedge p^* \vdash \perp \tag{5}$$

We are going to show that the demonstrated equation $1' = 0$ is equivalent to the last condition. Obviously condition (5) implies that $1 \wedge 0 \vdash \perp$ and using the duality of the linear negation we obtain also $\top \vdash 1 \vee 0$: hence $1' = 0$.
The viceversa is less obvious. Suppose that $1' = 0$ and $p \vdash 1$ and notice that $^*((p \wedge p^*)') \cdot (p \wedge p^*) \vdash \perp \vdash 0$. For if $q \vdash 1$ we can show that $^*(q') \vdash 1$ then $^*((p \wedge p^*)') \cdot (p \wedge p^*) \vdash 1 \cdot p = p$ and $^*((p \wedge p^*)') \cdot (p \wedge p^*) = (^*(p') \wedge p') \cdot (p \wedge p^*) \vdash p' \cdot 1 = p'$, hence $^*((p \wedge p^*)') \cdot (p \wedge p^*) \vdash p \wedge p' = \perp$.
But if $^*(q') \cdot q \vdash 0$ then $q = \perp$: we obtain immediately $^*(q') \vdash {}^*q$ hence, using duality of linear negation, $q \vdash q'$ and $q \vdash \perp$. □

Proposition 6.2 *Following axioms:*

$$\begin{array}{rcl} p \wedge p^* \wedge 1 & \vdash & \perp \\ p^* & \vdash & {}^*p \\ (p+q)' & = & p' \cdot q' \end{array}$$

are not derivable from

$$^*(p^*) = p = (^*p)^*$$

Proof. Consider the following relational monoid M:

$$\begin{array}{c|cc} & 1 & a \\ \hline 1 & 1 & a \\ a & a & \end{array}$$

where the composition aa is not defined. The composition is obviously associative since if $(xy)z$ is defined there must be at least two neuter elements among x, y, z, hence the above product is just $x(yz)$. Given a set X with a function $f : X \longrightarrow X$, we can define a 'functor' (preserving composition and neuter element) $X : M \to \mathcal{SET}$ being f the image $X(a)$ and the identity the image $X(1)$. Define now a (classical, the order being given by the identity) universe:

$$S = \{ < x, \mu > \mid x \in X \ , \ \mu \in M \}$$

with the ternary relation defined as:

$$\begin{array}{c} < t, \tau > | < l, \lambda > < r, \rho > \quad \text{iff} \\ t = l \ , \ X(\lambda)(l) = r \ , \ \tau = \lambda\rho \end{array}$$

and neuter element being

$$1 = \{ < x, 1 > \mid x \in X \}$$

The associativity condition holds since if $< t, \tau > | < x, \xi > < r, \rho >$ and $< x, \xi > | < l, \lambda > < g, \gamma >$ then $t = x = l$, $X(\lambda)(l) = g$ and $\tau = \xi\rho = (\lambda\gamma)\rho = \lambda(\gamma\rho)$.
Therefore $< t, \tau > | < l, \lambda > < g, \gamma\rho >$ and $< g, \gamma\rho > | < g, \gamma > < X(\gamma)(g), \rho >$ i.e. $< g, \gamma\rho > | < g, \gamma > < r, \rho >$ since $r = X(\xi)(l) = X(\lambda\gamma)(l) = X(\gamma)(X(\lambda)(l)) = X(\gamma)(g)$.
Also unity conditions hold: observe that if $< t, \tau > | < l, \lambda > < i, 1 >$ then $t = l$ and $\tau = \lambda 1 = \lambda$ and obviously $< t, \tau > | < t, \tau > < X(\tau)(t), 1 >$ holds.

Let be now $\overline{()} : M \to M$ the exchange between the elements of M. Define a function $R : S \to S$ as

$$R_{< s, \sigma >} = < X(\sigma)(s), \overline{\sigma} >$$

Then R is a bijection iff $X(a)$ is. Suppose that $< s', \overline{\sigma} >$ has a preimage $< s, \sigma >$, then $X(\sigma)(s) = s'$, i.e. for all σ $X(\sigma)$ is surjective; now suppose that $X(\overline{\sigma})(s) = X(\overline{\sigma})(s')$, then $R_{< s, \sigma >} = R_{< s', \sigma >}$ hence $< s, \sigma > = < s', \sigma >$ and $s = s'$. Since every $m \in M$ is an $\overline{m'}$ for some m', then every $X(\sigma)$ must be injective. Easy calculations show that if $X(a)$ is a bijection then R is.

Let be now $X(a)$ a bijection and $0 = 1$ a chosen linear false. Then the relation $\emptyset$ is defined as:

$$\begin{array}{c} < l, \lambda > \emptyset < r, \rho > \quad \text{iff} \\ \lambda\rho \neq 1 \ , \ X(\lambda)(l) = r \end{array}$$

The first part of the condition asserts that $\overline{\lambda} = \rho$, hence $< r, \rho >= R_{<l,\lambda>}$. If $< r, \rho >= R_{<l,\lambda>}$ then the relation $\emptyset$ holds since $\lambda\overline{\lambda} = a \neq 1$.

