Properties of relatively pseudocomplemented directoids

Ivan Chajda¹  Miroslav Kolařík²  Filip Švrček¹

Faculty of Sciences
Palacký University Olomouc, Czech Republic

¹Department of Algebra and Geometry
ivan.chajda, filip.svrcek@upol.cz

²Department of Computer Science
miroslav.kolarik@upol.cz

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Relatively pseudocomplemented lattices and semilattices play an important role in the investigation of intuitionistic logics and their reducts. They were intensively studied by G.T. Jones. The operation of relative pseudocomplementation serves as an algebraic counterpart of the intuitionistic connective implication.

To investigate some more general algebraic systems connected with non-classical logic (as e.g. BCK-algebras, BCI-algebras, etc.), we often study ordered sets which are not necessarily semilattices. However, a bit weaker structure was introduced by J. Ježek and R. Quackenbush as follows.
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Directoids

Definition

By a **directoid** is meant a groupoid $\mathcal{D} = (D; \sqcap)$ satisfying identities

(D1) $x \sqcap x = x$,
(D2) $x \sqcap y = y \sqcap x$,
(D3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$. 

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Properties of relatively pseudocomplemented directoids
Directoids are partially ordered sets

Every directoid $\mathcal{D} = (D; \sqcap)$ can be converted into an ordered set $(D; \leq)$ via

$$x \leq y \iff x \sqcap y = x.$$ 

Every downward directed ordered set $(D; \leq)$ can be organized into a directoid taking

$$x \sqcap y = y \sqcap x \in L(x, y) \iff x \parallel y,$$

$$x \sqcap y = y \sqcap x = x \iff x \leq y.$$
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$$x \sqcap y = y \sqcap x = x \iff x \leq y.$$
Let \((S; \land)\) be a \(\land\)-semilattice, \(a, b \in S\). By a relative pseudocomplement of \(a\) with respect to \(b\) in \(S\) we mean the greatest element among \(x \in S\) satisfying
\[
 a \land x \leq b.
\]

Or equivalently (But only if it exists!), \(a \star b\) is the greatest element among \(x \in S\) satisfying
\[
 a \land x = a \land b.
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Let \((S; \wedge)\) be a \(\wedge\)-semilattice, \(a, b \in S\). By a relative pseudocomplement of \(a\) with respect to \(b\) in \(S\) we mean the greatest element among \(x \in S\) satisfying
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\[ N_5 \]
There is a greatest $x \in N_5$ with $c \land x = c \land a$.
Relative pseudocomplementation on semilattices

There is a greatest $x \in N_5$ with $c \land x = c \land a$.
$c \ast a$ does not exist.
In general,

\[ x \leq y \not\Rightarrow x \sqcap z \leq y \sqcap z. \]

**D**:

There is the greatest \( x \in D \) such that \( a \sqcap x \leq b \).

But \( c \) is not the greatest \( x \in D \) such that \( a \sqcap x = a \sqcap b \).
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Definition (by I. Chajda)

Let $\mathcal{D} = (D; \sqcap)$ be a directoid and $a, b \in D$. An element $x$ is called a relative pseudocomplement of $a$ with respect to $b$ if it is a greatest element of $D$ such that

$$a \sqcap x = a \sqcap b.$$ 

It is denoted by $a \ast b$.

A directoid $\mathcal{D}$ is relatively pseudocomplemented if there exists $a \ast b$ for every $a, b \in D$. 
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A directoid $\mathcal{D}$ is relatively pseudocomplemented if there exists $a \ast b$ for every $a, b \in D$. 

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Properties of relatively pseudocomplemented directoids
Theorem

Let \((D; \sqcap)\) be a directoid and \(\ast\) a binary operation on \(D\). Then \(D = (D; \sqcap, \ast)\) is a relatively pseudocomplemented directoid if and only if it satisfies the following identities

\[
\begin{align*}
(S1) & \quad x \sqcap (x \ast y) = x \sqcap y \\
(S2) & \quad (x \ast y) \sqcap y = y \\
(S3) & \quad x \ast y = x \ast (x \sqcap y) \\
(S4) & \quad x \ast x = y \ast y.
\end{align*}
\]
Theorem

Let \( D = (D; \sqcap, *) \) be an algebra with two binary operations. Then \( D \) is a relatively pseudocomplemented directoid if and only if it satisfies the identities (D2), (D3), (S1), (S2) and (S3), i.e.

(D2) \( x \sqcap y = y \sqcap x \),

(D3) \( x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z \),

(S1) \( x \sqcap (x \ast y) = x \sqcap y \),

(S2) \( (x \ast y) \sqcap y = y \),

(S3) \( x \ast y = x \ast (x \sqcap y) \).

The identities (D2), (D3), (S1), (S2) and (S3) are independent.
Axiom system of RPCD

Theorem

Let $\mathcal{D} = (D; \sqcap, \star)$ be an algebra with two binary operations. Then $\mathcal{D}$ is a relatively pseudocomplemented directoid if and only if it satisfies the identities (D2), (D3), (S1), (S2) and (S3), i.e.

