Classifying Unification Problems in Distributive Lattices and Kleene Algebras

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(joint work with Simone Bova)
Preliminaries
Algebraic Unifiers


Given an algebraic language $\mathcal{L}$, a unification problem in the language $\mathcal{L}$ is a finite set of equations $S = \{(s_1, t_1), \ldots, (s_n, t_n)\} \subseteq \text{Term}_2^L$. 
Given an algebraic language $\mathcal{L}$, a **unification problem** in the language $\mathcal{L}$ is a finite set of equations $S = \{(s_1, t_1), \ldots, (s_n, t_n)\} \subseteq \text{Term}^2_\mathcal{L}$.

Given a unification problem $S$ and an equational theory $E$, an **algebraic $E$-unifier** for $S$ is pair $(h, P)$ where $P$ is a projective algebra in the equational class determined by $E$ and $h: Fp(S) \rightarrow P$ is a homomorphism.

If \((h_1, P_1), (h_2, P_2)\) are algebraic \(E\)-unifiers for \(S\), we say that \((h_1, P_1)\) is more general than \((h_2, P_2)\) \(((h_2, P_2) \preccurlyeq (h_1, P_1))\) if there exists a homomorphism \(f : P_1 \to P_2\) such that:

\[
\begin{align*}
Fp(S) & \xrightarrow{h_1} P_1 \\
 & \downarrow h_2 \\
 & \downarrow f \\
 & P_2
\end{align*}
\]
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Algebraic Unifiers

If \((h_1, P_1), (h_2, P_2)\) are algebraic \(E\)-unifiers for \(S\), we say that \((h_1, P_1)\) is **more general** than \((h_2, P_2)\) \(((h_2, P_2) \preccurlyeq (h_1, P_1))\) if there exists a homomorphism \(f: P_1 \to P_2\) such that

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\begin{array}{c}
Fp(S) \\ \downarrow \ h_1 \\
\downarrow \ h_2 \\
\downarrow \ f \\
P_1 \\ \downarrow \ f \\
P_2
\end{array}
\]

We denote by \(\mathcal{U}_E(S)\) the pre-ordered set of algebraic \(E\)-unifiers for \(S\).
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Unification types

A unification problem $S$ in an equational theory $E$ is said to have type:

$\mathcal{U}_E(S)$

1
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$\omega$
Preliminaries

Unification types

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Preliminaries

Unification types

A unification problem $S$ in an equational theory $E$ is said to have type:

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1

$\omega$

$\infty$

0

$\infty$

Nullarity

Classification

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Natural duality

(I) Duals of Projectives

(II): Nullarity

(III): Necessary Conditions for Nullarity

Classification
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- 1 if every unification problem $S$ has type 1,
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- $1$ if every unification problem $S$ has type $1$,
- $\omega$ if every unification problem $S$ has type $\omega$ and at least one $S$ has not unification type $1$, 
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- $0$ if at least one $S$ has unification type 0.
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Duals of Unifiers

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(I) Description of finitely generated projectives
(II) Analysis of the unification type
(III): Causes of nullarity
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Duals of Unifiers

\[ Fp(S) \xrightarrow{u} P \quad \leftrightarrow \quad I \xrightarrow{f} D(Fp(S)) \]
Unifiers through duality

Working strategy

(I) Description of finitely generated projective algebras. *Injective objects.*
Unifiers through duality

Working strategy

(I) Description of finitely generated projective algebras. *Injective objects.*

(II) Analysis of the unification type. *Examples*
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(I) Description of finitely generated projective algebras. *Injective objects.*

(II) Analysis of the unification type. *Examples*

(III) Classification of a given unification problem. *Analysis of the examples*
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A finite bounded distributive lattice $L$ is projective if and only if $\langle J(L), \leq \rangle$ is a lattice.
Bounded Distributive Lattices

(II) Analysis of the unification type

The unification problem \( S = \{ x \land y \approx z \lor t \} \) has nullary unification type.
Bounded Distributive Lattices

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The unification problem $S = \{ x \land y \approx z \lor t \}$ has nullary unification type.
Lemma

Let $S$ be a unification problem in the language of bounded lattices. If there exist $x, a, b, c, d, y \in J(F_p(S))$ satisfying:

(i) $x \leq a, b \leq c, d \leq y$, and

(ii) it does not exist $e \in J(F_p(S))$ such that $a, b \leq e \leq c, d$,

then the unification type of $S$ in the equational theory of distributive lattices is nullary.
Bounded Distributive Lattices

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Theorem

Let $S$ be a unification problem in the language of bounded lattices. Then the unification type of $S$ is:

Unitary if and only if $J(F_p(S))$ is a lattice,
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**Finitary** if and only if for every $x, y \in J(F_p(S))$ the interval $[x, y]$ is a lattice,
Theorem

Let $S$ be a unification problem in the language of bounded lattices. Then the unification type of $S$ is:

- **Unitary** if and only if $\mathcal{J}(F_p(S))$ is a lattice,
- **Finitary** if and only if for every $x, y \in \mathcal{J}(F_p(S))$ the interval $[x, y]$ is a lattice,
- **Nullary** otherwise.
Kleene Algebras

Definition

A Kleene algebra $A = (A, \wedge, \vee, \neg, 0, 1)$ is a bounded distributive lattice equipped with a unary operation, $\neg x$, satisfying:

1. $x = \neg \neg x$,
2. $x \wedge y = \neg (\neg x \vee \neg y)$,
3. $x \wedge \neg x \leq y \vee \neg y$.

0

1
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Definition
A structure $X = \langle X, \leq, \sim, Y, \tau \rangle$ is called a **Kleene space** if it satisfies the following conditions:

(i) $\langle X, \leq, \tau \rangle$ is a Priestley space,
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(i) \( \langle X, \leq, \tau \rangle \) is a Priestley space,
(ii) \( \sim \) is a closed binary relation, i.e., \( \sim \) is a closed subset of \( X^2 \),
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(iii) \( Y \) is a closed subset of \( X \), and

(iv) for every \( x, y, z \in X \):

(a) \( x \sim x \),

(b) if \( x \sim y \) and \( x \in Y \), then \( x \leq y \),

(c) if \( x \sim y \) and \( y \leq z \), then \( z \sim x \).
Kleene Algebras

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\[ K = \{ \leq, \sim, Y \} \]
Kleene Algebras

(I) Duals of Projectives

Theorem

Let $A$ be a finite Kleene Algebra. Then the following statements are equivalent:

(i) $A$ is projective,

(ii) $X_K(A) = \{X_A, \leq_A, \sim_A, Y_A, \tau_A\}$ satisfies the following conditions:

(a) $\langle X_A, \leq_A \rangle$ is a meet semi-lattice,
(b) $Y_A = \text{Max}(\langle X_A, \leq_A \rangle)$,
(c) $X_A$ is 2-conditionally complete,
(d) $x \sim_A y$ if and only if there exists $z \in X_A$ such that $x, y \leq z$. 

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Kleene Algebras

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Kleene Algebras

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Lemma

Let $S$ be a unification problem in the language of Kleene algebras. If there exist $x, a, b, c, d, y, z \in XK(F_p(S))$ satisfying:

(i) $x \leq a, b \leq c, d, c \leq y$ and $d \leq z$,

(ii) $y, z \in Y, and$

(iii) it does not exist $e \in XK(F_p(S))$ such that $a, b \leq e \leq c, d$,

then the unification type of $S$ is nullary.
Kleene Algebras

(III): Necessary Conditions for Nullarity

Lemma

Let $S$ be a unification problem in the language of kleene algebras with. If there exist $w, a, b, c, d, e, f, x, y, z \in XK(F_p(S))$ satisfying:

(i) $w \leq a, b, c; a \leq d, e; b \leq d, f; c \leq e, f; d \leq x; e \leq y; $ and $f \leq z$,

(ii) $x, y, z \in Y$, and

(iii) it does not exists $g \in XK(F_p(S))$ such that $a, b, c \leq g$,

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Theorem

Let $S$ be a unification problem. Then the unification type of $S$ over the equational theory of Kleene algebras is:

- **unitary** if and only if the set
  
  $$K = \{ x \in XK(Fp(S)) \mid \exists y \in Y, x \leq y \}$$

  is a 2-conditionally complete meet semilattice,

- **finitary** if and only if for $x \in K$ the set
  
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Thank you for your attention!

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