In this universe the conditions

$$\begin{array}{rcl} p \wedge p^* \wedge 1 & \vdash & \bot \\ (p+q)' & = & p' \cdot q' \end{array}$$

do not hold since they implies $0 = 1'$ where we have chosen $0 = 1$. To show that condition of cyclicity do not hold we must chose $X(a)$ in a such manner that R is not an involution; it is necessary and sufficient $X(a)$ not to be the identity map. $\square$

We can obtain the following

Corollary 6.3 *De Morgan laws and non contradiction principle cannot be derived from the following conditions*

$$\begin{array}{rcl} {}^*(p^*) & = p = & ({}^*p)^* \\ p^* & \vdash & {}^*p \end{array}$$

choosing $X(a)$ to be the identity map.

Proposition 6.4 *The following axioms:*

$$\begin{array}{rcl} p^* & \vdash & {}^*p \\ (p+q)' & = & p' \cdot q' \end{array}$$

are not derivable from following ones:

$$\begin{array}{rcl} {}^*(p^*) & = p = & ({}^*p)^* \\ p \wedge p^* \wedge 1 & \vdash & \bot \end{array}$$

Proof. We can construct an 'ad hoc' universe to prove the proposition. Chose a set S (ordered by identity), a fixed element $\iota \in S$ which will be the only member of the unitary attribute, and a bijection R such that $R_\iota = \iota$. The linear false well be the complement of unary attribute.
Axioms help us in filling the composition table:

$$\begin{array}{c|cccc} & \iota & \dots & \rho & \dots \\ \hline \iota & & & & \\ \vdots & & & & \\ \lambda & & & \tau & \\ \vdots & & & & \end{array}$$

Because of the unity condition $\exists\iota(\, \tau \mid \iota\rho \,,\, \iota \in 1\,)$ iff $\tau = \rho$ and its specular, fill boxes (λ, ι) and (ι, ρ) as usual only with λ and ρ respectively. Fill all the other boxes with all the elements but the unity. Add it only in boxes (λ, R_λ) because of the axiom $\exists\tau(\, \tau \mid \lambda\rho \,,\, \tau = \iota\,)$ iff $\rho = R_\lambda$. We have to prove only that this composition table is associative which is done observing that the associativity condition on singletons holds:

$$(\{\lambda\} \cdot \{\gamma\}) \cdot \{\rho\} = \{\lambda\} \cdot (\{\gamma\} \cdot \{\rho\})$$

If one of λ, γ, ρ is the unity the condition holds. If none of them is the unity observe that in two steps we always reach the top:

$$\top = \{L_\rho\} \cdot \{\rho\} \subseteq 0 \cdot \{\rho\} \subseteq (\{\lambda\} \cdot \{\gamma\}) \cdot \{\rho\}$$

and similarly

$$\top = \{\lambda\} \cdot \{R_\lambda\} \subseteq \{\lambda\} \cdot 0 \subseteq \{\lambda\} \cdot (\{\gamma\} \cdot \{\rho\})$$

In this universe conditions

$$\begin{array}{rcl} {}^*(p^*) & = p = & ({}^*p)^* \\ p \wedge p^* \wedge 1 & \vdash & \perp \end{array}$$

hold since we have constructed the decomposition relation in a such way that $\emptyset$ is representable by the isomorphism R which have as fixed points the unitary elements. However the other two

$$\begin{array}{rcl} p^* & \vdash & {}^*p \\ (p+q)' & = & p' \cdot q' \end{array}$$

do not hold if R is not an involution. We can chose $S = \{\iota, \alpha, \beta, \gamma\}$ and R to be the permutation $(\iota)(\alpha\,\beta\,\gamma)$. □

Proposition 6.5 *The De Morgan law:*

$$(p+q)' \quad = \quad p' \cdot q'$$

cannot be derived from following set of axioms:

$$\begin{array}{rcl} {}^*(p^*) & = p = & ({}^*p)^* \\ p \wedge p^* \wedge 1 & \vdash & \perp \\ p^* & \vdash & {}^*p \end{array}$$

Proof. We just give the de-composition table for a classical universe $< S, |, 1, 0, R >$ where

- $S = \{\iota, \alpha, \beta\}$
- $1 = \{\iota\}$
- $0 = \{\alpha, \beta\}$
- R is the permutation $(\iota)(\alpha\,\beta)$

$\mid$	ι	α	β
ι	ι	α	β
α	α	β	ι, α
β	β	ι, α	α, β

Observe that $\beta \mid \beta\beta$ but not $R_\beta \mid R_\beta R_\beta$ i.e. $\alpha \mid \alpha\alpha$. We leave the reader to check axioms are satisfied. □

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