(D2) $x \sqcap y = y \sqcap x$,
(D3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$,
(S1) $x \sqcap (x \star y) = x \sqcap y$,
(S2) $(x \star y) \sqcap y = y$,
(S3) $x \star y = x \star (x \sqcap y)$.

The identities (D2), (D3), (S1), (S2) and (S3) are independent.
1 Introduction

2 Relative pseudocomplement as a residuum

3 Congruence properties

4 References
Adjointness property for RPCD

In relatively pseudocomplemented \(\wedge\)-semilattices,

\[ a \wedge x \leq b \iff x \leq a^* b. \quad (APS) \]

In relatively pseudocomplemented directoids,

\[ a \sqcap x = a \sqcap b \implies x \leq a^* b, \quad (I) \]

but not converselly in general.
In relatively pseudocomplemented $\land$-semilattices,

$$a \land x \leq b \iff x \leq a \ast b.$$ (APS)

In relatively pseudocomplemented directoids,

$$a \sqcap x = a \sqcap b \implies x \leq a \ast b,$$ (I)

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Properties of relatively pseudocomplemented directoids
In relatively pseudocomplemented $\land$-semilattices,

$$a \land x \leq b \iff x \leq a \ast b.$$  \hspace{1cm} (APS)

In relatively pseudocomplemented directoids,

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a \wedge x \leq b \iff x \leq a \ast b. \quad (APS)
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In relatively pseudocomplemented directoids,
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a \sqcap x = a \sqcap b \implies x \leq a \ast b,
\]
but not conversely in general.
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\((APS)\)

In relatively pseudocomplemented directoids,

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In relatively pseudocomplemented \( \wedge \)-semilattices,

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a \wedge x \leq b \iff x \leq a \ast b.
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In relatively pseudocomplemented directoids,

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but not conversely in general.
D is RPCD and $a \star b =$
$\mathcal{D}$ is RPCD and $a * b =$
$\mathcal{D}$ is RPCD and $a \ast b =$
$D$ is RPCD and $a \ast b = d$. Then $x \leq a \ast b$, but $a \sqcap x \neq a \sqcap b$. 

$$a \ast b = d$$

$$c = a \sqcap x$$

$$0 = a \sqcap b = a \sqcap d$$
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$\mathcal{D}$ is RPCD and $a \ast b = d$. Then $x \leq a \ast b$, but $a \sqcap x \neq a \sqcap b$. 

$\mathcal{D}$ : 

\begin{align*}
a \ast b &= d \\
b &= c = a \sqcap x \\
0 &= a \sqcap b = a \sqcap d \\
x \leq a \ast b
\end{align*}
So the right hand side of (I) must be completed to obtain a condition in the form of an equivalence.

Theorem

Let \((D; \sqcap)\) be a directoid and \(\ast\) be a binary operation on \(D\). Then \(\mathcal{D} = (D; \sqcap, \ast)\) is a relatively pseudocomplemented directoid if and only if the following adjointness property holds

\[
(a \sqcap x) = (a \sqcap b) \iff x \leq a \ast b \quad \text{and} \quad a \sqcap (a \ast b) = a \sqcap x. \quad (APD)
\]
The right hand side of (I) must be completed to obtain a condition in the form of an equivalence.

**Theorem**

Let \((D; \sqcap)\) be a directoid and \(\ast\) be a binary operation on \(D\). Then \(D = (D; \sqcap, \ast)\) is a relatively pseudocomplemented directoid if and only if the following adjointness property holds:

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a \sqcap x = a \sqcap b \quad \text{iff} \quad x \leq a \ast b \quad \text{and} \quad a \sqcap (a \ast b) = a \sqcap x. \quad \text{(APD)}
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So the right hand side of \((I)\) must be completed to obtain a condition in the form of an equivalence.

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\begin{align*}
    a \sqcap x &= a \sqcap b \quad \text{iff} \quad x \leq a * b \quad \text{and} \quad a \sqcap (a * b) = a \sqcap x. \quad (APD)
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Outline

1. Introduction
2. Relative pseudocomplement as a residuum
3. Congruence properties
4. References
The terms

\[
\begin{align*}
t_0(x, y, z) & = x, \\
t_1(x, y, z) & = x \sqcap [((z \ast y) \sqcap (x \ast z)) \ast (x \ast y)], \\
t_2(x, y, z) & = x \sqcap (y \ast z), \\
t_3(x, y, z) & = z \sqcap [((z \ast x) \ast (x \ast y)) \ast (z \ast y)], \\
t_4(x, y, z) & = z
\end{align*}
\]

are Jónsson terms proving congruence distributivity of the variety of relatively pseudocomplemented directoids.
Variety of RPCD is CD

Theorem

The terms

\[ t_0(x, y, z) = x, \]
\[ t_1(x, y, z) = x \sqcap (((z*y) \sqcap (x*z)) * (x*y)), \]
\[ t_2(x, y, z) = x \sqcap (y*z), \]
\[ t_3(x, y, z) = z \sqcap (((z*x) * (x*y)) * (z*y)), \]
\[ t_4(x, y, z) = z \]

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t_4(x, y, z) &= z
\end{align*}

are Jónsson terms proving congruence distributivity of the variety of relatively pseudocomplemented directoids.
Variety of RPCD is WR and C3-P

Theorem

Variety of RPCD is weakly regular

\[ r_1(x, y) = (x \ast y) \sqcap (y \ast x) \]

and congruence 3-permutable

\[ p_0(x, y, z) = x, \]
\[ p_1(x, y, z) = x \sqcap (y \ast z), \]
\[ p_2(x, y, z) = z \sqcap (y \ast x), \]
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\[ r_1(x, y) = (x * y) \sqcap (y * x) \]

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  p_1(x, y, z) &= x \sqcap (y * z), \\
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The question of CP for the variety $\mathcal{V}$ of RPCD

- The problem is open.
- We are able to find proper subvarieties of $\mathcal{V}$ which are CP.
- The variety $\mathcal{R}$ of relatively pseudocomplemented semilattices.
- $\mathcal{W}$ . . . relatively pseudocomplemented directoids satisfying the identity

$$((x * y) * y) \sqcap x = x$$

($T$)
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$$\left( (x \ast y) \ast y \right) \sqcap x = x \quad (T)$$
Let \((D; \sqcap, *)\) be a relatively pseudocomplemented directoid, where \(D = \{0, c, a, b, 1\}\) and \(\sqcap, *\) are defined by the following Hasse diagram and the table:

\[
\begin{array}{c|cccccc}
* & 0 & c & a & b & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 \\
c & 0 & 1 & 1 & 1 & 1 \\
a & b & c & 1 & b & 1 \\
b & a & c & a & 1 & 1 \\
1 & 0 & c & a & b & 1 \\
\end{array}
\]

\((D; \sqcap, *) \in \mathcal{W}\), but \((D; \sqcap, *) \notin \mathcal{R}\) since \(a \sqcap b = c \neq a \sqcap b\).
Let \((D; \sqcap, \ast)\) be a relatively pseudocomplemented directoid, where \(D = \{0, c, a, b, 1\}\) and \(\sqcap, \ast\) are defined by the following Hasse diagram and the table:

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\end{array}
\]

\((D; \sqcap, \ast) \in \mathcal{W}, \text{ but } (D; \sqcap, \ast) \not\in \mathcal{R}\) since \(a \land b = c \neq a \sqcap b\).
\( D \subseteq V \); but \((a \ast b) \ast b = d \ast b = b \not\geq a\), so \( D \notin W \).
\[ D \in \mathcal{V}, \text{ but } (a \ast b) \ast b = d \ast b = b \not\geq a, \text{ so } D \notin \mathcal{W} \]
\[ D \in V, \text{ but } (a \ast b) \ast b = d \ast b = b \not\geq a, \text{ so } D \notin W \]
\[ \mathcal{D} : \]

\[ a \ast b = d \]

\[ c = a \cap x \]

\[ 0 = a \cap b = a \cap d \]

\[ \mathcal{D} \in \mathcal{V}, \text{ but } (a \ast b) \ast b = d \ast b = b \not\geq a, \text{ so } \mathcal{D} \not\in \mathcal{W} \]
The variety \( \mathcal{W} \) is CP

Theorem

The variety \( \mathcal{W} \) of the RCPD satisfying \((T)\) is congruence permutable and

\[
p(x, y, z) = ((x \ast y) \ast z) \sqcap ((z \ast y) \ast x)
\]

is its Maltsev term.
The variety $\mathcal{W}$ is CP

**Theorem**

The variety $\mathcal{W}$ of the RCPD satisfying (T) is congruence permutable and

$$p(x, y, z) = ((x \ast y) \ast z) \sqcap ((z \ast y) \ast x)$$

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The variety $\mathcal{W}$ is CP

**Theorem**

The variety $\mathcal{W}$ of the RCPD satisfying $(T)$ is congruence permutable and

$$p(x, y, z) = ((x * y) * z) \sqcap ((z * y) * x)$$

is its Maltsev term.
Subdirectly irreducible members of $\mathcal{V}$

- $\mathcal{D} \oplus 1 \ldots$ the directoid constructed from a directoid $\mathcal{D}$ with a greatest element $q$ by adding a new greatest element 1
- If $\mathcal{D}$ is a RPCD then also $\mathcal{D} \oplus 1$ is.

**Theorem**

For any relatively pseudocomplemented directoid $\mathcal{D}$, the directoid $\mathcal{D} \oplus 1$ is a subdirectly irreducible member of $\mathcal{V}$.

**Corollary**

Every finite chain considered as a RPCD is subdirectly irreducible.
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For any relatively pseudocomplemented directoid $D$, the directoid $D \oplus 1$ is a subdirectly irreducible member of $\mathcal{V}$.

**Corollary**

Every finite chain considered as a RPCD is subdirectly irreducible.
Are there any other SI members in $\mathcal{V}$?

Two non-trivial (congruence) partitions:

$\pi_1 = \{\{0, b\}, \{a\}, \{c, 1\}\}$

$\pi_2 = \{\{0, b\}, \{a, c, 1\}\}$
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